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On the probability of an undetected surface-breaking crack

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Abstract

Surface-breaking fatigue cracks are common defects in metal components subjected to cyclic loads. Such cracks tend to propagate in stress fields that are below the critical stress level for static loading. An important part of a damage tolerant design philosophy is the requirement that surface-breaking cracks should be detectable before they reach a critical depth. In this paper, we consider a surface-breaking crack in a two-dimensional geometry, whose original depth is defined by a probability density function. The increase of the crack depth with number of cycles is governed by Paris law, and the detectability depends on a probability of crack detection (POD). Based on this information we determine the probability that the crack depth will have exceeded a prescribed critical value at a specified number of cycles.

1 Introduction

Fatigue cracks in cyclically loaded bodies are often generated at a free surface, usually by nucleation from a pre-fracture defect such as a scratch, particularly if the loading generates tensile stresses parallel to the free surface. Once they have been generated, surface-breaking cracks propagate faster than internal cracks, due to larger values of the stress-intensity factors. It is, therefore, important to be able to detect a surface-breaking crack before it penetrates too deep into the material.

From the point of view of a damage tolerant design philosophy, surface-breaking cracks are not necessarily totally unacceptable, provided that information is available on their growth and their detectability so that appropriate

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action can be taken when their dimensions have reached undesirable magnitudes.

Suppose for conceptual purposes we consider a two-dimensional geometry, with a single crack normal to the free surface. If, ideally, its original depth and a growth law, as well as a perfect detection technique would be available, then the depth after any number of cycles could be determined with precision. Unfortunately, such an ideal situation, which allows deterministic calculations, does not exist. In reality, as shown in this paper, we can only obtain information on the depth of surface-breaking cracks by probabilistic considerations.

In view of the preceding discussion, we focus our attention in this paper on the growth of a macrocrack with a known probability density function for the depth a, and on quantifying the effects of the probability of inspection on the detection of a crack that has grown under continued cyclic loading. We are especially interested in finding the probability that a crack with a critical dimension greater than a predefined magnitude exists in the component after an inspection at a specified number of cycles.

To proceed, we need to assume a suitable model for the crack growth, which accounts for the variabilities observed in fatigue crack growth. Various such models are available in the literature (see e.g. Sobczyk and Spencer [1]). In the present case, for simplicity, we have represented the crack growth by a nonlinear differential equation with fixed parameters and have introduced the variabilities only through the random size of the macrocrack. We have modeled the capability of a specified monitoring technique using the probability of detection (POD) concept. A POD is a statistical representation of the probability that a given monitoring technique is able to detect a specific flaw in a given material or structure.

The general approach we have described to account for the inspections is similar to the one presented in Palmberg et. al [2]. In Ref. [2], the authors presented expressions for the crack size distribution after inspections assuming that a crack detected in an inspection is repaired. In contrast to the present case where the parameters in the crack growth rate can be randomized, Ref. [2] started with a power crack-growth rate, which is randomized by a stochastic process with a lognormal distribution. The approach of the present paper follows Ref. [3], where pre-crack fatigue damage and crack growth were placed in the context of structural health monitoring.

2 Two-dimensional surface-breaking crack

The geometry is a homogeneous and isotropic linearly elastic half-space, which is in a state of time-harmonic tension parallel to the free surface defined by

$$\sigma_X = \Delta\sigma \sin(\omega t) \tag{1}$$

The half-space contains a surface-breaking crack of depth a. The case of a crack in a half-space is a good approximation for a crack in a layer, when the layer thickness is much larger than the crack depth.

For a half-space the Mode-I stress-intensity is of the well known form

$$K = 1.12\sigma\sqrt{\pi a}. \tag{2}$$

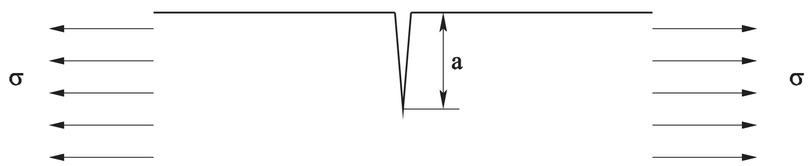


Figure 1: Surface-breaking crack in a tensile field

For cyclic loading, the commonly used crack growth law is Paris law (see Paris and Erdogen [4]), which is written as

$$\frac{da}{dN} = A\left(\Delta K\right)^m, \tag{3}$$

where N is the number of cycles, da/dN is the rate of crack growth, A and m are material parameters and ΔK is the amplitude of the stress intensity factor. For constant amplitude loading and after substitution of Equation (2), we can integrate Equation (3) ($m \neq 2$) to get

$$a_N^{1-m/2} = a_0^{1-m/2} + N\,A\left(1 - \frac{m}{2}\right)\left(1.12\Delta\sigma\sqrt{\pi}\right)^m, \tag{4}$$

where a_0 and a_N are the crack lengths at cycles $N = 0$ and N respectively. To account for the inherent uncertainty in the fatigue behavior, some of the quantities appearing in Equation (4) can be taken to be random with know probability density functions. For simplicity, we restrict ourselves to the case in which the only random quantity is the initial crack length a_0 with density given by $f_0(a_0)$. This follows naturally since the initial crack length is assumed to have come forth from the damage evolution and macrocrack initiation process described in Ref.[5].

The crack length after N cycles, a_N, is given by Equation (4) which can be rewritten as

$$a_N = \left(a_0^\beta + NC\right)^{1/\beta}, \tag{5}$$

where

$$C = A\beta\left(1.12\Delta\sigma\sqrt{\pi}\right)^m \tag{6}$$

$$\beta = \frac{1}{2}(2-m). \tag{7}$$

From Equation (5), we derive

$$a_0 = \left(a_N^\beta - NC\right)^{1/\beta} \equiv h(a_N). \tag{8}$$

The probability density function of a_N is given by (see Hahn and Shapiro [6])

$$f_N(a_N) = f_0[h(a_N)]\left|\frac{da_0}{da_N}\right|, \tag{9}$$

which may be written as

$$f_N(a_N) = f_0\left[\left(a_N^\beta - NC\right)^{1/\beta}\right]\left|a_N^{\beta-1}\left(a_N^\beta - NC\right)^{(1-\beta)/\beta}\right| \tag{10}$$

Note that the probability that there exists a crack with length $a_N > a_{cr}$ is given by

$$\Pr(a_N > a_{cr}; N) = \int_{a_{cr}}^{\infty} f_N(a_N)\, da_N \tag{11}$$

The effect of inspections on the probability that there exists an *undetected crack* with $a > a_{cr}$ is now discussed. We assume that the POD curve of the inspection technique is known. Typical POD curves are shown in Figure 2 where the letters 'A', 'B' and 'C' denote three different inspection techniques. A convenient expression for the POD curves is shown in the insert of Figure 2. The probability that the crack is *not* detected is then given by

$$PND(a) = 1 - \frac{\alpha\, a^\gamma}{1+\alpha\, a^\gamma} = \frac{1}{1+\alpha\, a^\gamma}. \tag{12}$$

It is evident that curve A represents the best technique and C the worst one.

Let us now consider the case of a first inspection at cycle N_1. Just prior to the inspection, the crack length density is given by $f_{N_1}(a_{N_1})$, which follows

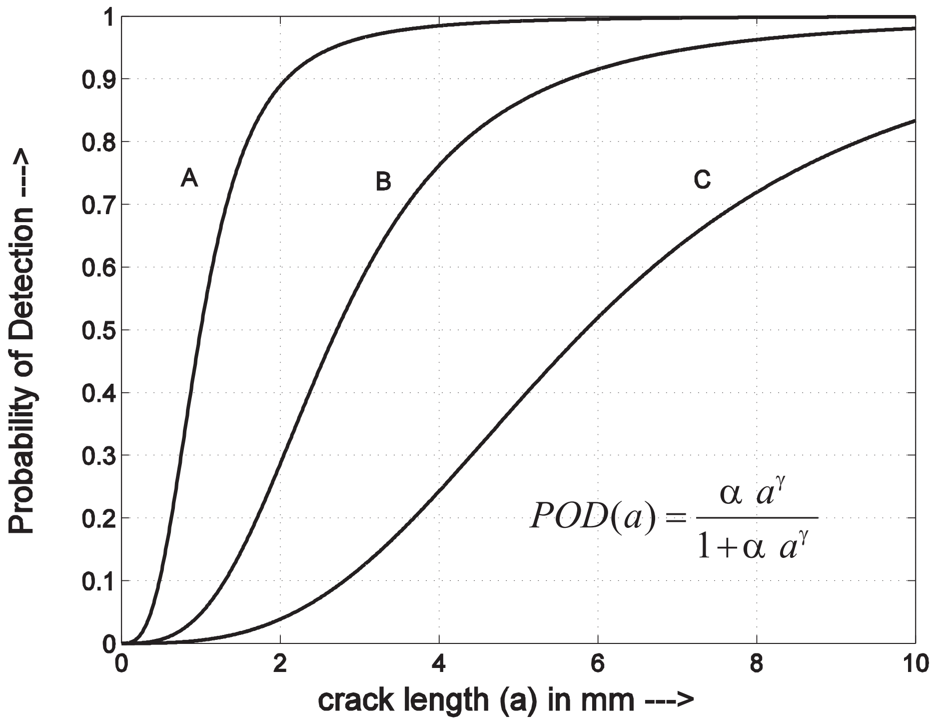


Figure 2: POD curves. A: $\alpha = 1.00\text{mm}^{-\gamma}$, $\gamma = 3.0$; B: $\alpha = 0.05\text{mm}^{-\gamma}$, $\gamma = 3.0$; C: $\alpha = 0.005\text{mm}^{-\gamma}$, $\gamma = 3.0$

from Equation (10) by replacing all N's with N_1's. Just after the inspection, the crack length density of the undetected crack is

$$f^{+}_{N_1}\left(a_{N_1}; N_1\right) = \frac{\overline{f^{+}}_{N_1}\left(a_{N_1}; N_1\right)}{\Pr\left(N_1\right)}, \tag{13}$$

where

$$\overline{f^{+}}_{N_1}\left(a_{N_1}; N_1\right) = PND\left(a_{N_1}\right) f_{N_1}\left(a_{N_1}\right). \tag{14}$$

In Equation (14), $PND\left(a_{N_1}\right)$ follows from Equation (12), and

$$\Pr(N_1) = \int_0^\infty PND\left(a_{N_1}\right) f_{N_1}\left(a_{N_1}\right) da_{N_1}. \tag{15}$$

It should be noted that $\Pr\left(N_1\right)$ is the probability that there exists an undetected crack after N_1 cycles.

To determine the crack length density at $N > N_1$, but before the second inspection at N_2 cycles, i.e. $N < N_2$, we follow the same steps as the ones leading to Equation (13). From Equations (12) and (13), we have

$$a_N = \left[a^{\beta}_{N_1} + \left(N - N_1\right) C\right]^{1/\beta}, \tag{16}$$

which can be solved for a_{N_1} as

$$a_{N_1} = \left(a_N^{\beta} - (N - N_1)\, C\right)^{1/\beta} \equiv h_1\left(a_N\right). \tag{17}$$

The crack length density of an undetected crack at cycle $N > N_1$ with $N < N_2$ is then given by

$$f_N\left(a_N; N > N_1\right) = \frac{\overline{f_N}\left(a_N; N > N_1\right)}{\Pr\left(N_1\right)}, \tag{18}$$

where,

$$\overline{f_N}\left(a_N; N > N_1\right) = \overline{f^+}_{N_1}\left[h_1\left(a_N\right); N_1\right] \left|\frac{da_{N_1}}{da_N}\right|. \tag{19}$$

Using

$$\begin{aligned} \frac{da_{N_1}}{da_N} &= \left[a_N^{\beta} - (N - N_1)\, C\right]^{(1-\beta)/\beta} a_N^{\beta-1} \\ &= \left[h_1\left(a_N\right)\right]^{1-\beta} a_N^{\beta-1}, \end{aligned} \tag{20}$$

we simplify Equation (19) to

$$\begin{aligned} \overline{f_N}\left(a_N; N > N_1\right) = PND\left[h_1\left(a_N\right)\right] f_0 \left\{\left[h_1\left(a_N\right)^{\beta} - N_1 C\right]^{1/\beta}\right\} \times \\ \left|h_1\left(a_N\right)^{\beta-1}\left[h_1\left(a_N\right)^{\beta} - N_1 C\right]^{(1-\beta)/\beta}\right| \left|h_1\left(a_N\right)^{1-\beta} a_N^{\beta-1}\right| \end{aligned} \tag{21}$$

which can be further simplified to

$$\begin{aligned} &\overline{f_N}\left(a_N; N > N_1\right) = \\ &PND\left[h_1\left(a_N\right)\right] f_0 \left[\left(a_N^{\beta} - NC\right)^{1/\beta}\right] \left|\left(a_N^{\beta} - NC\right)^{(1-\beta)/\beta} a_N^{\beta-1}\right|. \end{aligned} \tag{22}$$

The probability that there exists a crack with $a_N > a_{cr}$ at cycle N, which was undetected at the inspection at cycle N_1 can then be written as

$$\begin{aligned} \Pr(a_N > a_{cr}, N \geqslant N_1; ND) = \Pr(N_1) \int_{a_{cr}}^{\infty} f_N(a_N; N \geqslant N_1) da_N \\ = \int_{a_{cr}}^{\infty} PND\left[h_1\left(a_N\right)\right] f_0 \left[\left(a_N^{\beta} - NC\right)^{1/\beta}\right] \left|\left(a_N^{\beta} - NC\right)^{(1-\beta)/\beta} a_N^{\beta-1}\right| da_N \end{aligned} \tag{23}$$

After N cycles, the crack length density follows from Equation (18) as

$$f_N\left(a_N; N > N_1\right) = \frac{\overline{f_N}\left(a_N; N > N_1\right)}{\Pr\left(N_1\right)}, \tag{24}$$

where

$$\overline{f_N}\left(a_N; N > N_1\right) \tag{25}$$

is given by Equation (22).

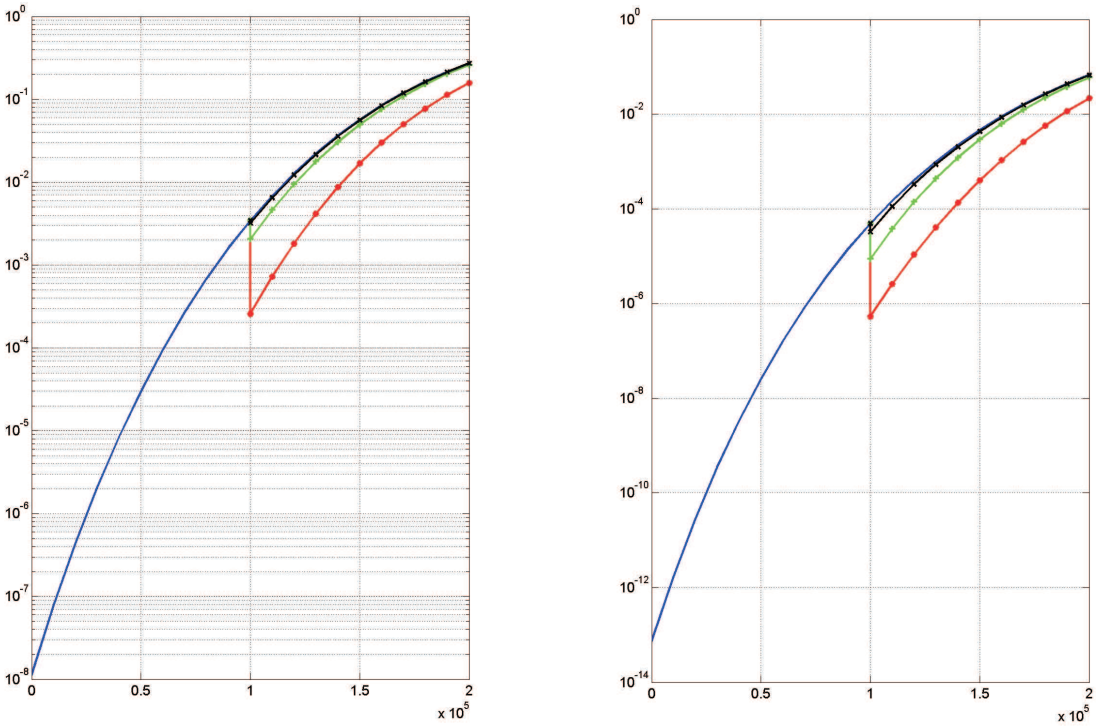
3 Numerical results

For the numerical calculations, the parameters in Paris law were chosen as (see Moran et. al. [7]): $m = 2.67,\ A = 5.069 \times 10^{-12},\ \Delta\sigma = 280$ MPa and $R = 0$. We also assume that the initial crack length a_0 has a lognormal distribution with density

$$f_0\left(a_0\right) = \frac{1}{a_0 \sigma \sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left(\ln\left(a_0\right) - \mu\right)^2\right]. \tag{26}$$

Here μ and σ are the mean and the standard deviation of the random variable $y = \ln\left(a_0\right)$. We assumed that the initial crack length a_0 has mean 0.250 mm and standard deviation 0.1 mm. In this example a_{cr} was chosen as $a_{cr} = 3$ mm. Results are given in Fig. 3 for the three monitoring techniques with POD's labeled 'A', 'B' and 'C' (see Figure 2). The probabilities of interest are evaluated using numerical integration. We first transform the integral defined over the infinite interval (a_{cr}, ∞) to a finite interval $(0, 1)$ using the transformation $x = e^{-(a - a_{cr})}$ and then use the adaptive quadrature routines described in Espelid [8] to evaluate the integral.

Results for three values of a_{cr} have been computed. The unmarked curves in Fig.3 plot $\Pr\left(a > a_{cr}; N\right)$ versus N, i.e., the probability that a crack with length $a > a_{cr}$ exists at cycle N. The curves have been calculated using Eq.(11). As N increases the probability
of a crack $a > a_{cr}$ increases. The curves, marked by symbols, represent $\Pr\left(a_N > a_{cr}, N \geqslant N_1; ND\right)$, i.e., the probability that an undetected crack of length $a_N > a_{cr}$, exists at cycle N, which has not been detected at the inspection at $N = N_1$=100,000 cycles. This probability has been calculated by using Eq. (23). The different symbols refer to the three POD's, as indicated in the figures. Since for $N > N_1$, the probability of an undetected crack $a_N > a_{cr}$ is smaller than the probability that a crack $a_N > a_{cr}$ exists at all, the marked curves are lower than the unmarked ones, with a step difference at $N = N_1$. Since there is a higher probability of $a_N > a_{cr}$ as N increases, there is also a higher probability of an undetected crack $a_N > a_{cr}$ but the difference decreases as N increases. It is noted that an inspection with a poor POD, such as represented by C, has very little positive influence. On the other hand, a good inspection technique such as given by A, can reduce the probability of an undetected crack $a_N > a_{cr}$ almost by an order of magnitude.



$a_{cr} = 2$ mm, Single inspection at 100,00 cycles.

$a_{cr} = 4$ mm, Single inspection at 100,000 cycles

Figure 3: $\Pr(a > a_{cr}; N)$ - solid line, vertical axis, and $\Pr(a_N > a_{cr}, N \geqslant N_1; ND)$-symbols, versus N- horizontal axis; x: POD C, +: POD B, dot: POD A.

4 Concluding comments

We have considered a surface-breaking crack in a cyclic tensile field. The original crack depth is defined by a probability density function. The growth of the crack with the number of cycles, N, is governed by Paris law. The detectability of the crack depends on a probability of detection (POD) function. Explicit formulas have been derived for the probability that there exists a crack $a_N > a_{cr}$, and for the probability that a crack $a_N > a_{cr}$ is not detected by an inspection after N_1 cycles. For two values of a_{cr} and three POD's, numerical results display the probability that a crack $a > a_{cr}$ is not detected.

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O verovatnoći neuočene površinske prskotine

Površinske prskotine usled zamora materijala su uobičajeni defekti u metalnim komponentama izloženim cikličnim opterećenjima. Ovakve prskotine teže da se šire u naponskim poljima koja su ispod kritičnog napona za statička opterećenja. Važan deo filozofije dizajniranja sa tolerisanjem oštećenja je zahtev da površinske prskotine moraju biti uočljive pre nego što dostignu kritičnu dubinu. U ovome radu, razmatramo površinsku prskotinu u dve dimenzije, čija je početna dubina definisana funkcijom gustine raspodele verovatnoće. Povećenje dubine prskotine sa brojem ciklusa opterećenja je definisano Parisovim zakonom, a uočljivost zavisi od verovatnoće uočavanja prskotine. Na osnovu ovih podataka odredjujemo verovatnoću da će dubina prskotine preći propisanu kritičnu vrednost pri odredjenom broju ciklusa opterećenja.

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