Incompressible laminar temperature boundary layer on a body of revolution - the adiabatic case

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Abstract

In the paper the universal governing equations of incompressible laminar temperature boundary layer on the sphere are obtained using the improved method of general similarity for the case of adiabatic boundary conditions. Universal solutions in one parametric approximation for $Pr = 1$ and $Pr = 0.72$ are obtained by numerical integration. Calculated universal functions for temperature boundary layer are presented graphically. As an example eigen-temperature of the sphere are calculated and discussed.

Nomenclature

a_0, b_0	constants
A	dimensionless displacement thickness
B	dimensionless momentum thickness
c	coefficient of thermal conductivity
ET	eigen-temperature
f_1	first form parameter
f_k	set of form parameters

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F	characteristic function
H	displacement/momentum thickness ratio
F	characteristic boundary layer function
K	dimensionless temperature
Pr	Prandtl number
T	temperature
T_∞	temperature of the outer flow (constant)
u, v	velocity components
U_∞	velocity of the outer potential flow
U_0	velocity far afore the body (constant)
x, y	coordinates
δ^*	displacement thickness
δ^{**}	momentum thickness
ζ	dimensionless friction factor
η	dimensionless transversal coordinate
θ_k	recurrent function
ν	kinematics viscosity
ψ	stream function
Φ	universal dimensionless stream function

1   Introduction

Flow around bodies of revolution is very often and important in engineering practice. Defined in meridian plane of a cylindrical coordinate system it is reliable model of fluid flow around drops and bubbles, missiles and planes, as well as flow in pipes and turbo machines. The concept of boundary layer can be applied for the flow around the body of revolution, taking into consideration curvature of the body in circumferential direction.

The case of the dynamic boundary layer on a body of revolution is well known in literature, due to its significance for engineering practice. Using Karman-Pohlhausen [1] method procedure for solving dynamic boundary layer equations on the body of revolution was developed by Schlichting [2], Truckenbrodt [3] and Parr [4]. It is interesting to mention that Saljnikov [5] used Goertler's approach to model treated problem.

The concept of universalisation of the laminar boundary layer equations, introduced by Loitskianskii [6], was applied by Bogdanova [7], for the case of flow around the body of revolution. Complete unversalisation of the equations, proposed by Saljnikov [8], of treated problem was done by Kukic [9], and Bachrun et all [10].

Temperature boundary layer on the axis symmetrical bodies is also interesting for engineering practice, firstly due to the problem of aerodynamic heating of the missiles. Since this problem was treated mainly experimentally, presented paper is attempted to treat it theoretically.

2 Universal equations of dynamic and thermal boundary layer

Starting from the equations for spacious boundary layer and considering posed geometry of the treated problem, presented on the Fig.1, Bogdanova [7] obtained governing equations (continuity and momentum) in the form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2}.$$

$$(1)$$

Temperature can be involved using energy equation in the form:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\nu}{\Pr}\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c}\left(\frac{\partial u}{\partial y}\right)^2. \tag{2}$$

Boundary conditions of the treated problem, considering Fig.1 are:

$$y = 0\,, \quad u = v = 0,\ \frac{\partial T}{\partial y} = 0$$

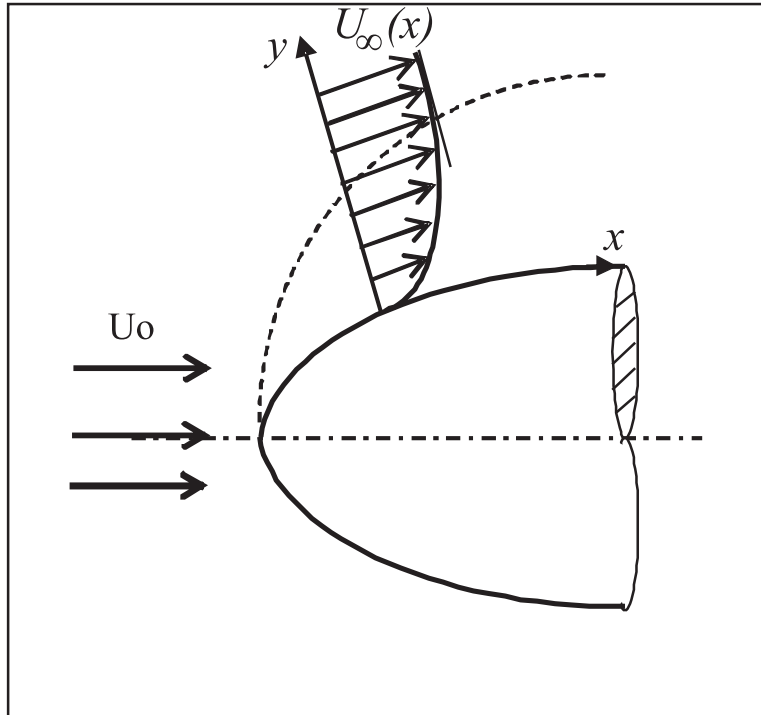$$y \to \infty\,,\ u = U_\infty\left(x\right),\ T = T_\infty \tag{3}$$

Figure 1: Boundary layer on a body of revolution

For the posed system of governing equations (1-3) it is interesting to find universal form so that such equations can be solved once for ever. The solutions of universal equations then can be applied to any particular case. Such approach is of certain theoretical interest, even that posed system can be solved directly for certain particular case. In order to obtain universal form of the governing equations (1-3) Bogdanova's [7] treatment, improved by more appropriate Saljnikov's [8] variables.

Namely, first at all, stream function for incompressible flow is introduced:

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x} \quad . \tag{4}$$

Transversal coordinate y can be transformed in dimensionless form:

$$\eta(x,y) = \frac{U_\infty(x)^{\frac{b_o}{2}} y}{\sqrt{a_0\nu \int\limits_0^x U_\infty(x)^{b_0-1}\, dx}}, \tag{5}$$

so that dimensionless stream function $\Phi(x,\eta)$ could be introduced, in the form:

$$\psi(x,y) = U_\infty(x)^{1-\frac{b_o}{2}} \Phi(x,\eta) \sqrt{a_0\nu \int\limits_0^x U_\infty(x)^{b_0-1}\, dx}. \tag{6}$$

According to the results of Saljnikov [8], values of the constants in (5) and (6) are a_0=0.4408 and b_0=5.714.

For the adiabatic case dimensionless temperature $K(x,\eta,Pr)$ could be introduced in the form:

$$T(x,y,\mathrm{Pr}) = T_\infty + \frac{U_\infty(x)^2}{2c} K(x,\eta,\mathrm{Pr}). \tag{7}$$

Second important transformation according to the Loitskianskii's method of general similarity [6] is in longitudinal direction. Instead of longitudinal coordinate x a set of form parameters f_k of Loitskianskii can be introduced:

$$f_k = U_\infty(x)^{k-1} \frac{d^k U_\infty(x)}{dx^k} \left(\frac{f_1}{\frac{dU_\infty(x)}{dx}}\right)^k, \quad k = 1,2,...\infty. \tag{8}$$

According to (5,6) the first form parameter of the Loitskianskii's set (8) is:

$$f_1 = \frac{a_0 B^2 U'_\infty(x)}{U_\infty(x)^{b_0}} \int\limits_0^\infty U_\infty(x)^{b_0-1} dx. \tag{9}$$

If the velocities of the potential outer flow $U_\infty(x)$ is arbitrary continual function that can be derived k-times, f_k is set of mutually independent functions, dependant on x. Derivatives on x can be replaced using operator:

$$\frac{\partial}{\partial x} = \sum_1^\infty \frac{\partial f_k}{\partial x} \frac{\partial}{\partial f_k} = \frac{U'_\infty}{U_\infty f_1} \sum_1^\infty \Theta_k \frac{\partial}{\partial f_k}, \tag{10}$$

and recurrent function:

$$\Theta_k = [k(f_1 + F) - f_1]f_k + f_{k+1}. \tag{11}$$

In (10) and (11) certain new functions are introduced:

- B - dimensionless momentum thickness:

$$B = \int\limits_0^\infty \frac{\partial \Phi}{\partial \eta}\left(1 - \frac{\partial \Phi}{\partial \eta}\right) d\eta; \tag{12}$$

- A - dimensionless displacement thickness:

$$A = \int\limits_0^\infty \left(1 - \frac{\partial \Phi}{\partial \eta}\right) d\eta; \tag{13}$$

- H - displacement/momentum thickness ratio:

$$H = \frac{\delta^*}{\delta^{**}} = \frac{\int\limits_0^\infty \left(1 - \frac{v_x}{U_\infty}\right) dx}{\int\limits_0^\infty \frac{v_x}{U_\infty}\left(1 - \frac{v_x}{U_\infty}\right) dx} = \frac{A}{B}; \tag{14}$$

- ζ - dimensionless friction factor:

$$\zeta = \left[\frac{\partial\left(u/U_\infty\right)}{\partial\left(y/\delta^{**}\right)}\right]_{y=0} = B[\Phi_{\eta\eta}\left(x,\eta\right)]_{\eta=0};\qquad(15)$$

- F - characteristic function:

$$F = 2[\zeta - (2+H)\,f_1].\qquad(16)$$

Introducing all above mentioned transformations governing set of the equations for dimensionless stream function $\Phi(f_k,\eta)$ and temperature $K(f_k,\eta)$ could be obtained in the universal form:

$$\Phi_{\eta\eta\eta} + \frac{f_1}{B^2}\left(1-\Phi_\eta^2\right) + \frac{a_0 B^2 + (2-b_0)\,f_1}{2B^2}\Phi\Phi_{\eta\eta} =$$
$$\qquad\qquad,\qquad(17)$$
$$= \frac{1}{B^2}\sum_{k=1}^{\infty}\Theta_k\left(\Phi_\eta\Phi_{\eta f_k} - \Phi_{\eta\eta}\Phi_{f_k}\right)$$

$$K_{\eta\eta} + \Pr\frac{a_0 B^2 + (2-b_0)\,f_1}{2B^2}\Phi K_\eta - \Pr\frac{2f_1}{B^2}\Phi_\eta K + 2\Pr\Phi_{\eta\eta}^2 =$$
$$\qquad\qquad.\qquad(18)$$
$$= \frac{\Pr}{B^2}\sum_{k=1}^{\infty}\Theta_k\left(\Phi_\eta K_{f_k} - \Phi_{f_k}K_\eta\right)$$

Corresponding boundary conditions are:

$$\eta = 0:\ \ \Phi = \Phi_\eta = 0\ ,\ \ K_\eta = 0$$
$$\qquad\qquad.\qquad(19)$$
$$\eta \to \infty\ :\ \ \Phi_\eta = 1\ ,\ \ K = 0$$

As it shown by Saljnikov [7] one-parametric approximation can be used. Namely, instead of the set of parameters (8), which generates the set of equations for each f_k, only the first one f_1 (9) can be used, generating only one set of two universal governing equations. Numerical experiments with second and higher order approximations have shown

negligible progress in accuracy in comparison with one-parametric approximation. In this way system of universal equation (17-18) could be written in one-parametric approximation. Namely, introducing in (17-18):

$$f_1 \neq 0,\, f_2 = f_3 = ... = 0 \,;\, \Theta_1 = f_1 F,\, \Theta_2 = \Theta_3 = ... = 0, \qquad (20)$$

universal set of the equations for dimensionless stream function $\Phi(f_1,\eta)$ and temperature $K(f_1,\eta)$ could be transformed in the form:

$$
\begin{aligned}
\Phi_{\eta\eta\eta} + \frac{f_1}{B^2}\left(1 - \Phi_\eta^2\right) + \frac{a_0 B^2 + (2 - b_0)\, f_1}{2B^2} \Phi\Phi_{\eta\eta} = \\
= \frac{f_1 F}{B^2}\left(\Phi_\eta \Phi_{\eta f_1} - \Phi_{\eta\eta}\Phi_{f_1}\right)
\end{aligned}
\qquad , \qquad (21)
$$

$$
\begin{aligned}
K_{\eta\eta} + \Pr \frac{a_0 B^2 + (2 - b_0)\, f_1}{2B^2}\Phi K_\eta - \Pr \frac{2f_1}{B^2}\Phi_\eta K + \\
+ 2\Pr \Phi_{\eta\eta}^2 = \frac{\Pr f_1 F}{B^2}\left(\Phi_\eta K_{f_1} - \Phi_{f_1} K_\eta\right)
\end{aligned}
\qquad . \qquad (22)
$$

Boundary conditions (19) are not affected with this transformation in longitudinal direction.

3   Results

Partial differential equations (21) and (22) for universal dimensionless stream function $\Phi(f_1,\eta)$ and dimensionless temperature $K(f_1,\eta)$, with corresponding boundary conditions (19) are nonlinear partial differential equations without analytic solution, so that numerical treatment of this system is considered necessary. It is evident that equation (21) for universal dimensionless stream function $\Phi(f_1,\eta)$ does not depend on dimensionless temperature $K(f_1,\eta)$, so that it can be solved firstly. Then, taking into consideration obtained results for $\Phi(f_1,\eta)$, equation for $K(f_1,\eta)$ could be solved.

It is important to mention that for $f_1=0$ both equations (21-22) are reduced to ordinary differential equations:

$$\Phi_{\eta\eta\eta} + \frac{a_0}{2}\Phi\Phi_{\eta\eta} = 0, \tag{23}$$

$$K_{\eta\eta} + \Pr\frac{a_0}{2}\Phi K_\eta + 2\Pr\Phi_{\eta\eta}^2 = 0. \tag{24}$$

On the arbitrary body of revolution this section, $f_1=0$, corresponds to the point of minimal pressure/maximal velocity in meridian plane. It evidently solution for flat plate and could be easily obtained. Thus, integration domain of the treated system (21-22) is divided in two parts – first one from $f_1=0$ towards stagnation point ($f_1¿0$), and second one from $f_1=0$ towards separation point ($f_1¡0$).

Numerical integration of the presented system of equation (21-22) was done using finite difference method. For universal dimensionless stream function $\Phi(f_1,\eta)$ at first, order of the equation (21) was reduced introducing $\phi(f_1,\eta) = \Phi_\eta(f_1,\eta)$. Discretisation of the equation (22) was done applying implicit scheme, central for η, backward for f_1. As a result of such procedure, tridiagonal system of algebraic equations was obtained. It was solved using iterative Gaussian elimination procedure. Integration was done in semi-domains ($-0.82\leq f_1 \leq 0$; $0\leq\eta\leq12$) and ($0\leq f_1 \leq+0.82$; $0\leq\eta\leq12$) with steps in longitudinal $\Delta f_1 =0.001$and transversal $\Delta\eta =0.01$ direction. Results for dynamic boundary layer achieved in presented research are with good agreement with previously ones obtained by Kukic [9] and Bachrun et all [10].

It is important to mention that $K(f_1,\eta)$ is not completely universal, since it depends on Prandtl number Pr. In the case for $Pr=1$ for dimensionless temperature function $K(f_1,\eta)$ well known theoretical result could be obtained in the form:

$$K = 1 - \Phi_\eta^2. \tag{25}$$

These results are presented graphically. On the Fig.2 profiles of dimensionless temperature $K(f_1,\eta)$ are presented for different sections of the boundary layer, from stagnation point ($f_1=+0.08$) via point of minimal pressure/maximal velocity ($f_1=0$) and towards separation point ($f_1=-0.08$). Distribution of the dimensionless temperature $K(f_1,\eta)$ along boundary layer is given on the Fig.3. Dimensionless temperature $K(f_1,\eta)$
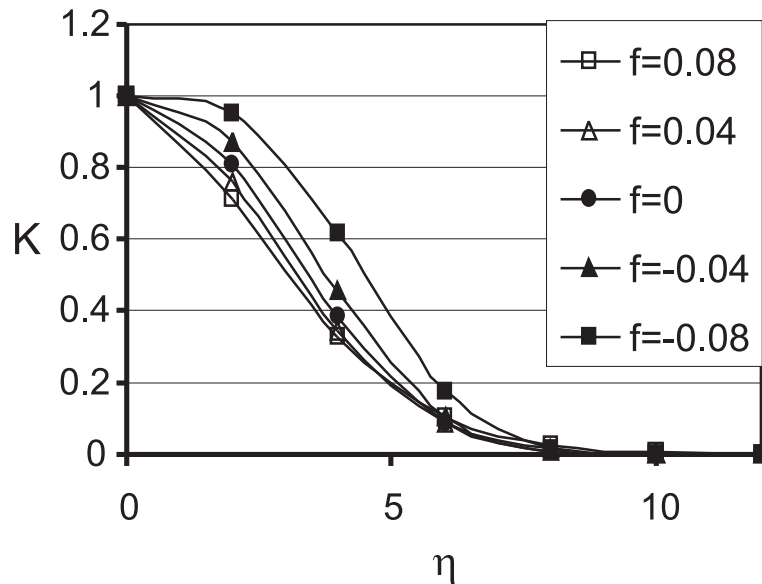
Figure 2: Dimensionless temperature profiles across boundary layer for $Pr = 1$

for Pr$\neq$1 could be obtained in the same way as $\Phi(f_1,\eta)$. It is interesting to mention that complete procedure was accomplished using MS-Excel. Integration was done in the same semi-domains $(-\ -0.82 \leq f_1 \leq 0; 0 \leq \eta \leq 12)$ and $(0 \leq f_1 \leq +0.82; 0 \leq \eta \leq 12)$ with coarser steps in longitudinal $\Delta f_1 = 0.01$ and transversal $\Delta \eta = 0.1$ direction. On the Fig. 4 profiles of dimensionless temperature $K(f_1,\eta)$ are given for different sections of the boundary layer, from stagnation point $(f_1=+0.08)$ towards separation point $(f_1=-0.08)$, and the Fig.5 presents distribution of the dimensionless temperature $K(f_1,\eta)$ along boundary layer. From the Figs.4 and 5 is obvious that function $K(f_1,\eta)$ has greatest values on the surface of the body $(\eta=0)$ and that its values decrease approaching outer border of the boundary layer. Complete numerical results for dy-

namic and temperature boundary layer, for $Pr=1$ and for air, $Pr=0.72$ are given in [11].
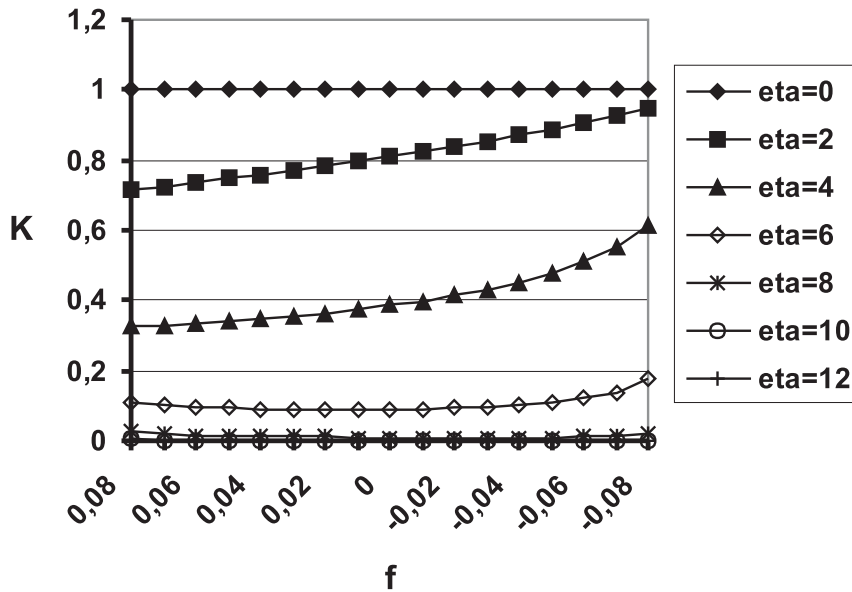

Figure 3: Dimensionless temperature distribution along boundary layer for $Pr = 1$

4 Example

Obtained universal numerical results can be applied to calculate thermal boundary layer on particular body of revolution. As it is shown by Saljnikov [8] for further application of universal function most important is the function $f_1/(B^2)$:

$$\frac{f_1}{B^2} = \frac{a_0 U'_\infty (x)}{U_\infty (x)^{b_0}} \int_0^\infty U_\infty (x)^{b_0-1} dx \qquad (26)$$
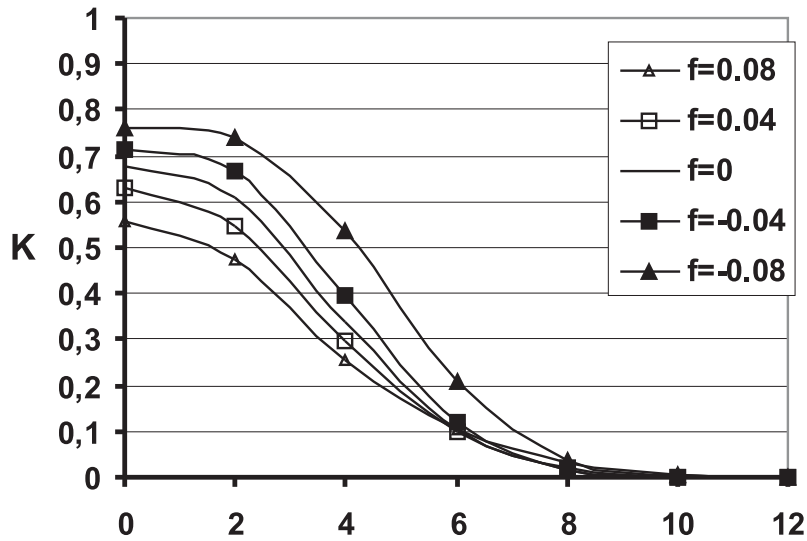
Figure 4: Dimensionless temperature profiles across boundary layer for $Pr = 0.72$

Let us take a sphere as particular example. In this case velocity distribution, according to [12], is given as:

$$U_\infty (x) = U_0 \tfrac{3}{2} \sin \varphi. \tag{27}$$

where U_0 is velocity of the undisturbed uniform stream far afore the body.

For treated adiabatic case it is most interesting to calculate eigen-temperature of the adiabatic surface $ET(x)$. Assuming equation (7) it could be defined as:

$$ET(x) = \frac{T(x, 0, \mathrm{Pr}) - T_\infty}{U_0^2 / 2c} = \frac{U_\infty (x)^2}{U_0^2} K(x, 0, \mathrm{Pr}). \tag{28}$$

Since the obtained results for dimensionless temperature $K(x, \eta, Pr)$ are not completely universal one must assume particular value of Prandtl
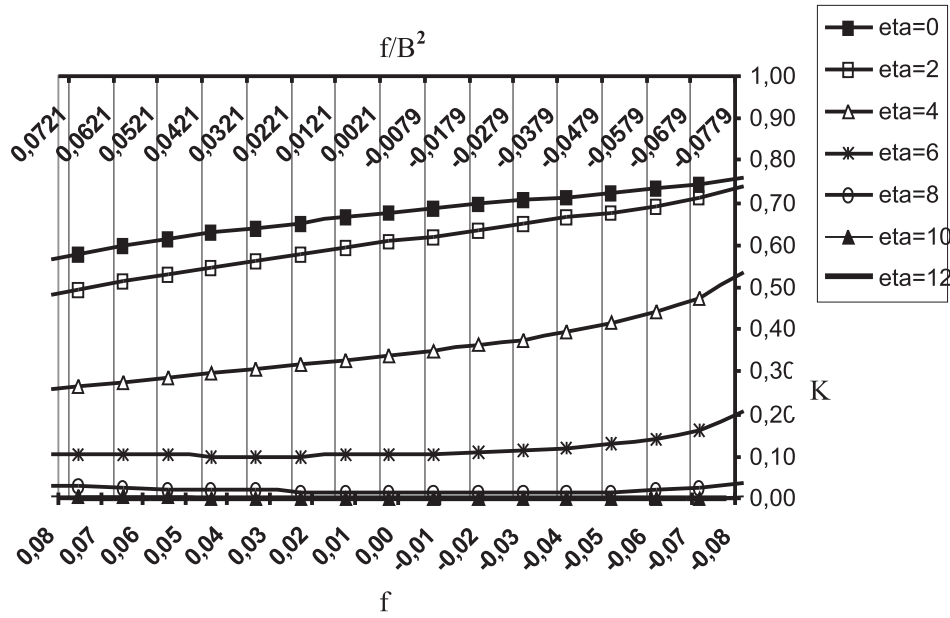
Figure 5: Dimensionless temperature distribution along boundary layer for $Pr = 0.72$

number. For this example calculations would be done for air, $Pr=0.72$. First step of the computing procedure is to calculate the function $f_1/(B^2)$ (26) for sphere, using particular velocity distribution (27). These results are presented graphically on the Fig.6. Second step is to compare obtained values of the function $f_1/(B^2)$ with universal results and to find, by interpolation, corresponding values of function f_1 and universal functions. In presented case it concerns universal function K for the surface of the body, i.e. for $\eta=0$. When these values are found, the eigen-temperature ET of the body is calculated (third step of the computing procedure) using equation (28). Results are presented graphically on the Fig.6. Cumulative effect of the viscous friction is manifested by monotonously increase of the eigen-temperature ET. Stagnation temperature on the body surface:

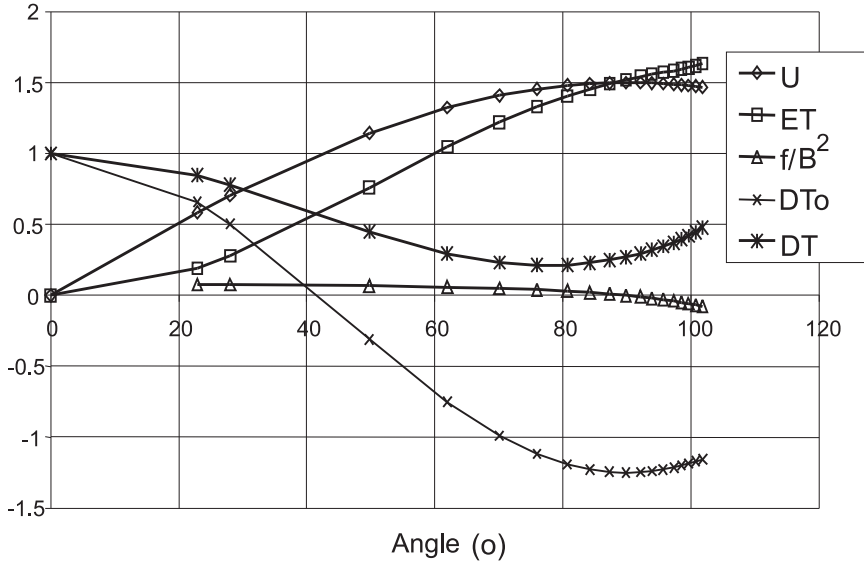$$DT_0 = T_0 - T_\infty = \frac{U_\infty (x)^2}{2c}, \tag{29}$$

Figure 6: Velocity U_∞ (marked as U), function f_1/B^2, eigen-temperature ET , stagnation temperature difference $DT_0 = T_0 - T_\infty$, and surface temperature difference $DT = T - T_\infty = T_0 - T_\infty + ET$ on the sphere for $Pr = .072$

is calculated also and presented on the Fig.6 as temperature difference DT_0. Typical effect of aerodynamic heating is obvious from the graph of DT_0. Body surface temperature comprehending coupled effects of stagnation and friction can be estimated as:

$$DT = T_0 - T_\infty + ET = DT_0 + ET. \qquad (30)$$

From its graph, presented on the Fig.6, it is obvious that zone around stagnation point is thermally loaded. Effects of friction are manifested downwash the boundary layer.

It is evident, from the $ET(x)$ diagram, that boundary layer is calculated from rather high values of the angle ϕ ($\sim 23^0$). It seems that in treated particular case of the sphere boundary layer theory fails in relatively wide neighborhood of the stagnation point. This fact can be explained since the sphere is a blunt body, with comparatively big surface exposed normally to the outer stream. Stagnation point region of the sphere with impulsive changes of velocity and pressure is big in com-

parison with its total surface. For some other body of revolution, with more stream like form, the stagnation point region would be comparatively smaller, and the universal solutions results could be applied closer to the stagnation point. On the other side it seems that separation point position ($\sim102^0$) is in good agreement with literature.

5   Conclusions

In the presented paper, the governing equations of the incompressible laminar thermal boundary layer, for adiabatic boundary conditions, on the body of revolution are obtained in the universal form, using the general similarity Loitskianskii's [6] method, with Saljnikov's [7] modifications of transversal coordinate. Results for universal dimensionless stream function $\Phi(x,\eta)$ and temperature $K(x,\eta,Pr)$ are achieved numerically, using finite difference method. Universal solutions of the dimensionless temperature $K(x,\eta,Pr)$ for $Pr=1$ and $Pr=0.72$ are presented graphically. They are very convenient for practical calculations of dynamic and temperature boundary layer. Correlation of the particular case, with known velocity distribution of the outer flow $U_\infty(x)$ along the surface concerned, with universal solutions is done by means of the function f_1/B^2. Calculations of eigen-temperature and surface temperature on the sphere are performed for $Pr=0.72$, and presented graphically. Inconvenience of the sphere for laminar boundary layer calculation around the stagnation point is discussed.

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Nestišljivi laminarni temparaturski na osno simetričnom telu - adijabatski slučaj

UDK 532.526

U radu su dobijene univerzalne jednačine nestišljivog temperaturskog sloja na sferi. U tom cilju se koristi poboljšani metod sličnosti za slučaj adijabatskih granišnih uslova. Numeričkom integracijom u jednoparametarskoj aproksimaciji su dobijena univerzalna rešenja za $Pr = 1$ i $Pr = 0.72$. Univerzalne funkcije izračunate za temperaturski granični sloj su prikazane grafički. U okviru jednog primera je izračunata i diskutovana sopstvena temperatura na sferi.