**Theoret. Appl. Mech.**, Vol.35, No.1-3, pp. 119–136, Belgrade 2008

# On Tensors of Elasticity

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Dedicated to the memory of prof. Dušan Krajčinović.

**Abstract**

An objective of this paper is to reconcile the "symmetry" approach with the "symmetry groups" approach as these two different points of view presently coexist in the literature. Here we will be concerned exclusively with linearly elastic materials. The starting point for an analysis of the inherent symmetry of elastic materials is the notion of a symmetry transformation.Particularly, we paid attention to the compliance tensor for cubic and hexagonal crystals.

**Keywords:** linearly elastic materials, symmetry, tensors of elasticity, compliance tensor, cubic and hexagonal crystals.

## 1 Introduction

We say the body is **linearly elastic** if for each $\mathbf{x} \in B$ there exists a linear transformation $\mathbb{C}_{\mathbf{x}}$ from the space of all tensors into the space of all symmetric tensors such that

$$\mathbf{T}(\mathbf{x}) = \mathbb{C}_{\mathbf{x}}[\mathbf{E}(\mathbf{x})], \tag{1}$$

where $\mathbf{E}$ is **strain tensor**. By definition

$$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \tag{2}$$

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where $\mathbf{u}$ and $\nabla\mathbf{u}$ are the **displacement vector** and the **displacement gradient**, respectively.

We call $\mathbb{C}_{\mathbf{x}}$ the **elasticity tensor** for $\mathbf{x}$ and the function $\mathbb{C}$ on $B$ with values $\mathbb{C}_{\mathbf{x}}$ the **elasticity field**. In general, $\mathbb{C}_{\mathbf{x}}$ depends on $\mathbf{x}$; if, however, $\mathbb{C}_{\mathbf{x}}$ and the density $\varrho(\mathbf{x})$ are independent of $\mathbf{x}$, we say that $\mathcal{B}$ is **homogeneous**.

Since, $\mathbf{T}$ and $\mathbf{E}$ are symmetric, the elasticity tensor has the following properties:

$$C_{ijkl} = C_{jikl} = C_{ijlk}. \tag{3}$$

We call the 36 numbers $C_{ijkl}$ **elasticities**.

Further, we say that $\mathbb{C}$ is symmetric if

$$\mathbf{A}\cdot \mathbb{C}[\mathbf{B}] = \mathbf{B}\cdot \mathbb{C}[\mathbf{A}]$$

for every pair of **symmetric tensors A** and **B**, **positive semi-definite** if

$$\mathbf{A}\cdot \mathbb{C}[\mathbf{A}] \geq 0$$

for every symmetric tensor $\mathbf{A}$, and **positive definite** if

$$\mathbf{A}\cdot \mathbb{C}[\mathbf{A}] > 0$$

for every non-zero symmetric tensor $\mathbf{A}$. Of course, $\mathbb{C}$ is symmetric iff its components obey

$$C_{ijkl} = C_{klij}. \tag{4}$$

Then $\mathbb{C}$ has 21 distinct elasticities, which corresponds to the most asymmetric elastic solid, namely to **triclinic crystals**.

If the elasticity tensor is invertible, than its inverse

$$\mathbb{K}_{\mathbf{x}} = \mathbb{C}_{\mathbf{x}}^{-1} \tag{5}$$

is called the **compliance tensor**, defining the relation

$$\mathbf{E}(\mathbf{x}) = \mathbb{K}_{\mathbf{x}}[\mathbf{T}(\mathbf{x})] \tag{6}$$

between the strain $\mathbf{E}(\mathbf{x})$ and the stress $\mathbf{T}(\mathbf{x})$ at $\mathbf{x}$.

Note that $\mathbb{C}$ is invertible whenever it is positive definite.

## 2 Material symmetry

For crystals with higher symmetry, the number of elasticities can be reduced further, the exact number being dependent on the material symmetries present in the crystal.

Material symmetry is exhibited in the sense that particular changes of reference configuration exist which leave the stress at $\mathbf{x}$ arising from an arbitrary deformation invariant. The larger the collection of such transformations the greater the degree of symmetry possessed by the material.
Thus we have motivated the following **definition:**

> The **symmetry group** $g_{\mathbf{x}}$ for the material at $\mathbf{x}$ is the set of all **orthogonal** tensors $\mathbf{Q}$ that obey
>
> $$\mathbf{Q}\mathbb{C}_{\mathbf{x}}[\mathbf{E}]\mathbf{Q}^T = \mathbb{C}_{\mathbf{x}}[\mathbf{Q}\mathbf{E}\mathbf{Q}^T] \tag{7}$$
>
> for every symmetric tensor $\mathbf{E}$. We say that the material at $\mathbf{x}$ is **isotropic** if the symmetry group $g_{\mathbf{x}}$ equals the orthogonal group, **anisotropic** if $g_{\mathbf{x}}$ is a **proper** subgroup of the orthogonal group.

Clearly, $g_{\mathbf{x}}$ always contains the two-element group $\{-\mathbf{I}, \mathbf{I}\}$ as a subgroup. It can be seen (Spencer [1971]) that $g_{\mathbf{x}}$ is the direct product of this two-element group and a group $g_o$ which consists only of proper orthogonal transformations, i.e. rotations. Consequently the type of anisotropy is characterized by the type of the group $g_o$.

Although there is an infinite number of types of rotation groups $g_o$, twelve of them seem to exhaust the kinds of symmetries occurring in theories proposed up to now as being appropriate to describe the behavior of real anisotropic solids. Particularly transverse isotropy is appropriate to real materials having a laminated or a bounded structure.

These thirty-two crystal classes are grouped into following six systems:

(i) *Triclinic system*,

(ii) *Monoclinic system*,

(iii) *Rhombic system*,

(iv) *Tetragonal system*,

(v) *Hexagonal system*,

(vi) *Cubic system*.

## 2.1 Isotropy

There are no isotropic tensors of the first order. The isotropic tensors of second, third and higher order can be constructed only by tensors $\delta_{ij}$, Kronecker delta, and $e_{ijk}$, Ricci tensor of alternation. Obviously, tensors

$$\delta_{i_1 i_2}\delta_{i_3 i_4}\ldots\delta_{i_{r-1} i_r}$$

and

$$e_{i_1 i_2 i_3}\delta_{i_4 i_5}\dots\delta_{i_{r-1} i_r}$$

are of even and odd order, respectively. They are isotropic, as well as any of their **isomers**, i.e. tensors which differ from original one by the arrangements of its indices. Therefore, any linear combination of such isomers is an isotropic tensor. It can also be proved that any isotropic tensor can be represented by linear combination of some isomers (Gurevich [1948], Spencer [1971]). For example,

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}, \tag{8}$$

and

$$\begin{aligned} c_{ijklmn} = {} & \lambda_1\delta_{ij}\delta_{kl}\delta_{mn} + \lambda_2\delta_{ij}\delta_{km}\delta_{ln} + \lambda_3\delta_{ij}\delta_{kn}\delta_{lm} + \\ & + \lambda_4\delta_{ik}\delta_{jl}\delta_{mn} + \lambda_5\delta_{ik}\delta_{jm}\delta_{ln} + \lambda_6\delta_{ik}\delta_{jn}\delta_{lm} + \lambda_7\delta_{il}\delta_{jk}\delta_{mn} + \\ & + \lambda_8\delta_{il}\delta_{jm}\delta_{kn} + +\lambda_9\delta_{il}\delta_{jn}\delta_{km} + \lambda_{10}\delta_{im}\delta_{jk}\delta_{ln} + \lambda_{11}\delta_{im}\delta_{jl}\delta_{kn} + \\ & + \lambda_{12}\delta_{im}\delta_{jn}\delta_{kl} + \lambda_{13}\delta_{in}\delta_{jk}\delta_{lm} + \lambda_{14}\delta_{in}\delta_{jl}\delta_{km} + \lambda_{15}\delta_{in}\delta_{jm}\delta_{kl} \end{aligned} \tag{9}$$

are general forms of isotropic tensors of fourth and sixth order, respectively. Isotropic tensors of eight and higher even order can be constructed in the same way. However, in these cases their isomers are not mutually independent. More precisely, the number of independent isomers, $L_r$, is less then the number of all their possible isomers

$$N_r = \frac{r!}{2^n n!},$$

where $r = 2n$. The following table illustrate it for some $r$

| | | | | | | |
|---|---|---|---|---|---|---|
| $r$ | $=$ | 2 | 4 | 6 | 8 | 10 |
| $N_r$ | $=$ | 1 | 3 | 15 | 105 | 945 |
| $L_r$ | $=$ | 1 | 3 | 15 | 91 | 603. |

In order to calculate $L_r$ the method of theory of group representation is used (Ljubarskii [1957]). Because of huge number of $L_r$, for practical purposes in continuum mechanics, we usually confine our attention to $r = 2, 4, 6$. Particularly, when dealing with elasticity tensors of second and third order we make use of their symmetric properties, i.e. (3) and

$$c_{ijklmn} = c_{jiklmn} = c_{klijmn} = c_{ijklnm} = c_{ijmnkl}, \tag{10}$$

so that their representation becomes quite simple:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{11}$$

$$
\begin{aligned}
c_{ijklmn} = {} & a\delta_{ij}\delta_{kl}\delta_{mn} + \\
& + b\left(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} + \delta_{im}\delta_{kl}\delta_{jn}\right. \\
& + \left.\delta_{in}\delta_{kl}\delta_{jm} + \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{il}\delta_{jk}\delta_{mn}\right) + \\
& + c\left(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{il}\delta_{jm}\delta_{kn}\right. \\
& + \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{im}\delta_{jl}\delta_{kn} + \\
& + \left.\delta_{in}\delta_{jk}\delta_{lm} + \delta_{in}\delta_{jl}\delta_{km}\right).
\end{aligned} \tag{12}
$$

It is easy to see that

$$
\lambda = c_{1122}, \qquad \mu = \frac{1}{2}(c_{1111} - c_{1122})
$$

and

$$
a = c_{112233}, \quad b = \frac{1}{2}(c_{112222} - c_{112233}),
$$
$$
c = \frac{1}{8}(c_{111111} + 2c_{112233} - 3c_{112222}).
$$

The constants $\lambda$, $\mu$ and $a, b, c$ are invariant with respect to the choice of the coordinate system. It is, therefore, appropriate to call them the **universal constants** for the isotropic materials. Contrary to them, elastic constants $c_{ijkl}$ (or $c_{ijklmn}$) change their values under arbitrary orthogonal coordinate transformations; also the number of constants, required to specify the elastic property of a crystal changes from coordinate system to coordinate system. These two aspects are a rather severe handicap in the treatment of various problems and may explain, in part at least, why the theory of cubic crystals in the elastic domain did not enjoy a development comparable to that of the classical or isotropic theory of elasticity. Because of that we have been for a long time in need to find universal constants for all crystal classes.

## 3 Invariant elastic constants for crystals

It was Thomas [1966] who obtained invariant constants $\lambda$, $\mu$, $\alpha$ for cubic crystals similar to Lames $\lambda$, $\mu$ for isotropic solids. He has proposed the following expression

$$
c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \alpha n_{ai}n_{aj}n_{ak}n_{al}, \tag{13}
$$

where $n_{ai}$ are the components of the unit vectors $\mathbf{n}_a$ $(a = 1, 2, 3)$ which represent the **crystallographic directions** of cubic crystal in an arbitrary Cartesian system (The crystallographic axes represent in direction and magnitude the three non-parallel edges of the unit cell of a crystal. The unit vectors along these axes are referred to as crystallographic directions (Nye [1960])). Srinivasan & Nigam [1969], proposed a procedure how to derive invariant constants of $C_{ijkl}$ for all other crystal classes. The procedure is, as suggested by Synge & Schild [1969], in some sense, based on the representation of tensors in **anholonomic coordinate systems**. For simplicity, the unit vectors $\mathbf{n}_a$ $(a = 1, 2, 3)$ along crystallographic directions are chosen as anholonomic basis. Generally, they are not orthogonal. In order to make this manuscript self-contained we proceed in explaining this procedure.

Given a vector $\mathbf{n}$. Then

$$\mathbf{n} = n_i \mathbf{e}_i = n_a \mathbf{n}_a;$$

$\mathbf{e}_i$ and $\mathbf{n}_a$ $(i, a = 1, 2, 3)$ are two systems of basis vectors, respectively. Usually we take $\mathbf{e}_i$ orthonormal. Then

$$n_i = n_{ai} n_a, \qquad n_{ai} = \mathbf{n}_a \cdot \mathbf{e}_i,$$

where there is summation over $a$; $a$ is not tensor index.

Let $\mathbf{m}_a$ be reciprocal basis to the basis vectors $\mathbf{n}_a$. Then

$$\mathbf{n}_a \cdot \mathbf{m}_b = \delta_{ab} \quad \Rightarrow \quad n_{ai} m_{bi} = \delta_{ab}.$$

Note that there is no distinction between contravariant and covariant indices since we are working in Cartesian frames of references. Then we obtain

$$n_a = m_{ai} n_i.$$

But $n_a$ do not depend on the choice of coordinate system with respect to the indices $i$. Therefore $n_a$ are invariant and behave as scalars with respect to any such coordinate transformation. The same approach can be applied to any tensor. Srinivasan & Nigam [1969] stated that this idea can be useful in finding the invariant dielectric constants, piezo-electric and photo-elastic coefficients. Because of that they confine they application of the procedure to the tensors of second, third and fourth order. See also Cowin & Mehrabadi [1987], Jarić [1989], Ting [1996], Jarić [1998].

We are interested in the fourth rang tensor $C_{ijkl}$. We write

$$C_{ijkl} = n_{ai} n_{bj} n_{ck} n_{dl} A_{abcd}.$$

Note that $A_{abcd}$ possesses the same symmetric properties as $C_{ijkl}$. In the above form the scalars $A_{abcd}$ are the 21 invariant elastic constants for the **triclinic crystal** (no axes or plane of symmetry). But from this expression it is possible to obtain expressions in the case of crystals belonging to other systems by imposing the various point group symmetries on $C_{ijkl}$. This is done by keeping the coordinates unchanged and transforming only the vectors $\mathbf{n}_a$ according to the symmetries present in the crystal whereas in dealing with elastic symmetry it has been customary to transform coordinates. We shall illustrate it in case of cubic crystals.

Here and further we shall use the following notation and definitions:

$Orth$ - the set of all orthogonal tensors,

$Orth^+$ - the set of all rotations,

$\mathbb{A}, \mathbb{B}, \ldots$ - 4-tensors in three-dimensional Euclidean real space $R_3$,

$\mathbb{Q} = \mathbf{Q} \times \mathbf{Q}$ - the orthogonal 4-tensor; $\mathbf{Q} \in Orth^+$; " $\times$ " - Kronecker product,

$\mathbb{I}$ - identity 4-tensor, $\mathbb{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

In the case of the cubic crystals crystallographic directions $\mathbf{n}_a$ are orthonormal. The atomic array is unchanged

(1) by inversion about the plane defined by $\mathbf{n}_a$. These will take $\mathbf{n}_a \Rightarrow -\mathbf{n}_a$.

(2) by $\pi/2$ rotations about each of $\mathbf{n}_a$.

These symmetry conditions leads to the vanishing of 12 constants. The remaining 9 constants are

$$\begin{aligned} A_{1111} &= A_{2222} = A_{3333} = A, \\ A_{1122} &= A_{2233} = A_{1133} = B, \\ A_{1212} &= A_{2323} = A_{1313} = C. \end{aligned}$$

Then

$$\mathbb{C} = A \sum_{a=1}^{3} \mathbb{N}_a + B \sum_{a<b}^{3} \mathbb{N}_{ab} + C \sum_{a<b}^{3} \mathbb{M}_{ab}, \tag{14}$$

where

$$\begin{aligned} \mathbb{N}_a \ &\Rightarrow \quad N_{a_{ijkl}} = n_{ai} n_{aj} n_{ak} n_{al}, \qquad \text{(no sum over } a\text{)}, \\ \mathbb{N}_{ab} \ &\Rightarrow \quad N_{ab_{ijkl}} = n_{ai} n_{aj} n_{bk} n_{bl} + n_{bi} n_{bj} n_{ak} n_{al}, \qquad \text{(no sum over } a \text{ and } b\text{)}, \\ \mathbb{M}_{ab} \ &\Rightarrow \quad M_{ab_{ijkl}} = n_{ai} n_{bj} n_{ak} n_{bl} + n_{ai} n_{bj} n_{bk} n_{al} + \\ &\qquad + n_{bi} n_{aj} n_{ak} n_{bl} + n_{bi} n_{aj} n_{bk} n_{al}, \quad \text{(no sum over } a \text{ and } b\text{)}. \end{aligned}$$

We can simplify expressions and calculations making use of $\mathbf{n}_a \Rightarrow n_{ai} n_{aj}$ (no sum over $a$) and $\mathbf{n}_{ab} \Rightarrow n_{ai} n_{bj}$. Then

$$\begin{aligned} \mathbb{N}_a &= \mathbf{n}_a \otimes \mathbf{n}_a, \\ \mathbb{N}_{ab} &= \mathbf{n}_a \otimes \mathbf{n}_b + \mathbf{n}_b \otimes \mathbf{n}_a, \\ \mathbb{M}_{ab} &= \mathbf{n}_{ab} \otimes \mathbf{n}_{ab} + \mathbf{n}_{ab} \otimes \mathbf{n}_{ba} + \mathbf{n}_{ba} \otimes \mathbf{n}_{ab} + \mathbf{n}_{ba} \otimes \mathbf{n}_{ba}. \end{aligned}$$

Also

$$\mathbf{I} = \sum_a \mathbf{n}_a \ \Rightarrow \quad \mathbf{I}\otimes\mathbf{I} = \sum_{a,b} \mathbf{n}_a \otimes \mathbf{n}_b = $$
$$= \sum_a \mathbf{n}_a \otimes \mathbf{n}_a + \sum_{a<b} \left(\mathbf{n}_a \otimes \mathbf{n}_b + \mathbf{n}_b \otimes \mathbf{n}_a\right),$$

i.e.

$$\mathbf{I}\otimes\mathbf{I} = \sum_a \mathbb{N}_a + \sum_{a<b} \mathbb{N}_{ab}. \tag{15}$$

Next, from

$$\mathbb{I} \ \Rightarrow \quad \mathbb{I}_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

we have

$$\begin{aligned}
2\mathbb{I}_{ijkl} &= \left(\sum_a n_{ai}n_{ak}\right)\left(\sum_b n_{bj}n_{bl}\right) + \left(\sum_a n_{ai}n_{al}\right)\left(\sum_b n_{bj}n_{bk}\right) = \\
&= \sum_{a,b}\left(n_{ai}n_{bj}n_{ak}n_{bl} + n_{ai}n_{bj}n_{bk}n_{al}\right) = \\
&= 2\sum_a n_{ai}n_{aj}n_{ak}n_{al} + \\
&+ \sum_{a<b}\left(n_{ai}n_{bj}n_{ak}n_{bl} + n_{ai}n_{bj}n_{bk}n_{al} + n_{bi}n_{aj}n_{ak}n_{bl} + n_{bi}n_{aj}n_{bk}n_{al}\right) = \\
&= 2\sum_a N_{a_{ijkl}} + \sum_{a<b} M_{ab_{ijkl}},
\end{aligned}$$

i.e.

$$2\mathbb{I} = 2\sum_a \mathbb{N}_a + \sum_{a<b} \mathbb{M}_{ab}. \tag{16}$$

¿From (14), (15) and (16) we finally obtain

$$\mathbb{C} = \lambda\mathbf{I}\otimes\mathbf{I} + 2\mu\mathbb{I} + \alpha\mathbb{N}, \tag{17}$$

where

$$\lambda = B, \quad \mu = C, \quad \alpha = A - B - 2C$$

and

$$\mathbb{N} = \sum_a \mathbb{N}_a = \sum_a \mathbf{n}_a \otimes \mathbf{n}_a \quad \Rightarrow \quad \mathbb{N}_{ijkl} = n_{ai}n_{aj}n_{ak}n_{al}. \tag{18}$$

Note that $\mathbb{N}$ is symmetric tensor with respect to all of its indices.

In the same way Srinivasan & Nigam [1969] obtained expressions in the case of crystals belonging to other systems by imposing the various point group symmetries on $C_{ijkl}$.

# 4 On compliance tensor

Here we state (7) in the restricted form:

We say that $\mathbb{Q}$ is symmetry transformation of $\mathbb{C}$ if

$$\mathbb{Q}\mathbb{C} = \mathbb{C}\mathbb{Q} \tag{19}$$

for some $\mathbf{Q} \in Orth^+$. It is easy to show that all such $\mathbf{Q} \in Orth^+$ define an **isotropy group** of $\mathbb{C}$. More precisely, we say that $\mathbb{C}$ is an **isotropic tensor** if $g_C = Orth^+$; otherwise $\mathbb{C}$ is an **anisotropic** tensor and then $g_C \subset Orth^+$.

Let $\mathbb{C}$ be **invertible**. Let $\mathbb{K}$ be the corresponding **compliance tensor**, i.e.

$$\mathbb{C}\mathbb{K} = \mathbb{K}\mathbb{C} = \mathbb{I}. \tag{20}$$

Then from (19) and (20) we have the following (Gurtin [1972])

**Proposition 1** *The isotropy group $g_C$ of tensor $\mathbb{C}$ is also the isotropy group of its inverse (compliance) tensor $\mathbb{K}$.*

**Corollary 1** *The representation tensors forms of tensors $\mathbb{C}$ and $\mathbb{K}$ is the same.*

The above proposition and its corollary enable one to find the explicit form of $\mathbb{K}$ if the form of $\mathbb{C}$ is known. The way to find it is an algebraic one: we simply have to find the corresponding coefficients of representation of $\mathbb{K}$ making use of (20) as unique functions of the coefficients of $\mathbb{C}$. We proceed to demonstrate it in the case when $\mathbb{C}$ represents the elasticity tensor for cubic crystals and hexagonal crystals.

First, we start with cubic crystals, i.e. with (17)

$$\mathbb{C} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I} + \alpha \mathbb{N}.$$

Since $\mathbb{A}\mathbb{B} = \mathbb{D}$, or in componental form $\mathbb{D}_{ijkl} = (\mathbb{A}\mathbb{B})_{ijkl} = \mathbb{A}_{ijpq}\mathbb{B}_{pqkl}$, the reader can easily verify that the following multiplication table holds:

$$\begin{array}{c|ccc} & \mathbb{I} & \mathbf{I}\otimes\mathbf{I} & \mathbb{N} \\ \hline \mathbb{I} & \mathbb{I} & \mathbf{I}\otimes\mathbf{I} & \mathbb{N} \\ \mathbf{I}\otimes\mathbf{I} & \mathbf{I}\otimes\mathbf{I} & 3\mathbf{I}\otimes\mathbf{I} & \mathbf{I}\otimes\mathbf{I} \\ \mathbb{N} & \mathbb{N} & \mathbf{I}\otimes\mathbf{I} & \mathbb{N} \end{array} \tag{21}$$

**Lemma 1** *$\mathbf{I} \otimes \mathbf{I}$, $\mathbb{I}$ and $\mathbb{N}$ are linearly independent.*

The proof is trivial.

Thus, the 4-tensors $\mathbf{I} \otimes \mathbf{I}$, $\mathbb{I}$ and $\mathbb{N}$ are the basis of the representation of $\mathbb{C}$ and its compliance tensor $\mathbb{K}$.

**Proposition 2**

$$\mathbb{K} = p\mathbf{I} \otimes \mathbf{I} + 2q\mathbb{I} + r\mathbb{N}, \tag{22}$$

*where*

$$p = -\frac{\lambda}{(2\mu+\alpha)(3\lambda+2\mu+\alpha)}, \quad q = \frac{1}{4\mu}, \quad r = -\frac{\alpha}{2\mu(2\mu+\alpha)} \tag{23}$$
$$\mu \neq 0, \quad 2\mu+\alpha \neq 0, \quad 3\lambda+2\mu+\alpha \neq 0.$$

*Proof.* By Corollary, $\mathbb{K}$ must have the same form as $\mathbb{C}$. Therefore we have only to find the value of $p$, $q$ and $r$ from (20). Again, making use of the table and after some arrangement of the terms we obtain

$$[(3\lambda+2\mu+\alpha)p + 2\lambda q + \lambda r]\mathbf{I} \otimes \mathbf{I} + 4\mu q\mathbb{I} + [2\alpha q + (2\mu+\alpha)r]\mathbb{N} = \mathbb{I}.$$

By Lemma 1 we have the set of linear equations

$$\begin{aligned} (3\lambda+2\mu+\alpha)p + 2\lambda q + \lambda r &= 0, \\ 4\mu q &= 1, \\ 2\alpha q + (2\mu+\alpha)r &= 0 \end{aligned} \tag{24}$$

and from them (23). ■

**Remark 1.** Several symmetric relationships between constants $\lambda$, $\mu$, $\alpha$ and $p$, $q$, $r$ can be derived.

I. Because of (20), (17) and (22) the set of equations (24) is symmetric with respect to strict interchange of $\lambda$, $\mu$, $\alpha$ and $p$, $q$, $r$ (i.e. $\lambda \Leftrightarrow p$, $\mu \Leftrightarrow q$, $\alpha \Leftrightarrow r$). Thus

$$\begin{aligned} (3\lambda+2\mu+\alpha)p + 2\lambda q + \lambda r &= 0, \\ 4\mu q &= 1, \\ 2r\mu + (2q+r)\alpha &= 0. \end{aligned}$$

Then from (23) we have at once

$$\lambda = -\frac{p}{(2q+r)(3p+2q+r)}, \quad \mu = \frac{1}{4q}, \quad \alpha = -\frac{r}{2q(2q+r)}, \tag{25}$$
$$q \neq 0, \quad 2q+r \neq 0, \quad 3p+2q+r \neq 0.$$

II. It is easy to verify the following symmetric relations

$$\begin{aligned} 4\mu q &= 1, \\ (2\mu+\alpha)(2q+r) &= 1, \\ (3\lambda+2\mu+\alpha)(3p+2q+r) &= 1. \end{aligned} \tag{26}$$

III. In the linear theory of elasticity the stored energy $\Sigma$ is defined by

$$\Sigma = \frac{1}{2}\mathbf{e}\cdot \mathbb{C}[\mathbf{e}],$$

where $\mathbf{e}$ is the infinitesimal strain. If we write $\varepsilon$ for the traceless part of $\mathbf{e}$, i.e. $I_\varepsilon = \operatorname{tr}\varepsilon = 0$, then

$$\varepsilon = \frac{1}{3}I_\varepsilon \mathbf{I} - \mathbf{e}, \qquad I_e = \operatorname{tr}\mathbf{e},$$

so that

$$\operatorname{tr}\mathbf{e}^2 = \frac{1}{3}I_e^2 + \operatorname{tr}\varepsilon^2.$$

Then (see Appendix)

$$\Sigma = \frac{1}{6}\,(3\lambda + 2\mu + \alpha)\, I_e^2 + \frac{1}{2}(2\mu + \alpha)\sum_a \varepsilon_{aa}^2 + 2\mu \sum_{a<b} \varepsilon_{ab}^2. \tag{27}$$

There is physical reason to require that $\Sigma$ be a positive definite form, for then in any given small strain from an unstressed state, the stress must do positive work. Hence, from (27) we conclude that this will be the case iff

$$\begin{gathered} \mu > 0, \\ 2\mu + \alpha > 0, \\ 3\lambda + 2\mu + \alpha > 0, \end{gathered} \tag{28}$$

or, equivalently,

$$\begin{gathered} q > 0, \\ 2q + r > 0, \\ 3p + 2q + r > 0, \end{gathered} \tag{29}$$

which follows from (26). The same conclusion will follow from the positive definiteness of $\Sigma = \frac{1}{2}\mathbf{t}\cdot \mathbb{K}[\mathbf{t}]$, where $\mathbf{t}$ is stress tensor.

Another, very important, conclusion follows from (29) and (23):

The existence of compliance tensor does not ensure the positive definitness of the strain energy.

**Remark 2.** Jarić [1998] has shown that $\mathbb{C}$ is an isotropic tensor in $E_n$, $n \geq 2$, iff

$$\mathbb{C} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I}. \tag{30}$$

By the same procedure it is easy to derive the compliance tensor $\mathbb{K}$ for this general case. Then

| | $\mathbb{I}$ | $\mathbf{I}\otimes\mathbf{I}$ |
|---|---|---|
| $\mathbb{I}$ | $\mathbb{I}$ | $\mathbf{I}\otimes\mathbf{I}$ |
| $\mathbf{I}\otimes\mathbf{I}$ | $\mathbf{I}\otimes\mathbf{I}$ | $3\mathbf{I}\otimes\mathbf{I}$ |
| $\mathbb{N}$ | $\mathbb{N}$ | $\mathbf{I}\otimes\mathbf{I}$ |

**Proposition 3**

$$\mathbb{K} = \sigma\mathbf{I}\otimes\mathbf{I} + 2\tau\mathbb{I}, \tag{31}$$

*where*

$$\sigma = -\frac{\lambda}{2\mu(n\lambda+2\mu)}, \quad \tau = \frac{1}{4\mu}, \tag{32}$$
$$\mu \neq 0, \quad n\lambda + 2\mu \neq 0.$$

*Proof.* Again from (20) and the table in this case we obtain

$$4\mu\tau = 1,$$
$$\sigma(n\lambda+2\mu) + 2\lambda\tau = 0,$$

and from this (32). ■

In the theory of elasticity two cases are of importance:
$n = 2$:

$$\tau = \frac{1}{4\mu}, \qquad \sigma = -\frac{\lambda}{4\mu(\lambda+\mu)}, \tag{33}$$

and
$n = 3$:

$$\tau = \frac{1}{4\mu}, \qquad \sigma = -\frac{\lambda}{2\mu(3\lambda+2\mu)}. \tag{34}$$

This follows also from (23) when $\alpha = 0$.

We shall proceed one step further applying the same procedure for

## 4.1 Hexagonal system

In hexagonal crystals the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is 120° and those between $\mathbf{n}_3$ and $\mathbf{n}_1$, and $\mathbf{n}_3$ and $\mathbf{n}_2$ is 90°. Also $\mathbf{n}_3$ will be a six-fold axes of rotation. In this case the relation

$$n_{1i}n_{1j} + n_{2i}n_{2j} + \frac{1}{2}\left(n_{1i}n_{2j} + n_{2i}n_{1j}\right) + \frac{3}{4}n_{3i}n_{3j} = \frac{3}{4}\delta_{ij}$$

hold. Then

$$C_{ijkl} = \lambda_1 \delta_{ij}\delta_{kl} + \frac{1}{2}\lambda_2 \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \lambda_3 n_{3i}n_{3j}n_{3k}n_{3l} + \\ + \lambda_4 \left(n_{3i}n_{3j}\delta_{kl} + n_{3k}n_{3l}\delta_{ij}\right) + \\ + \lambda_5 \left(n_{3i}n_{3k}\delta_{jl} + n_{3i}n_{3l}\delta_{jk} + n_{3j}n_{3k}\delta_{il} + n_{3j}n_{3l}\delta_{ik}\right),$$

or

$$\mathbb{C} = \lambda_1 \mathbf{I}\otimes\mathbf{I} + \lambda_2 \mathbb{I} + \lambda_3 \mathbb{N}_3 + \lambda_4 \left(\mathbf{n}_3 \otimes \mathbf{I} + \mathbf{I}\otimes\mathbf{n}_3\right) + \lambda_5 \mathbb{M}, \tag{35}$$

$$\mathbb{M} \quad \Rightarrow \quad M_{ijkl} = n_{3i}n_{3k}\delta_{jl} + n_{3i}n_{3l}\delta_{jk} + n_{3j}n_{3k}\delta_{il} + n_{3j}n_{3l}\delta_{ik}.$$

Obviously its compliance tensor $\mathbb{K}$ is given by the expression

$$\mathbb{K} = p_1 \mathbf{I}\otimes\mathbf{I} + p_2 \mathbb{I} + p_3 \mathbb{N}_3 + p_4 \left(\mathbf{n}_3 \otimes \mathbf{I} + \mathbf{I}\otimes\mathbf{n}_3\right) + p_5 \mathbb{M}. \tag{36}$$

Before we proceed further here we give the multiplication table of $\mathbf{I}\otimes\mathbf{I}$, $\mathbb{N}_3$, $\mathbf{n}_3\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{n}_3$ and $\mathbb{M}$:

| | $\mathbf{I}\otimes\mathbf{I}$ | $\mathbb{N}_3$ |
|---|---|---|
| $\mathbb{I}\otimes\mathbb{I}$ | $3\mathbf{I}\otimes\mathbf{I}$ | $\mathbf{I}\otimes\mathbf{n}_3$ |
| $\mathbf{N}_3$ | $\mathbf{n}_3\otimes\mathbf{I}$ | $\mathbb{N}_3$ |
| $\mathbf{n}_3\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{n}_3$ | $3\mathbf{n}_3\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{I}$ | $\mathbb{N}_3+\mathbf{I}\otimes\mathbf{n}_3$ |
| $\mathbb{M}$ | $4\mathbf{n}_3\otimes\mathbf{I}$ | $4\mathbb{N}_3$ |

| | $\mathbf{n}_3\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{n}_3$ | $\mathbb{M}$ |
|---|---|---|
| $\mathbb{I}\otimes\mathbb{I}$ | $3\mathbf{I}\otimes\mathbf{n}_3+\mathbf{I}\otimes\mathbf{I}$ | $4\mathbf{I}\otimes\mathbf{n}_3$ |
| $\mathbf{N}_3$ | $\mathbb{N}_3+\mathbf{n}_3\otimes\mathbf{I}$ | $4\mathbb{N}_3$ |
| $\mathbf{n}_3\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{n}_3$ | $3\mathbb{N}_3+\mathbf{I}\otimes\mathbf{I}+\mathbf{n}_3\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{n}_3$ | $4(\mathbb{N}_3+\mathbf{I}\otimes\mathbf{n}_3)$ |
| $\mathbb{M}$ | $4(\mathbb{N}_3+\mathbf{n}_3\otimes\mathbf{I})$ | $2(4\mathbb{N}_3+\mathbb{M})$ |

Again from $\mathbb{CK} = \mathbb{I}$, we obtain, after some lengthy calculation and arrangement of terms,

$$\begin{gathered}\mathbf{I}\otimes\mathbf{I}\left[p_1(3\lambda_1+\lambda_2+\lambda_4)+p_2\lambda_1+p_4(\lambda_1+\lambda_4)\right]+ \\ +\mathbb{I}p_2\lambda_2+ \\ +\mathbb{N}_3\left[p_2\lambda_3+p_3(\lambda_2+\lambda_3+\lambda_4+4\lambda_5)+p_4(\lambda_3+3\lambda_4+4\lambda_5)+4p_5(\lambda_3+\lambda_4+2\lambda_5)\right]+ \\ \mathbb{M}\left[p_2\lambda_5+p_5(\lambda_2+2\lambda_5)\right]+ \\ \mathbf{n}_3\otimes\mathbf{I}\left[p_1(\lambda_3+3\lambda_4+4\lambda_5)+p_2\lambda_4+p_4(\lambda_2+\lambda_3+\lambda_4+4\lambda_5)\right]+ \\ +\mathbf{I}\otimes\mathbf{n}_3\left[p_2\lambda_4+p_3(\lambda_1+\lambda_4)+p_4(3\lambda_1+\lambda_2+\lambda_4)+4p_5(\lambda_1+\lambda_4)\right]=\mathbb{I}.\end{gathered} \tag{37}$$

We note that the last term can be written as

$$\mathbf{I}\otimes\mathbf{n}_3\left[\lambda_1(p_3+3p_4+4p_5)+\lambda_2 p_4+\lambda_4(p_2+p+3+p_3+p_4+4p_5)\right],$$

i.e. the last two terms can be obtained from each other when we strictly interchange the coefficients $\lambda_\sigma \Leftrightarrow p_\sigma$ $(\sigma = 1,2,3,4,5)$. The same symmetric property is satisfied by the other coefficients as can be very easily verified. Thus from (37) we can write the expression for

$$\mathbb{KC} = \mathbb{I},$$

when we strictly interchange the coefficients $\lambda_\sigma \Leftrightarrow p_\sigma$ $(\sigma = 1,2,3,4,5)$. The only difference appears in the coefficients $\mathbf{n}_3 \otimes \mathbf{I}$ and $\mathbf{I} \otimes \mathbf{n}_3$, i.e. they are mutually interchanged. Thus

$$\begin{gathered}
\mathbf{I} \otimes \mathbf{I}[\cdots] + \mathbb{I}[\cdots] + \mathbb{N}_3[\cdots] + \mathbb{M}[\cdots] + \\
+\mathbf{n}_3 \otimes \mathbf{I}[\lambda_1(p_3 + 3p_4 + 4p_5) + \lambda_2 p_4 + \lambda_4(p_2 + p_3 + p_4 + 4p_5)] + \\
\mathbf{I} \otimes \mathbf{n}_3\,[\lambda_2 p_4 + \lambda_3(p_1 + p_4) + \lambda_4(3p_1 + p_2 + p_4) + 4\lambda_5(p_1 + p_4)] = \mathbb{I}.
\end{gathered}$$

Next, we can prove that linear operators

$$\mathbf{I} \otimes \mathbf{I},\ \mathbb{I},\ \mathbb{N}_3,\ \mathbf{I} \otimes \mathbf{n}_3,\ \mathbf{n}_3 \otimes \mathbf{I},\ \mathbb{M}$$

are linearly independent. Therefore from

$$\mathbb{I}: \quad p_2\lambda_2 = 1$$

we have at once

$$p_2 = \frac{1}{\lambda_2}, \tag{38}$$

and from

$$\mathbb{M}: \quad p_2\lambda_5 + p_5(\lambda_2 + 2\lambda_5) = 0,$$

$$p_5 = -\frac{\lambda_5}{\lambda_2(\lambda_2 + 2\lambda_5)}. \tag{39}$$

Moreover, the following symmetric relations holds

$$(p_2 + 2p_5)(\lambda_2 + 2\lambda_5) = 1.$$

Further, the remaining set of equations

$$\begin{aligned}
\mathbf{I} \otimes \mathbf{I}: &\quad p_1(3\lambda_1 + \lambda_2 + \lambda_4) + p_2\lambda_1 + p_4(\lambda_1 + \lambda_4) = 0, \qquad (40)\\
\mathbb{N}_3: &\quad p_2\lambda_3 + p_3(\lambda_2 + \lambda_3 + \lambda_4 + 4\lambda_5) + \\
&\quad + p_4(\lambda_3 + 3\lambda_4 + 4\lambda_5) + 4p_5(\lambda_3 + \lambda_4 + 2\lambda_5) = 0,\\
\mathbf{n}_3 \otimes \mathbf{I}: &\quad p_1(\lambda_3 + 3\lambda_4 + 4\lambda_5) + p_2\lambda_4 + p_4(\lambda_2 + \lambda_3 + \lambda_4 + 4\lambda_5) = 0,\\
\mathbf{I} \otimes \mathbf{n}_3: &\quad p_2\lambda_4 + p_3(\lambda_1 + \lambda_4) + p_4(3\lambda_1 + \lambda_2 + \lambda_4) + 4p_5(\lambda_1 + \lambda_4) = 0,
\end{aligned}$$

has to be solved for $p_1$, $p_3$ and $p_4$. Again, we remind the reader that the last equation can be written as

$$\lambda_1(p_3 + 3p_4 + 4p_5) + \lambda_2 p_4 + \lambda_4(p_2 + p_3 + p_4 + 4p_5) = 0,$$

which is obviously symmetric to the third one. From ($40_1$) and ($40_3$), making use of (38), we obtain

$$p_1 = \frac{1}{\lambda_2 \Delta}\left[\lambda_4^2 - \lambda_1(\lambda_2 + \lambda_3 + 4\lambda_5)\right], \tag{41}$$

$$p_4 = \frac{1}{\lambda_2 \Delta}\left[\lambda_1(\lambda_3 + 4\lambda_5) - \lambda_4(\lambda_2 + \lambda_4)\right], \tag{42}$$

where

$$\Delta = (2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 4\lambda_5) - 2(\lambda_1 + \lambda_4)^2.$$

Further, from ($40_2$) and ($40_4$), taking into account (38) and (39), we obtain (42) and $p_3$.

$$\begin{aligned} p_3 = &- \frac{\lambda_4}{\lambda_2(\lambda_1 + \lambda_4)} + \frac{4\lambda_5}{\lambda_2(\lambda_2 + 2\lambda_5)} - \\ &- \frac{(3\lambda_1 + \lambda_2 + \lambda_4)[-\lambda_4(\lambda_2 + \lambda_4) + \lambda_1(\lambda_3 + 4\lambda_5)]}{\lambda_2(\lambda_1 + \lambda_4)[-2(\lambda_1 + \lambda_4)^2 + (2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \lambda_3 + 4\lambda_4 + 4\lambda_5)]}. \end{aligned}$$

We close this subsection with remark that different approach can be found in Walpole [1984]. Also, we note that the determination of third-order elastic coefficients (elastic tensor of six order) has be done by Fumi [1952], [1987] and Brugger [1965]. In our opinion this topic deserves further investigation because of it importance.

# 5 Conclusion

Here we present a new approach in order to derive compliance tensor for cubic and hexagonal crystals. This new approach is an algebraic one and consists of the set of linear equations with respect to the material coefficients of compliance tensor. It appears that the relation between material constants for elastic and compliance tensor is symmetric for crystal classes considered here. So derived compliance tensor for isotropic materials can be regarded as a special case. The compliance tensor for other crystal classes can be derived in the same way. Then the procedure is lengthy one because of the number of material coefficients for these crystal classes.

# References

[1] Bruger, K., *Determaination of Third-Order Elastic Coefficients in Crystals*, J. Appl. Physics, Vol. 36, No. 3, pp.368-774, March (1965).

[2] Cowin, S. C., and Mehrabadi, M. M., *On the identification of material symmetry for anisotropic elastic material*, Q.J.Mech.Appl.Math., **40**, pp. 451-476, (1987).

[3] Fumi G. Fausto, *Third-Order Elastic Coefficients of Crystals*, Physical Review, Vol 83, no.4, 1274, (1951).

[4] Fumi G. Fausto, *Third-Order Elastic Coefficients in Trigonal and Hexagonal Crystals*, Physical Review, Vol 86, no.4, 561, (1952).

[5] Gurevich, G.B., *Basics of Algebraic Invariants Theory* (in Russian), Gostehizdat, (1948).

[6] Gurtin, M.E., *The linear theory of elasticity*, Handbuch der Physik $\mathbf{6a}_2$, Berlin, Springer-Verlag (1972).

[7] Jarić, J., *On the conditions of the existence of a plane of symmetry for anisotropic elastic material*, Mechanics Research Communications, Vol.21, No. 2, 153-174 (1994). g‘aa

[8] Spencer, A. J.M., *Theory of Invariants*, Ch.3 in "Continuum Phisics", Vol.**1**, (A.C.Eringen, ed.) Academic Press, New York, London (1971).

[9] Srinivasan, T.P., & Nigam, S.D., *Invariant Elastic Constants for Crysrals*, J. Math. Mech., Vol. 19., No. 5, (1969).

[10] Synge, J.L., and Schild, A., *Tensor calculus*, University of Toronto Press, Toronto, (1969).

[11] Thomas, T.Y., *On the stress-strain relations for cubic crystals*, Proc. Nat. Acad. Sci., **55**, pp. 235-239 (1966).

[12] Ting, T.C.T., *Anisotropic Elasticity*, Oxford Scientific Publications, (1996).

[13] Walpole, L.J., *Fourth rank tensors of the thirty two crystal classes*, Proc. R. Soc. Lond. A **391**, pp. 149-179, (1984).

# Appendix

In deriving (27) we make use of

$$\operatorname{tr}\varepsilon^2 = \varepsilon_{ij}\varepsilon_{ij} = \sum_a \varepsilon_{aa}^2 + 2\sum_{a<b}\varepsilon_{ab}^2,$$

and

$$\mathbf{e}\mathbb{N}[\mathbf{e}] = \mathbf{e}\cdot\left(\sum_{a=1}^{3}\mathbf{n}_a \otimes \mathbf{n}_a\right)[\mathbf{e}] = \sum_{a=1}^{3}(\operatorname{tr}\mathbf{n}_a\mathbf{e})^2.$$

But

$$\mathbf{n}_a\mathbf{e} = \frac{1}{3}I_e\mathbf{n}_a + \mathbf{n}_a\varepsilon \;\Leftrightarrow\; \operatorname{tr}\mathbf{n}_a\mathbf{e} = \frac{1}{3}I_e + \operatorname{tr}\mathbf{n}_a\varepsilon \;\rightarrow\; (\operatorname{tr}\mathbf{n}_a\mathbf{e})^2 =$$
$$= \frac{1}{9}I_e^2 + \frac{2}{3}I_e\operatorname{tr}\mathbf{n}_a\varepsilon + (\operatorname{tr}\mathbf{n}_a\varepsilon)^2$$

so that

$$\mathbf{e}\cdot\mathbb{N}[\mathbf{e}] = \sum_{a=1}^{3}(\operatorname{tr}\mathbf{n}_a\mathbf{e})^2 = \frac{1}{3}I_e^2 + \sum_{a=1}^{3}\sum_{a=1}^{3}(\operatorname{tr}\mathbf{n}_a\varepsilon)^2 = \frac{1}{3}I_e^2 + \sum_{a=1}^{3}\varepsilon_{aa}^2.$$

Submitted on March 2008.

# O tenzoru elastičnosti

U radu se, za analizu simetrije tenzora elastičnosti i njemu inverznog tenzora koristi Propozicija Grupa izotropije tenzora elasičnosti **C** je takođe izotropna grupa njemu inverznog tenzora popustivosti **K**. Kao posledica ove propozicije sledi da su tenzori **C** i **K** istog funkcionalnog oblika. Ova dva fundamentalna stava se koriste u slučaju kubnog i heksagonalnog kristala za njihovo određivanje. Isti postupak se može primeniti i za ostale kristalne klase.

doi:10.2298/TAM0802119J Math.Subj.Class.: 74B05