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Mixed convection flow past a horizontal plate

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Abstract

The mixed convection flow past a horizontal plate being aligned through a small angle of attack to a uniform free stream will be considered in the limit of large Reynolds number and small Richardson number. Even a small angle of inclination of the wake is sufficient for the buoyancy force to accelerate the flow in the wake which causes a velocity overshoot in the wake. Moreover a hydrostatic pressure difference across the wake induces a correction to the potential flow which influences the inclination of the wake. Thus the wake and the correction of the potential flow have to be determined simultaneously. However, it turns out that solutions exist only if the angle of attack is sufficiently large. Solutions are computed numerically and the influence of the buoyancy on the lift coefficient is determined.

Keywords: mixed convection, wake flow, boundary layer theory.

1 Introduction

The effect of weak buoyancy on the laminar flow past a horizontal plate which is aligned under a small angle of attack ϕ to the oncoming free stream will be investigated in the limit of large Reynolds numbers Re (see figure 1). Since the gravity force is almost perpendicular to the main flow direction buoyancy influences the flow field only indirectly by a (nonuniform) hydrostatic pressure distribution.

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The influence of buoyancy on the potential flow can be characterized by the Richardson number defined in terms of the total heat flux $\dot{Q}$ per unit depth of the plate (see Schneider 2005)

$$\mathrm{Ri} = \frac{\mathrm{g}\beta\dot{\mathrm{Q}}}{\rho \mathrm{c_p} \mathrm{u}_\infty^3} = \frac{\mathrm{Gr}}{\mathrm{Re}^{5/2}}\frac{1}{\mathrm{Pr}}\frac{\mathrm{Nu}}{\mathrm{Re}^{1/2}} \tag{1}$$

where $\mathrm{Gr} = \mathrm{g}\beta\Delta\mathrm{T}\mathrm{L}^3/\nu^2$, $\mathrm{Re} = \mathrm{u}_\infty\mathrm{L}/\nu$, $\mathrm{Nu} = \dot{\mathrm{Q}}/\mathrm{k}\Delta\mathrm{T}$, $\mathrm{Pr} = \rho\mathrm{c_p}\nu/\mathrm{k}$ are the Grashof, Reynolds, Nusselt and Prandtl number and β, c_p, μ, k, ρ are the isothermal expansion coefficient, the isobaric heat capacity, viscosity, thermal conductivity and density of the fluid. The plate of length L is assumed to be isothermal with the plate temperature $T_p = T_\infty + \Delta T$. The temperature of the ambient fluid is T_∞ and u_∞ is the velocity of the oncoming parallel flow.

Several authors (see Schlichting & Gersten 2000 for an overview) considered this indirect buoyancy effect in the boundary layer assuming that the flow past a finite plate will be similar to the flow past a semi-infinite plate. This is indeed the case if symmetric flow conditions are present, i. e. one side of the plate is heated, while the other one is cooled. However, more interesting is the case of a plate which is on both sides cooled or heated. Very often the parameter $K = \mathrm{Gr}\mathrm{Re}^{-5/2}$ (c.f. Schneider & Wasel 1984) has been used to characterize the influence of buoyancy onto the boundary layer flow assuming Pr and $\mathrm{Nu}\mathrm{Re}^{1/2}$ to be order one.

In a recent paper Schneider 2005 showed that for the flow past a finite plate the outer (potential) flow field is markedly influenced by buoyancy. In order to simplify the problem he neglected the viscous boundary layer and wake, by setting the Prandtl Pr number to zero. Considering a large Peclet number Pe temperature and density perturbations are limited to a thin thermal boundary layer and wake, respectively. An essential assumption to determine the perturbation of the outer flow field is the validity of the Kutta condition. Thus a vortex distribution on the wake and the plate has been introduced to compensate the hydrostatic pressure differences at the trailing edge and across the wake.

The goal of the paper is to describe the global flow field. Due to the inclination of the wake the flow in the wake is accelerated or decelerated by the tangential (to the wake) component of the hydrostatic pressure gradient. This effect has been neglected by Schneider 2005 limiting his analysis to Richardson numbers $\mathrm{Ri} \ll \mathrm{Re}^{-1/4}$. For technical reasons Schneider 2005 considered the flow problem in a channel with a width of the order $O(\mathrm{Ri}^{-\mathrm{n}})$, $n > 0$.

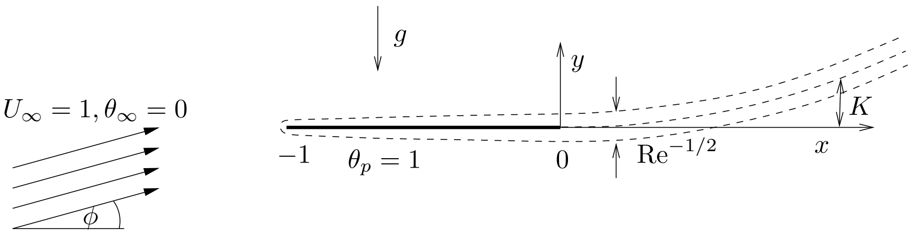

Figure 1: Mixed convection fbw past a horizontal plate

In section 2 we introduce the governing equations and derive the boundary-layer equations. They are formulated in local coordinates around the centerline of the wake. The position of the centerline has to be determined by the first order correction of the potential fbw. Thus the wake and the potential fbw field have to be determined simultaneously. The potential fbw correction consists of two contributions: one due to the angle of attack ϕ and a second one due to buoyancy differences across the wake.

Accordingly two dimensionless coupling parameters for the the angle of attack ϕ the buoyancy parameter K and the Reynolds number are introduced: The dimensionless parameter $\lambda = \phi K \sqrt{\mathrm{Re}}$ describes the effect of buoyancy in the far wake. Similarity solutions for the velocity and temperature profile in the far wake exist only for positive values of λ. The reduced buoyancy parameter $\kappa = K\mathrm{Re}^{1/4}$ is a measure for the hydro-static pressure difference across the wake. The ratio $K/\phi = \kappa^2/\lambda$ measures the influence of the hydrostatic pressure differences onto the potential fbw.

In section 3 results regarding the form of the wake, the velocities in the wake and the resulting lift force on the plate are presented and discussed.

2 Governing equations

We consider a two dimensional incompressible fbw using the Boussinesq approximation. The origin of the coordinate system is assumed to be at the trailing edge of a horizontal plate. The oncoming parallel fbw has an inclination ϕ to the horizontal x-axis. All lengths are made dimensionless with the plate length L, velocities are nondimensionalized by the velocity u_∞ of the unperturbed flow.

Temperature differences are scaled by the difference ΔT of the plate temperature and the temperature of the ambient fluid. Thus the governing equations read

$$uu_x + vu_y = -p_x + \mathrm{Re}^{-1}(u_{xx} + u_{yy}), \tag{2}$$

$$uv_x + vv_y = -p_y + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\theta + \mathrm{Re}^{-1}(v_{xx} + v_{yy}), \tag{3}$$

$$u\theta_x + v\theta_y = \frac{1}{\mathrm{Pr}\,\mathrm{Re}}(\theta_{xx} + \theta_{yy}), \tag{4}$$

$$u_x + v_y = 0, \tag{5}$$

subjected to the boundary conditions

$$u(x,0) = v(x,0) = 0, \quad \theta(x,0) = 1, \quad -1 < x < 0, \tag{6}$$

$$u(x,y) \to 1, \quad v(x,y) \to \phi, \quad \theta(x,y) \to 0, \quad x^2 + y^2 \to \infty. \tag{7}$$

2.1 The boundary layer and wake

Since the solution of the potential equation cannot satisfy the no slip boundary conditions a boundary layer of thickness $O(\mathrm{Re}^{-1/2})$ forms along the plate. In the wake after the plate the boundary layer approximation is valid too. Thus we can discuss the boundary layer and wake together. However, the position of the wake is not known a priori. Thus we define the vertical boundary layer coordinate $\bar{y}$ as the scaled distance from *center line* $y = y_w(x) = \phi\bar{y}_w(x)$ of the wake:

$$\bar{y} = (y - \phi\bar{y}_w(x))\sqrt{\mathrm{Re}}. \tag{8}$$

Along the plate $0 < x < 1$ the centerline of the boundary layer lies of course in the plate, thus we set $\bar{y}_w(x) = 0$ for $-1 < x < 0$.

The vertical velocity component $\bar{v}_w$ in the wake is referred to the vertical velocity at the center line:

$$\bar{v}_w(x,\bar{y}) = (v - u(x,\phi\bar{y}_w)K\bar{y}_w')\sqrt{\mathrm{Re}}. \tag{9}$$

The horizontal velocity component $\bar{u}_w$ and the pressure $\bar{p}_w$ in the wake and boundary layer are defined by

$$u(x,y) = \bar{u}_w(x,\bar{y}), \quad p(x,y) = K\bar{p}_w(x,\bar{y}). \tag{10}$$

Inserting into the governing equation we obtain for the leading order terms

$$\bar{u}_w \bar{u}_{w,x} + \bar{v}_w \bar{u}_{w,\bar{y}} = +\phi K \sqrt{\mathrm{Re}} \bar{y}_w' \bar{\theta}_w + \bar{u}_{w,\bar{y}\bar{y}}, \tag{11}$$

$$\bar{p}_{w,\bar{y}} = \bar{\theta}_w, \tag{12}$$

$$\bar{u}_w \bar{\theta}_{w,x} + \bar{v}_w \bar{\theta}_{w,\bar{y}} = \frac{1}{\mathrm{Pr}} \bar{\theta}_{w,\bar{y}\bar{y}}, \tag{13}$$

with the matching conditions $\bar{u}_w = 1$, $\bar{\theta} = 0$ for $\bar{y} \to \pm\infty$ and the boundary conditions

$$\begin{array}{ll} \bar{u}_w(x,0) = \bar{v}_w(x,0) = 0, \quad \bar{\theta}(x,0) = 1, & -1 < x < 0, \\ \bar{u}_{w,\bar{y}}(x,0) = \bar{v}_w(x,0) = \bar{\theta}_{w,\bar{y}}(x,0) = 0, & x > 0. \end{array} \tag{14}$$

Note that the hydrostatic pressure gradient has a component in the main flow direction $\phi K \sqrt{\mathrm{Re}} \bar{y}_w' \bar{\theta}_w$ which is proportional to the inclination $\phi \bar{y}_w'$ of the wake and to the density perturbation in the wake $K\bar{\theta}_w$ and inversely proportional to the thickness of the wake $\sqrt{\mathrm{Re}}$. Considering the limit $K \to 0$, $\phi \to 0$, $\mathrm{Re} \to \infty$ a coupling between these three parameters is necessary. Thus we introduce the reduced buoyancy parameter and reduced inclination parameter by

$$\kappa = K \mathrm{Re}^{1/4}, \quad \lambda = \phi \mathrm{K} \sqrt{\mathrm{Re}}. \tag{15}$$

At the plate the inclination of the center line vanishes and equations (11), (13) reduce to boundary layer equations for forced convection flow along a plate. Their solution is given by the well known Blasius similarity solution (c.f. Schlichting & Gersten 2000):

$$\bar{u}(x,\bar{y}) = F_B'(\zeta), \quad \bar{\theta} = D_B(\zeta), \quad \zeta = \frac{\bar{y}}{\sqrt{x+1}}, \tag{16}$$

where F_B, the Blasius function, and D_B are the solutions of the similarity equations

$$2F_B''' + F_B F_B'' = 0, \quad F_B(0) = F_B'(0), \quad F_B'(\infty) = 1, \tag{17}$$

$$\frac{2}{\mathrm{Pr}} D_B'' + F_B D_B' = 0, \quad D_B(0) = 1, \quad D_B(\infty) = 0. \tag{18}$$

Considering the wake $x > 0$ we transform the wake equations (11)-(13) to the following variables which are appropriate to discuss the limiting behavior for $x \to \infty$.

$$\psi = (x+1)^{3/5}F(x,\eta), \quad \bar{\theta}_w = (x+1)^{-3/5}D(x,\eta), \\ \text{with} \quad \eta = \bar{y}(x+1)^{-2/5}, \tag{19}$$

where ψ is a stream function. Note that the horizontal velocity $\bar{u}_w = (x+1)^{1/5}F'$ will grow unbounded for $x \to \infty$ if F' tends to a non-vanishing limit. Thus we expect (due to the scaling) a velocity overshoot in the wake.

We obtain the transformed wake equations:

$$F''' + \frac{3}{5}F''F - \frac{1}{5}(F')^2 + \lambda \bar{y}'_w D = (x+1)(F'F'_x - F''F_x), \tag{20}$$

$$\frac{1}{\mathrm{Pr}}D'' + \frac{3}{5}(FD)' = (x+1)(F'D_x - D'F_x), \tag{21}$$

subject to the boundary conditions

$$F(x,0) = F''(x,0) = D'(0), \quad F'(x,\infty) = \frac{1}{(x+1)^{1/5}}, \quad D(0,\infty) = 0, \tag{22}$$

and at the "initial conditions" at the trailing edge $x = 0$,

$$F(0,\eta) = F_B(\eta), \quad D(0,\eta) = D_B(\eta). \tag{23}$$

Here and in the following we denote derivatives with respect to η with a prime. Integrating the energy boundary-layer equation (13) with respect to $\bar{y}$ we obtain that the enthalpy flux in the wake is constant

$$\dot{H} = \int_{-\infty}^{\infty} \bar{u}_w \bar{\theta}_w \, \mathrm{d}\bar{y} = \int_{-\infty}^{\infty} F'D \, \mathrm{d}\eta = 2\int_0^{\infty} F'_B \Theta_B \, \mathrm{d}\zeta = \frac{\mathrm{Nu}}{\mathrm{Pr}\sqrt{\mathrm{Re}}}. \tag{24}$$

Integrating the degenerated momentum equation (12) with respect to the vertical direction, we conclude that across the wake there is a pressure difference Δp_w given by:

$$\Delta \bar{p}_w(x) = \bar{p}_w(x,\infty) - \bar{p}_w(x,-\infty) = \int_{-\infty}^{\infty} \bar{\theta}_w \, \mathrm{d}y =: \gamma_w(x). \tag{25}$$

Discussing the potential flow we will interpret $\gamma_w(x)$ as a vortex distribution along the center line of the wake. Given the center line of the wake $\bar{y}_w(x)$ we can integrate the wake equations with a usual marching technique. However, the center line is not known a priori. The derivative $\bar{y}'_w(x)$ is equal to the v-component of the first correction of the outer (potential) flow field evaluated at the x-axis. Thus the potential flow correction and the wake equations have to be solved simultaneously.

2.2 The limiting behavior of the wake

The transformed wake equations (20)-(22) are in a form such that the limiting behavior can be deduced just by setting derivatives with respect to x equal to zero and take the limit $x \to \infty$. Assuming that the far (potential) fbw fi eld is given by the asymptotic boundary condition (7) the scaled inclination of the wake $\bar{y}_w'$ tends to 1. Then for $\lambda > 0$ we obtain similarity equations for the asymptotic fbw and temperature profi le. Using the transform

$$\begin{gathered} F(x,\eta) \sim a\hat{F}(\hat{\eta}), \quad D \sim c\hat{D}(\hat{\eta}), \qquad \hat{\eta} = b\eta, \\ a = b = \left(\tfrac{\lambda \dot{H}}{2}\right)^{1/5}, \quad c = \tfrac{\dot{H}^{4/5}}{2^{4/5}\lambda^{1/5}}, \end{gathered} \tag{26}$$

the similarity equations can be normalized to:

$$\hat{F}''' + \frac{3}{5}\hat{F}''\hat{F} - \frac{1}{5}\hat{F}'\hat{F}' + \hat{D} = 0, \quad \frac{1}{\mathrm{Pr}}\hat{D}' + \hat{F}\hat{D} = 0, \tag{27}$$

$$\hat{F}(0) = \hat{F}''(0) = \hat{f}'(\infty) = 0, \quad \int_0^\infty \hat{F}'\hat{D}\,\mathrm{d}\hat{\eta} = 1. \tag{28}$$

A numerical solution of the similarity equations is shown in fi gure 2. It is a jet like profi le. Due to the scaling (19) we expect the following asymptotic behavior for the velocity and temperature profi le in the wake, respectively.

$$\bar{u}_w \sim (x+1)^{1/5} ab\hat{F}'(b\eta) + .., \tag{29}$$

$$\bar{\theta}_w \sim \frac{1}{(1+x)^{3/5}} c\hat{D}(b\eta) + \ldots \quad . \tag{30}$$

Thus in the wake the maximum velocity is proportional to $\lambda^{2/5}x^{1/5}$. The width of the far wake is proportional to $x^{1/5}/\lambda^{1/5}$. Although the temperature perturbation decreases like $\lambda^{-1/5}x^{-3/5}$ it is wide enough such that the resulting buoyancy force accelerates the fbw in the wake. As a consequence the hydrostatic pressure difference across the wake decays to zero for $x \to \infty$.

$$\gamma_w = \int_{-\infty}^{\infty} \bar{\theta}_w \,\mathrm{d}y \sim \frac{c}{(x+1)^{1/5}} \int_{-\infty}^{\infty} \hat{D}(\hat{\eta})\,\mathrm{d}\hat{\eta}. \tag{31}$$

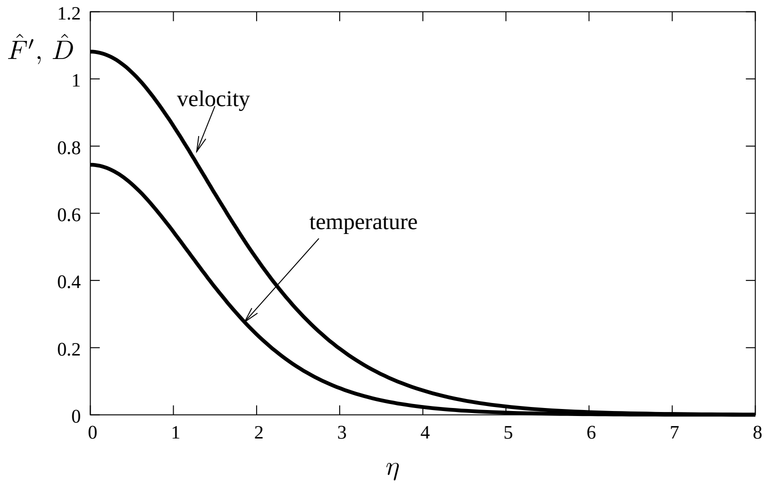

Figure 2: Similarity solution, temperature and velocity profi le in the far wake for Pr=1

2.3 The potential flow

We expand the potential fbw fi eld in terms of the buoyancy parameter K using the notation of complex functions of a complex variable $z = x+iy$, c.f. Schneider 1978.

$$u - iv \sim 1 - i\phi\sqrt{\frac{z}{z+1}} + K(u_1 - iv_1) \tag{32}$$

The fi rst correction gives the perturbation of the fbw fi eld due to the angle of attack ϕ. Here and in the following we will assume ϕ small and of comparable size than the buoyancy parameter $K = O(\mathrm{Re}^{-1/4})$. The second term in the expansion takes buoyancy effects into account and is therefore of order K.

Boundary conditions for the potential fbw correction $u_1 - iv_1$ are given at the plate

$$v_1(x, 0) = -0, \quad -1 < x < 0 \tag{33}$$

and along the wake where the pressure has jump a discontinuity given by (25). Using the linearized Bernoulli equation we have,

$$-u_1(x, 0+) + u_1(x, 0-) = \gamma_w(x). \tag{34}$$

Following Schneider (2005) we represent the potential fbw correction in terms of a vortex-distribution along the x-axis. Note the the deviation of the center line of the wake is small (of order K) on the scales of the original coordinates x, y which justifies to place the vortex distribution along the x-axis instead on the center line of the wake. Thus we have

$$u_1 - iv_1 = -\frac{1}{2\pi}\int_{-1}^{\infty}\gamma(\xi)\frac{y + i(x-\xi)}{(x-\xi)^2 + y^2}\,\mathrm{d}\xi, \tag{35}$$

with

$$\gamma(x) = \begin{cases} \gamma_p(x) & -1 < x < 0 \\ \gamma_w(x) & x > 0 \end{cases}. \tag{36}$$

Thus the jump condition for the horizontal velocity along the $x-$axis (34) is satisfied. It remains to determine the vortex distribution $\gamma_P(x)$ along the plate. From equation (33) we obtain the integral equation

$$\int_{-1}^{0}\frac{\gamma_p(\xi)\,\mathrm{d}\xi}{x-\xi} = -\int_{0}^{\infty}\frac{\gamma_w(\xi)\,\mathrm{d}\xi}{x-\xi} \tag{37}$$

with the solution, cf. Schneider (1978)

$$\gamma_p(x) = -\frac{1}{\pi}\sqrt{-\frac{x}{x+1}}\int_0^{\infty}\frac{\gamma_w(\xi)}{x-\xi}\sqrt{\frac{\xi+1}{\xi}}\mathrm{d}\xi, \quad -1 < x < 0. \tag{38}$$

Thus we obtain v_1

$$v_1(x) = \frac{1}{2\pi}\sqrt{\frac{x}{x+1}}\int_0^{\infty}\frac{\gamma_w(\xi)}{x-\xi}\sqrt{\frac{\xi+1}{\xi}}\,\mathrm{d}\xi, \quad x > 0, \tag{39}$$

and finally the scaled inclination of the wake is given by

$$\bar{y}_w'(x) = \sqrt{\frac{x}{x+1}} + \frac{\kappa^2}{\lambda}v_1(x). \tag{40}$$

Thus the boundary-layer (wake) equation (20) has the form

$$\begin{aligned} F''' + \tfrac{3}{5}F''F - \tfrac{1}{5}(F')^2 + \left(\lambda\sqrt{\tfrac{x}{x+1}} + \kappa^2 v_1(x)\right) D = \\ (x+1)(F'F_x' - F''F_x'). \end{aligned} \tag{41}$$

Finally we have obtained the wake-equations (41), (22), (23) which have to be solved simultaneously with the inclination of the wake (39) where the vortex distribution $\gamma_w(x)$ is given by (25).

3 Results

3.1 Numerical solution

For a given set of parameters (Pr, λ, κ) we pursue the following solution strategy. First we assume a vertical velocity distribution $v_1^{(0)}$ and solve the wake equations starting at the trailing edge ($x = 0$) by a marching technique. Since we expect the velocity and temperature profiles to converge only like $x^{-1/5}$ to their limiting similarity profiles we have to integrate over large distances. Thus we increase the step size in x-direction after each step by a constant factor, say $f = 1.011$. On the other hand we want to resolve the profiles near the trailing edge accurately. Thus we start there with a step size of $\Delta x = 10^{-7}$ Taking $N_x = 4000$ steps in $x-$direction the last grid point is of the order 10^{13}. The wake equation are discretized in x-direction by a simple first order difference scheme. Thus we get at each grid point a system of ordinary differential equations which is solved by a well proven ODE solver, COLPAR (Ascher et al. 1981).

Thus we obtain a first guess for the velocity and temperature profiles and the vortex distribution in the wake. Then we have to evaluate the integral (39) for an improved guess for v_1. In order to evaluate the integral (39) we replace $\gamma_w(\xi)\sqrt{\xi+1}$ by piecewise linear functions such that the integral can be integrated exactly. With a new guess for $v_1(x)$ we integrate the wake equations again and repeat the process until convergence is obtained. Usually it takes only 3 to 5 iterations.

Note that in the case $\lambda = 1$, $\kappa = 0$ the iteration is not necessary. The inclination of the wake is solely determined by the angle of attack ϕ. In that case the hydrostatic pressure differences across the wake are too small to influence the potential field significantly. However, buoyancy is limitted to the flow behavior in the wake. The flow is acelerated and a velocity overshoot develops for $x \to \infty$.

The case when λ and κ are of the same order is of much more interest. Thus in the following examples we fix the values for $\lambda = 1$ and the Prandtl number $\mathrm{Pr} = 0.71$ (air) and vary κ starting from $\kappa = 0$. However, convergence could not be obtained for $\kappa > 0.914$.

3.2 The vortex distribution in the wake

In figure 3 the vortex distribution $\gamma_w(x)$ is shown for different values of κ. At the trailing edge γ_w has the prescribed value $\gamma_w = 2\int_0^\infty D_B \, \mathrm{d}\eta$ Then it decays

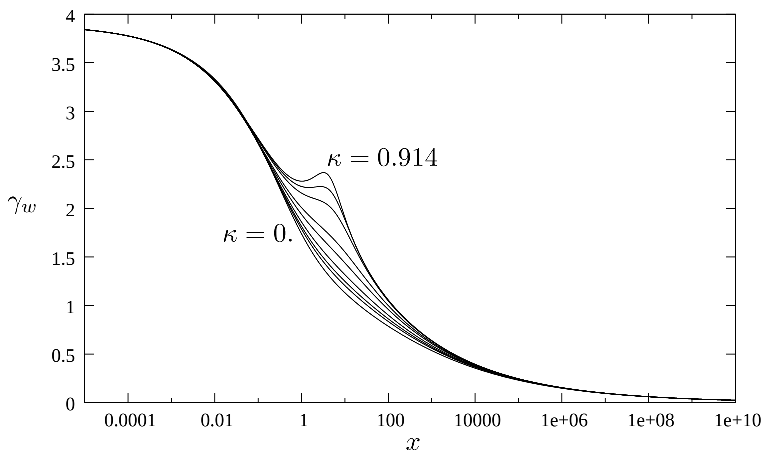

Figure 3: Vortex distribution along the wake, $\mathrm{Pr} = 0.71$, $\lambda = 1$, κ =0, 0.5, 0.6, 0.7, 0.8, 0.85, 0.9, 0.91, 0.914

monotonically like $x^{-1/5}$ to zero for $\kappa = 0$. For small values of κ there are only small deviations in the range from 1 to 100. About $\kappa = 0.7$ this deviation becomes markedly pronounced, (cf. $\kappa = 0.85$). For $\kappa = 0.91$ the vortex distribution γ_w has a plateau at $x = 10$ and at $\kappa = 0.914$ is has even a local maximum. It turns out that the solution is here very sensible to even very small perturbations in κ. The described solution method fails for $\kappa > 0.914$.

3.3 Local behavior near trailing edge

Although the boundary layer equations are valid along the plate and in the wake their solution has a singularity at the the trailing edge due to the change of the boundary conditions. At the plate the no slip boundary condition for the velocity and a Dirichlet condition for the temperature hold. In the wake all quantities, like velocity, shear rate $\partial \bar{u}_w / \partial \bar{y}$, temperature and heat flux $\partial\, \bar{\theta}_w / \partial \bar{y}$ have to be continuous. It has been shown that for the velocities and temperature the following asymptotic representation holds (c.f. Sychev et al. 1998, pp. 103)

$$\begin{gathered} \bar{u}(x, \bar{y}) = f_B'(\bar{y}) + x^{1/3} \left(\hat{f}'(\zeta) - k_1 \left(f_B''(0)\zeta - f_B''(y) \right) \right) + ..., \\ \zeta = \tfrac{\bar{y}}{x^{1/3}}, \end{gathered} \tag{42}$$

$$\bar{\theta}(x,\bar{y}) = D_B(\bar{y}) + x^{1/3}\left(\hat{D}'(\zeta) - k_1\left(D'_B(0)\zeta - D'_B(y)\right)\right) + ..., \qquad (43)$$

where k_1 is a given constant given in (3.1.16) in Sychev et al. 1998. As a consequence we obtain for the vortex distribution in the wake:

$$\gamma_w(x) = \int_{-\infty}^{\infty} \theta(x,\bar{y})\,\mathrm{d}\bar{y} \sim \gamma_{w,0} + \gamma_{w,1}x^{1/3} \qquad (44)$$

with

$$\gamma_{w,0} = 2\int_0^{\infty} D_B(\bar{y})\,\mathrm{d}\bar{y}, \quad \gamma_{w,1} = -2k_1 D_B(0) = -2k_1. \qquad (45)$$

Considering that v_1 vanishes at the plate we conclude that the velocity field is given locally by

$$u_1 - iv_1 \sim= -\frac{\gamma_0}{2} - \gamma_1|z|^{1/3}e^{i(\varphi-\pi)/3}, \qquad (46)$$

with $\varphi = \arctan y/x$, $|z| = \sqrt{x^2+y^2}$. Thus we obtain

$$u_1(x,0) = \begin{cases} -\frac{\gamma_w}{2} & \sim & -\frac{\gamma_{w,0}}{2} - \frac{\gamma_{w,1}}{2}|x|^{1/3} & x>0 \\ -\frac{\gamma_P}{2} & \sim & -\frac{\gamma_{w,0}}{2} - \gamma_{w,1}|x|^{1/3} & x<0 \end{cases}, \qquad (47)$$

$$v_1 \sim -\frac{\sqrt{3}}{2}\gamma_{w,1}x^{1/3}, \quad x>0. \qquad (48)$$

In figure 4 the local behavior of the vortex distribution near the trailing edge ($x=0$) is shown for $\lambda = 1$, $\kappa = 0$ is shown. The vortex is continuous there satisfying the Kutta condition, but the derivative is obviously singular as expected.

3.4 The wake

In the wake after plate there is a deficit $\dot{I}$ in the momentum flux due to the no slip boundary condition at the plate. Integrating the boundray-layer (wake) equations we obtain the balance equation for the momentum flux deficit

$$\frac{\mathrm{d}}{\mathrm{d}x}\dot{I} := \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\infty} \bar{u}_w(\bar{u}_w - 1)\,\mathrm{d}\bar{y} = \frac{\lambda \bar{y}'_w \gamma_w}{2}, \quad x>0. \qquad (49)$$

In a non-bouyant wake ($\lambda = 0$) the momemtum deficit would be constant along the wake. The width of the wake would increase and the velocity profile would tend to the unperturbed velocity profile with increasing distance from the plate.

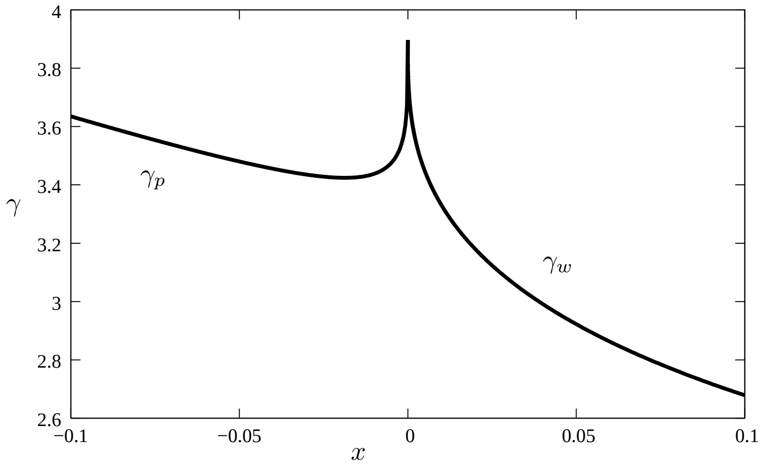

Figure 4: Local behavior of vortex distribution near trailing edge, $\mathrm{Pr} = 0.71$, $\lambda = 1$, $\kappa = 0.0$

Here the situation is different. The buoyancy force due to a slight inclination of the wake gives a contribution to the momentum flux balance. This can lead, as in the present case, to a velocity overshoot in the wake. In figure 5 we have shown the horizonal velocity component in the center of the wake. In the near wake $x \ll 1$ the velocity is not much influenced by buoyancy. It first recovers from the velocity deficit. About $x = 1$, for $\kappa = 0$ it has the value of the outside potential flow. Further downstream buoyancy accelerates the flow and a velocity overshoot forms. For $\kappa > 0$ the induced potential flow deforms the wake such that the center velocity is reduced compared to the case $\kappa = 0$. For $\kappa = 0.914$ a plateau forms.

The influence of the vortex distribution γ_w onto the form of the wake can be seen in figures 6 and 7. The first one shows the induced vertical velocity component v_1 at the wake. Starting at zero from the trailing edge it attains a positive maximum and then decreases rapidly to a negative minimum and finally increases slowly to its limiting value zero at infinity.

In figure 7 the (scaled) inclination $\tilde{y}_w'$ of the wake is shown. For $\kappa = 0$ it is the inclination of the wake after a plate with the small angle of attack ϕ. It is not affected by buoyancy. Shortly after the plate buoyancy tends to bend the wake upwards, but after $x \sim 0.1$ buoyancy tends to bend the wake downwards. For $\kappa = 0.91$ the wake has around $x = 1$ a section with negative inclination! Near

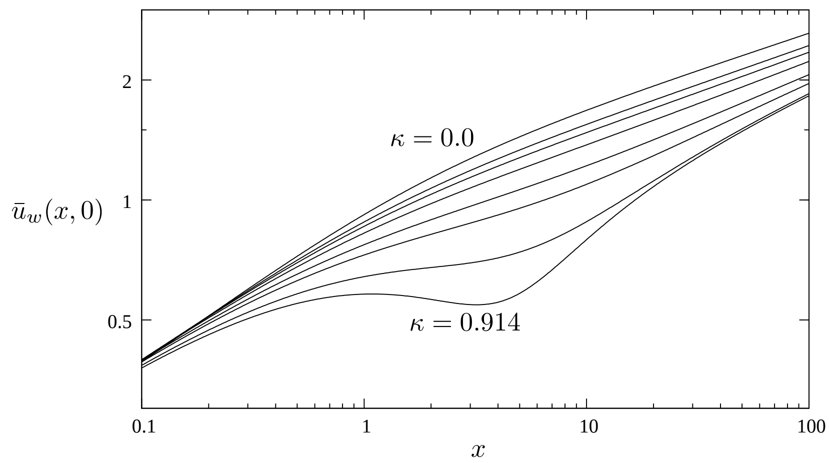

Figure 5: Velocity at the center of the wake, $\mathrm{Pr} = 0.71$, $\lambda = 1$, κ =0, 0.5, 0.6, 0.7, 0.8, 0.85, 0.9, 0.914

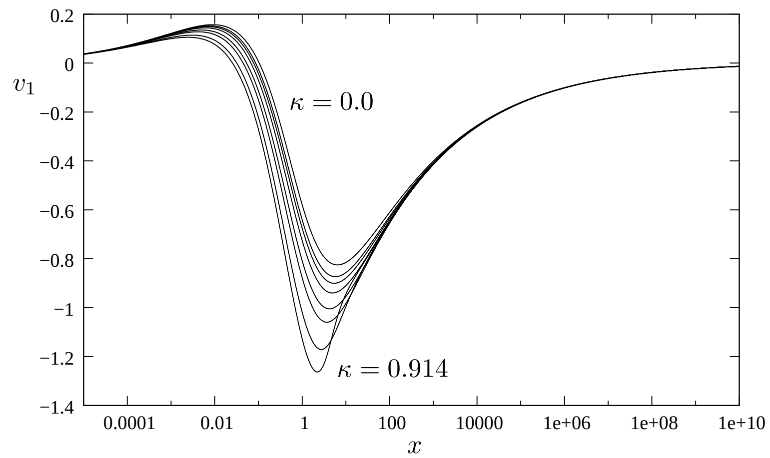

Figure 6: Vertical velocity component v_1 at the centerline of wake, $\mathrm{Pr} = 0.71$, $\lambda = 1$, κ =0, 0.5, 0.6, 0.7 ,0.8, 0.85, 0.9, 0.914

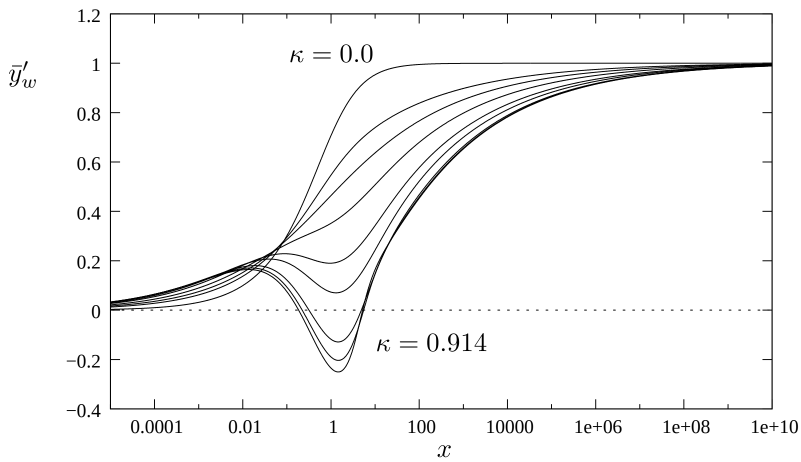

Figure 7: Inclination of wake, $\mathrm{Pr} = 0.71$, $\lambda = 1$, κ =0, 0.5, 0.6, 0.7 ,0.8, 0.85, 0.9, 0.91, 0.914

this limiting value of κ one sees that the inclination is very sensitive to κ. This seems to be an indication that around $\kappa = 0.914$ a bifurcation or a singularity occurs.

3.5 Vortex distribution along the plate

Finally we will discuss the resulting lift force onto the plate. The total vortex strength along the plate is given by

$$\Gamma_P(x) = \phi\gamma_\phi(x) + K\gamma_p(x) = -\phi\left(2\sqrt{\frac{-x}{x+1}} - \frac{\kappa^2}{\lambda}\gamma_p(x)\right). \tag{50}$$

In figure 8 we plot the buoyancy induced vortex strenght $\gamma_P(x)$ scaled with $\sqrt{x+1}$. It is positive for all values of κ and x.

The pressure due to the potential flow with circulation gives rise to a normal force acting on the plate, a lift force. Accounting for the contributions from the upper and lower surfaces of the plate and referring the lift force to the free stream stagnation pressure and the plate area we obtain for the lift coefficient C_L

$$\begin{aligned} C_L = -4\int_{-1}^{0} p(x,0+)\,\mathrm{d}x = -2\int_{-1}^{0}\Gamma_p(x)\,\mathrm{d}x = \\ \phi\left(2\pi - \tfrac{2\kappa^2}{\lambda}\int_{-1}^{0}\gamma_p(x)\,\mathrm{d}x\right). \end{aligned} \tag{51}$$

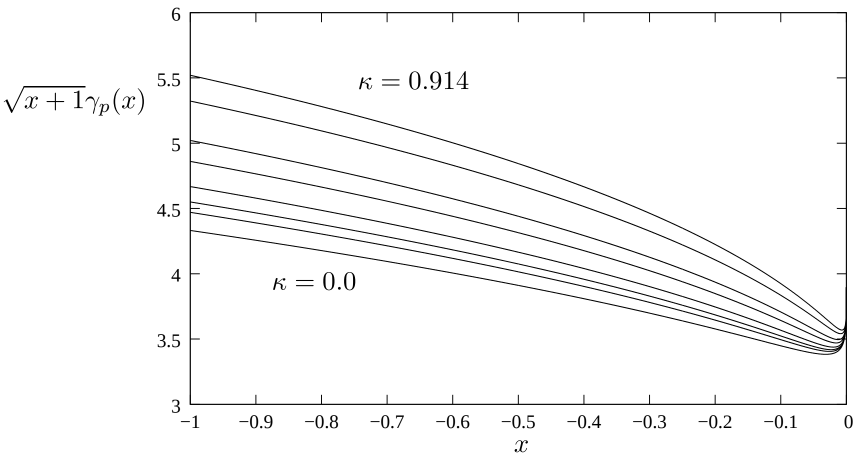

Figure 8: Vortex distribution along the plate, $\mathrm{Pr} = 0.71$, $\lambda = 1$, κ =0, 0.5, 0.6, 0.7 ,0.8, 0.85, 0.9, 0.914

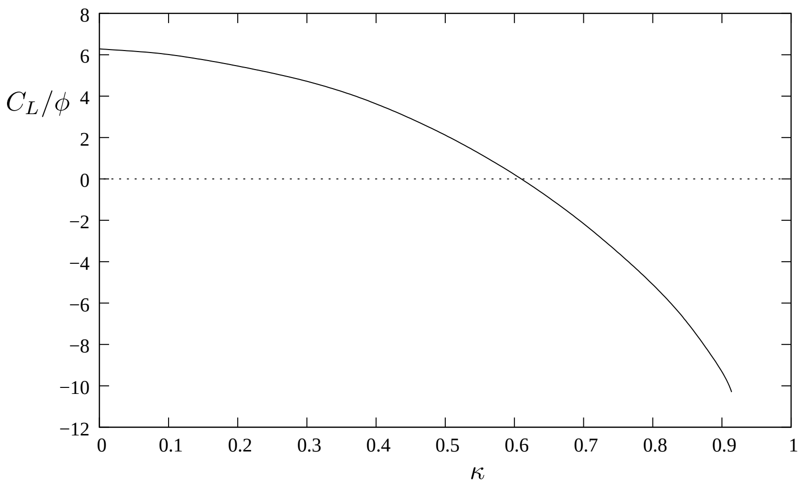

Figure 9: Lift coeffi cient, $\mathrm{Pr} = 0.71$, $\lambda = 1$

In figure 9 the lift coefficient C_L is shown as a function of κ. Buoyancy reduces the lift force. For $\kappa = 0.6$ the resulting lift force is zero and for larger values of κ a negative lift is obtained. The result is in accordance with Schneider (2005) (eq. 14) who also obtains a negative buoyancy induced lift.

4 Conclusions

The present study shows the interaction of the flow past a horizontal plate under a small angle of attack and buoyancy. Two dimensionless parameters λ, κ have been identified. The first one is a measure for the velocity overshoot in the far wake and the second one, more precisely the ratio κ^2/λ measures the influence of the hydrostatic pressure perturbation onto the potential field around the plate.

Most surprising solutions exist only for $\lambda > 0$ and κ less a than critical value.

For $\kappa > 0$ the buoyancy effects are not limited to the boundary layer and wake, where it leads to a velocity overshoot. A potential flow correction is induced by the hydrostatic pressure difference across the wake which reduces the lift force (or in extreme cases even reverses the direction of the lift force). We note that according to the present analysis no solution exists if the is oncoming flow is exactly horizontal. As we have remarked earlier Schneider (2005) has considered this case for buoyancy values so small that the inclination of the wake becomes negligible. Since in that case the vortex distribution will not decay for $x \to \infty$ inducing unbounded vertical velocities he placed the heated plate in a channel of width $b \sim \mathrm{Ri}^{-\mathrm{n}}$ for some positive constant n. This procedure worked to guarantee a solution. It would be of interest if this concepts works in the present case, where the inclination of the wake is taken into account.

Of the same interest is the question how the solution breaks down at $\lambda = 1$ $\kappa \sim 0.914$. Is there a bifurcation point? Are there locally multiple solutions?

In a forthcoming paper we will investigate the local behavior of the flow near the trailing edge with triple deck methods which has been presented first in Steinrück 2004.

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References

[1] U. Ascher, J. Christiansen and R. D., Russel, Collaction software for boundary-value ODEs, *ACM Trans. Math. Software* **7** (1981), 209-222.

[2] H. Schlichting and K Gersten, *Boundary Layer Theory*, 8^{th} Ed., Springer, Berlin /Heidelberg 2000.

[3] W. Schneider, *Mathematische Methoden der Strömungsmechanik*, Vieweg-Verlag, Braunschweig, 1978.

[4] W. Schneider, Lift, thrust and heat transfer due to mixed convection flow past a horizontal plate of finite length. *J. Fluid Mech.* **529** (2005), 51-69.

[5] W. Schneider and M. G. Wasel, Breakdown of the boundary-layer approximation for mixed convection above a hrorizontal plate. Int. J. Heat Mass Transfer **28** (1985), 2307-2313.

[6] H. Steinrück and Lj. Savić, The trailing edge problem for mixed convection flow past a horizontal plate, in Gutkowski, W, Kowaleski, T. A. (editors) *Proceedings of the 21st International Congress of Theoretical and Apllied Mechanics*,, IPPT PAN, Warszawa, 2004, Paper-Nr. FM2S_12030.

[7] W. V. Sychev, A. I. Ruban, Vic. V. Sychev, Vic. V. and G. L., Korolev, *Asymptotic theory of separated flows*, Cambridge University Press, Cambridge, 1998.

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Mešovito konvekciono tečenje preko horizontalne ploče

UDK 532.526

Slučaj opstrujavanja ravne ploče (konače dužine), koja se nalazi pod malim napadnim uglom i čija se temperatura razlikuje od temperature slobodne struje, bi´ce posmatran u graničnom procesu velikih vrednosti Re (Reynolds) i malih vrednosti Ri (Richardson) brojeva. Čak i male vrednosti nagibnog ugla traga su dovoljne da potisak ubrza strujanje u zoni iza ploče, tako da ´ce brzina u tragu biti ve´ca od brzine slobodne struje.

Pored toga, gradijent hidrostatičkog pritiska u tragu indukuje korekciju potencijalne struje, koja dovodi do pojave i promene nagiba traga. S toga se rešenje (trag i korekcija potencijalne struje) mora odrediti simultano. Ispostavlja se da ono postoji samo u slučaju da je ugao nagiba ploče dovoljno veliki.

Rešavanje je izvedeno numerički, a posebno je naglašen uticaj potiska na koefi cijent uzgona.