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Different approaches to Kovalevskaya top

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Abstract

In this paper we study the equations of motion of a rigid body around a fixed point in the case of Kovalevskaya. We give interpretation of the "mysterious substitution of variables" in Sofia Kovalevskaya's paper. The present paper also connects integration procedure proposed by Golubov with the original equations obtained by S. Kovalevskaya.

Keywords: Kovalevskaya top, integrable case, integration procedure

1 Introduction

It is well known that Kovalevskaya top is one of three famous integrable cases of spinning top. This paper deals with a systematic approach to the integration of this case. Sofia Kovalevskaya integrates the problem in terms of hyperelliptic quadratures after a complicated and mysterious substitution of variables, [7]. Here we present possible explanation for this substitution and explicit formulas for the solutions obtained by adapting Golubov's method [1] to Kovalevskaya's equations from [7]. This problem was studied in detail in thesis [11].

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The organization of this paper is as follows: Sections I contains formulation of the problem and basic facts concerning Kovalevskaya's case. Section II is devoted to change of variables in the paper of Kovalevskaya. It contains Weil's interpretation of Euler's results [4], [5] [9], [10] and paraphrase of these classical results given by Jurdjevic in [6] leading to the differential equations that appear in the paper of Sofia Kovalevskaya. Section III explains procedure for integration proposed by Golubov, but adapted to equations that appear in Kovalevskaya's paper.

2 Formulation of the problem

A spinning top is by definition a rigid body that rotates in a constant gravitational field. The most fascinating integrable case of spinning top was discovered by Sofia Kovalevskaya in 1889 under the conditions that $I_1 = I_2 = 2I_3, I_3 = 1, z_0 = y_0 = 0$. With (I_1, I_2, I_3) we denote principal moments of inertia, (x_0, y_0, z_0) is center of mass, $c = Mgx_0$ and (p, q, r) is vector of angular velocity. Finally with γ, γ', γ'' we denote cosine of angles between z axes of fixed coordinate system and axes of coordinate system that is attached to the top and whose origin coincides with the fixed point. Then the equations of motion take the following form:

$$\begin{aligned} 2\dot{p} &= qr, & \dot{\gamma} &= r\gamma' - q\gamma'', \\ 2\dot{q} &= -pr - c\gamma'', & \dot{\gamma'} &= p\gamma'' - r\gamma, \\ \dot{r} &= c\gamma', & \dot{\gamma''} &= q\gamma - p\gamma'. \end{aligned} \tag{1}$$

Sofia Kovalevskaya begins her investigations with solving problem for what values of $(I_1, I_2, I_3, x_0, y_0, z_0)$ Euler-Poisson's equations admit Laurent series solutions of complex time for their general solution. Such a formulation of the problem gave one new approach to a larger class of Hamiltonian systems on Lie groups ([6], [8], [2]).

2.1 Constants of motion

System (1) has three well known independent constants of motion:

$$2(p^2 + q^2) + r^2 = 2c\gamma + 6l_1 \text{ - energy },$$
$$2(p\gamma + q\gamma') + r\gamma'' = 2l \text{ - angular momentum in the direction of gravity},$$
$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1 \text{ - length of the gravity vector}.$$

The extra constant of motion in the case of S. Kovalevskaya is

$$[(p+\imath q)^2 + c(\gamma + \imath\gamma')][(p-\imath q)^2 + c(\gamma - \imath\gamma')] = k^2. \qquad (2)$$

Next, if we denote with

$$x_1 = p + \imath q, \qquad y_1 = \gamma + \imath\gamma',$$

$$x_2 = p - \imath q, \qquad y_2 = \gamma - \imath\gamma',$$

$$\xi_1 = (p+\imath q)^2 + c(\gamma + \imath\gamma') = x_1^2 + cy_1,$$

$$\xi_2 = (p-\imath q)^2 + c(\gamma - \imath\gamma') = x_2^2 + cy_2,$$

then integral (2) becomes

$$\xi_1\xi_2 = k^2. \qquad (3)$$

Rewrite $r^2, r\gamma'', \gamma''^2$ using x_1, x_2, ξ_1, ξ_2. From constants of motion and

$$p^2 + q^2 = x_1x_2,$$

$$2c\gamma = \xi_1 + \xi_2 - x_1^2 - x_2^2,$$

$$2c\imath\gamma' = \xi_1 - \xi_2 - x_1^2 + x_2^2,$$

we get

$$r^2 = 6l_1 - (x_1+x_2)^2 + \xi_1 + \xi_2,$$

$$cr\gamma'' = 2lc + x_1x_2(x_1+x_2) - x_2\xi_1 - x_1\xi_2,$$

$$c^2\gamma''^2 = c^2 - k^2 - x_1^2x_2^2 + x_2^2\xi_1 + x_1^2\xi_2.$$

Further S. Kovalevskaya denotes with: $E = 6l_1 - (x_1+x_2)^2$, $F = 2cl + x_1x_2(x_1+x_2)$ i $G = c^2 - k^2 - x_1^2x_2^2$.

Then we get

$$r^2 = E + \xi_1 + \xi_2,$$

$$cr\gamma'' = F - x_2\xi_1 - x_1\xi_2,$$

$$c^2\gamma''^2 = G + x_2^2\xi_1 + x_1^2\xi_2.$$

Replacing these expressions into identity $r^2c^2\gamma''^2 = (cr\gamma'')^2$ it turns into

$$(E + \xi_1 + \xi_2)(G + x_2^2\xi_1 + x_1^2\xi_2) - (F - x_2\xi_1 - x_1\xi_2)^2 = 0. \qquad (4)$$

After an algebraic transformation (4) becomes

$$[(\xi_1+\xi_2)(x_2^2\xi_1+x_1^2\xi_2)-(x_2\xi_1+x_1\xi_2)^2]+[(\xi_1+\xi_2)G+$$

$$(x_2^2\xi_1+x_1^2\xi_2)E+2(x_2\xi_1+x_1\xi_2)F]+EG-F^2=0. \tag{5}$$

Using identity $(\xi_1+\xi_2)(x_2^2\xi_1+x_1^2\xi_2)-(x_2\xi_1+x_1\xi_2)^2=\xi_1\xi_2[x_1^2+x_2^2-2x_1x_2]$ from (5) we get

$$\xi_1(G+x_2^2E+2x_2F)+\xi_2(G+x_1^2E+2x_1F)+EG-F^2+k^2(x_1-x_2)^2=0. \tag{6}$$

Next S. Kovalevskaya denotes with:

$$R(x_1)=x_1^2E+2x_1F+G=x_1^2(6l_1-(x_1+x_2)^2)+$$

$$2x_1(2cl+x_1x_2(x_1+x_2))+c^2-k^2-x_1^2x_2^2$$

$$=[-(x_1+x_2)^2x_1^2+2x_1^2x_2(x_1+x_2)-x_1^2x_2^2]+6l_1x_1^2+4clx_1+c^2-k^2$$

$$=-[(x_1+x_2)x_1-x_1x_2]^2+6l_1x_1^2+4clx_1+c^2-k^2$$

$$=-x_1^4+6l_1x_1^2+4clx_1+c^2-k^2.$$

Note here that even if coefficients E, F, G depend on x_1 and x_2 expression $Ex_1^2+2Fx_1+G$ depends just on x_1.

Let us also introduce

$$R(x_1,x_2)=Ex_1x_2+F(x_1+x_2)+G=[6l_1-(x_1+x_2)^2]x_1x_2+$$

$$[2cl+x_1x_2(x_1+x_2)](x_1+x_2)+[c^2-k^2-x_1^2x_2^2]$$

$$R_1(x_1,x_2)=EG-F^2=-6l_1x_1^2x_2^2-(c^2-k^2)(x_1+x_2)^2-$$

$$4lc(x_1+x_2)x_1x_2+6l_1(c^2-k^2)-4l^2c^2.$$

Then (6) becomes

$$R(x_2)\xi_1+R(x_1)\xi_2+EG-F^2+k^2(x_1-x_2)^2=0. \tag{7}$$

In order to express ξ_1 and ξ_2 as functions of x_1, x_2 after short calculation and with help of identity $R(x_1)R(x_2) - R^2(x_1, x_2) = (EG - F^2)(x_1 - x_2)^2$, S. Kovalevskaya gets quadratic equation

$$R(x_2)\xi_1^2 + \xi_1(R_1(x_1, x_2) + k^2(x_1 - x_2)^2) + R(x_1)k^2 = 0.$$

As a next step she made the following "mysterious" substitution of variables:

$$s_1 = \frac{R(x_1, x_2) - \sqrt{R(x_1)R(x_2)}}{2(x_1 - x_2)^2} + \frac{1}{2}l_1,$$

$$s_2 = \frac{R(x_1, x_2) + \sqrt{R(x_1)R(x_2)}}{2(x_1 - x_2)^2} + \frac{1}{2}l_1. \tag{8}$$

She omits any explanation concerning the above formulas and we will now try to put some light on them.

3 About the substitution of variables $s_{1,2} = \big(R(x_1, x_2) \mp \sqrt{R(x_1)R(x_2)}\big)(x_1 - x_2)^{-2}/2 + l_1/2$

With $P(x) = A + 4Bx + 6Cx^2 + 4Dx^3 + Ex^4$ we will denote general polynomial of degree four and with $P_1(x, y) = A + 2B(x + y) + 3C(x^2 + y^2) + 2Dxy(x+y) + Ex^2y^2$ a particular form that satisfies $P_1(x, x) = P(x)$.

Theorem 1. The unique form that satisfies $P_1^2(x, y) + (x - y)^2\hat{P}(x, y) = P(x)P(y)$ is $\hat{P}(x, y) = -4B^2 + 4(AD - 3BC)(x + y) + 2(AE + 2BD - 9C^2)(x^2 + y^2) + (BE - 3CD)xy(x + y) - 4D^2x^2y^2 - (AE + 4BD - 9C^2)(x - y)^2$.

Proof: By differentation of $F(x, y) = P(x)P(y) - P_1^2(x, y)$ we see that $F(x, y)$ must be in the form of $(x - y)^2\hat{P}(x, y)$, and by further calculations one finds that $\hat{P}(x, y)$ matches with the expression given in the statement of theorem. For more about this procedure, see [6].

Let $P_\theta(x, y) = P_1(x, y) - \theta(x - y)^2$ denote the most general biquadratic form that satisfies $P_\theta(x, x) = P(x)$ where θ is an arbitrary parameter. Then, $\Phi_\theta(x, y) = -(x - y)^2\theta^2 + 2P_1(x, y)\theta + \hat{P}(x, y)$ is the unique form that satisfies $P_\theta^2(x, y) + (x - y)^2\Phi_\theta(x, y) = P(x)P(y)$.

Φ_θ can be written as $\Phi_\theta(x,y) = a_\theta(x)y^2 + 2b_\theta(x)y + c_\theta(x) = a_\theta(y)x^2 + 2b_\theta(y)x + c_\theta(y)$ where is

$$a_\theta(x) = (2E\theta - 4D^2)x^2 + (4D\theta + 4BE - 12CD)x + AE - (\theta - 3C)^2,$$

$$b_\theta(x) = (2D\theta+2BE-6CD)x^2+(\theta^2-9C^2+AE+4BD)x+2B\theta+2AD-3BC,$$

$$c_\theta(x) = (AE - (\theta - 3C)^2)x^2 + (4B\theta + 4AD - 12BC)x + 2A\theta - 4B^2.$$

If with $G_\theta(x)$ we denote discriminant $b_\theta^2 - a_\theta c_\theta$, then $G_\theta(x) = p(\theta)P(x)$ where is $p(\theta) = 2\theta(\theta-3C)^2 + 2\theta(4BD - AE) + 4B^2E + 4AD^2 - 24BCD$.

Polynomial $p(\theta)$ can be linked with cubic elliptic curve

$$\Gamma = \{(\xi, \eta) : \eta^2 = 4\xi^3 - g_2\xi - g_3\},$$

where g_2 and g_3 are the invariants of curve $C = \{(x,u) : u^2 = P(x)\}$ explicitly given with $g_2 = AE - 4BD + 3C^2$ and $g_3 = ACE + 2BDC - AD^2 - B^2E - C^3$. If we take $\theta = 2(\xi + C)$ and $\eta^2 = \dfrac{p}{4}$ then we obtain $\eta^2 = 4\xi^3 - g_2\xi - g_3$.

Theorem 2. For each number θ, $\Phi_\theta(x,y) = 0$ is a solution of one of differential equations $\dfrac{dx}{\sqrt{P(x)}} \pm \dfrac{dy}{\sqrt{P(y)}} = 0, \quad (*)$. Conversely, for every solution $y(x)$ of either equation $(*)$ there exists a number θ such that $\Phi_\theta(x, y(x)) = 0$.

Proof: For the proof one can see [6],[9],[4].

In papers [9] and [10] Weil pointed out that Γ and C may be regarded as two components of a commutative algebraic group (G(C),+). He describes next mappings: $(M,N) \to M+N$ from $C\times C$ into Γ and $(M,P) \to M+P$ from $C\times\Gamma$ into C. With each solution $\Phi_\theta(x,y) = 0$ Weil associates two transformations from C into C, depending whether they change $\dfrac{dx}{u}$ into $\dfrac{dy}{v}$ or into $-\dfrac{dy}{v}$. For each point $M = (x,u)$ on C define two points $N_1 = (y_1, v_1)$ and $N_2 = (y_2, v_2)$ on C by formulas:

$$y_1 = \frac{1}{a_\theta(x)}(-b_\theta(x) + 2\eta u), \quad v_1 = -\frac{1}{2\eta}(b_\theta(y_1) + a_\theta(y_1)x)$$

and

$$y_2 = \frac{1}{a_\theta(x)}(-b_\theta(x) - 2\eta u), \quad v_2 = -\frac{1}{2\eta}(b_\theta(y_2) + a_\theta(y_2)x)$$

Then, it follows that

$$\frac{dy_1}{dx} = -\frac{\frac{\partial \Phi_\theta}{\partial x}}{\frac{\partial \Phi_\theta}{\partial y}} = \frac{2\eta v_1}{2\eta u}$$

and

$$\frac{dy_2}{dx} = -\frac{\frac{\partial \Phi_\theta}{\partial x}}{\frac{\partial \Phi_\theta}{\partial y}} = -\frac{2\eta v_2}{2\eta u}.$$

Using those mappings and group structure $G(C)$ we obtain that for value $\theta_1 = \dfrac{P_1(a,b) + \sqrt{P(a)P(b)}}{(b-a)^2}$ form $\Phi_{\theta_1}(x,y)$ is the solution of $\dfrac{dx}{\sqrt{P(x)}} - \dfrac{dy}{\sqrt{P(y)}} = 0$ and for $\theta_2 = \dfrac{P_1(a,b) - \sqrt{P(a)P(b)}}{(b-a)^2}$ form $\Phi_{\theta_2}(x,y)$ is the solution of $\dfrac{dx}{\sqrt{P(x)}} + \dfrac{dy}{\sqrt{P(y)}} = 0$. Using these results we can now state theorem which makes clear some formulas that appeared in the Kovalevskaya's paper [7] without any explanation.

Theorem 3. Let $O_1 = (\zeta_1, \eta_1)$ and $O_2 = (\zeta_2, \eta_2)$ denote points of C related by the formulas $O_1 = N - M$, $O_2 = N + M$. Then,

$$\frac{d\zeta_1}{\eta_1} = -\frac{dx}{u} + \frac{dy}{v}, \quad \frac{d\zeta_2}{\eta_2} = \frac{dx}{u} + \frac{dy}{v}.$$

Proof: For the proof one can see [6].

Sofia Kovalevskaya claims that

$$\frac{ds_1}{\sqrt{S(s_1)}} = \frac{dx_1}{\sqrt{R(x_1)}} + \frac{dx_2}{\sqrt{R(x_2)}}, \quad \frac{ds_2}{\sqrt{S(s_2)}} = -\frac{dx_1}{\sqrt{R(x_1)}} + \frac{dx_2}{\sqrt{R(x_2)}} \tag{9}$$

where is $S(s) = 4s^3 - g_2 s - g_3$ and $g_2 = k^2 - c^2 + 3l_1^2$, $g_3 = l_1(k^2 - c^2 - l_1^2) + l^2c^2$ are invariants of the curve $C = \{(x,u) | u^2 = R(x) = -x^4 + 6l_1x^2 + 4lcx + c^2 - k^2\}$. That is exactly application of Theorem 3, since $s_1 = \zeta_2$ and $s_2 = \zeta_1$. To show this claim, we just need to notice that instead of general polynomial $P(x)$ here we take $R(x)$, so $C = l_1$. From (8) we get

$$s_1 = \frac{-x_1^2x_2^2 + 3l_1(x_1^2+x_2^2) + 2lc(x_1+x_2) + c^2 - k^2 - 3l_1(x_1-x_2)^2 - \sqrt{P(x_1)P(x_2)}}{2(x_1-x_2)^2} + \frac{1}{2}l_1$$

$$= \frac{P_1(x_1,x_2) - 3l_1(x_1-x_2)^2 - \sqrt{P(x_1)P(x_2)}}{2(x_1-x_2)^2} + \frac{1}{2}l_1$$

$$= \frac{P_1(x_1,x_2) - \sqrt{P(x_1)P(x_2)}}{2(x_1-x_2)^2} - \frac{3l_1}{2} + \frac{l_1}{2}$$

$$= \frac{\theta_2}{2} - l_1$$

$$= \frac{2(\zeta_2 + C)}{2} - l_1$$

$$= \zeta_2$$

By analogous derivation, we get $s_2 = \zeta_1$. Next, we will derive system of differential equations for s_1 and s_2.

Let $(x_1, \sqrt{R(x_1)})$ and $(x_2, \sqrt{R(x_2)})$ denote points on C and $(s_1, \sqrt{S(s_1)})$, $(s_2, \sqrt{S(s_2)})$ points on Γ.

From $2\imath\dot{x}_1 = rx_1 + c\gamma''$ we get

$$-4(\frac{dx_1}{dt})^2 = r^2x_1^2 + 2rc\gamma''x_1 + c^2\gamma''^2.$$

Now, remember that $r^2 = E + \xi_1 + \xi_2$, $cr\gamma'' = F - x_2\xi_1 - x_1\xi_2$ and $c^2\gamma''^2 = G + x_2^2\xi_1 + x_1^2\xi_2$, so we get

$$-4(\frac{dx_1}{dt})^2 = R(x_1) + \xi_1(t)(x_1(t) - x_2(t))^2.$$

By analogous derivation is

$$-4(\frac{dx_2}{dt})^2 = R(x_2) + \xi_2(t)(x_1(t) - x_2(t))^2$$

and

$$4\frac{dx_1}{dt}\frac{dx_2}{dt} = R(x_1, x_2).$$

Next we get

$$\frac{-4}{S_1}\left(\frac{ds_1}{dt}\right)^2 = -\frac{4}{R(x_1)}\left(\frac{dx_1}{dt}\right)^2 - \frac{4}{R(x_2)}\left(\frac{dx_2}{dt}\right)^2 - \frac{8}{\sqrt{R(x_1)R(x_2)}}\frac{dx_1}{dt}\frac{dx_2}{dt}$$

$$= 2 + (x_1 - x_2)^2(\frac{\xi_1}{R(x_1)} + \frac{\xi_2}{R(x_2)}) - \frac{2R(x_1, x_2)}{\sqrt{R(x_1)R(x_2)}}$$

$$= 2 - \frac{(x_1 - x_2)^2}{R(x_1)R(x_2)}(R_1(x_1, x_2) + k^2(x_1 - x_2)^2) - \frac{2R(x_1, x_2)}{\sqrt{R(x_1)R(x_2)}}$$

$$= \frac{R(x_1)R(x_2) + R^2(x_1, x_2) - 2R(x_1, x_2)\sqrt{R(x_1)}\sqrt{R(x_2)} - k^2(x_1 - x_2)^4}{R(x_1)R(x_2)}$$

$$= \frac{(x_1 - x_2)^4}{R(x_1)R(x_2)}\left[\left(\frac{R(x_1, x_2) - \sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1 - x_2)^2}\right)^2 - k^2\right]$$

Replacing $\frac{R(x_1, x_2) + (-1)^i\sqrt{R(x_1)}\sqrt{R(x_2)}}{2(x_1 - x_2)^2} = s_i - \frac{l_1}{2}$ and denote with $k_1 = \frac{l_1}{2} + \frac{k}{2}$ and $k_2 = \frac{l_1}{2} - \frac{k}{2}$ previous expression becomes

$$-\frac{1}{S_1}\left(\frac{ds_1}{dt}\right)^2 = \frac{(x_1 - x_2)^4}{R(x_1)R(x_2)}\left(\frac{R(x_1, x_2) - \sqrt{R(x_1)}\sqrt{R(x_2)}}{2(x_1 - x_2)^2} - \frac{k}{2}\right)$$

$$\left(\frac{R(x_1, x_2) - \sqrt{R(x_1)}\sqrt{R(x_2)}}{2(x_1 - x_2)^2} + \frac{k}{2}\right)$$

$$= \frac{(x_1 - x_2)^4}{R(x_1)R(x_2)}(s_1 - k_1)(s_1 - k_2).$$

Also, $s_2 - s_1 = \dfrac{\sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1-x_2)^2}$ so finally we get

$$\frac{ds_1}{dt} = \frac{\sqrt{\Phi(s_1)}}{s_1 - s_2}$$

where $\Phi(s_1) = -(4s_1^3 - g_2 s_1 - g_3)(s_1 - k_1)(s_1 - k_2)$.

In the same way

$$\frac{ds_2}{dt} = \frac{\sqrt{\Phi(s_2)}}{s_2 - s_1}.$$

Finally, system of equations for s_1 and s_2 is

$$\frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} = 0, \qquad \frac{s_1 ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} = dt. \tag{10}$$

4 Integration procedure for Kovalevskaya top

Recall from (8) that s_1 and s_2 are the roots of quadratic equation

$$(x_1 - x_2)^2 (s - \frac{l_1}{2})^2 - R(x_1, x_2)(s - \frac{l_1}{2}) - \frac{1}{4} R_1(x_1, x_2) = 0. \tag{11}$$

Here we need to notice that Golubov in [1] s_1 and s_2 obtains as roots of equation

$$(x_1 - x_2)^2 (s - 3l_1)^2 - 2R(x_1, x_2)(s - 3l_1) - R_1(x_1, x_2) = 0. \tag{12}$$

Since we explained possible motivation for change of variable Sofia Kovalevskaya used in her famous paper we will now adapt method proposed by Golubov in [1] on equations given by S. Kovalevskaya in [7]. More about this adaptation one can find in [11]. Denote left side of (10) with

$$Q(w, x_1, x_2) = (x_1 - x_2)^2 w^2 - R(x_1, x_2) w - \frac{1}{4} R_1(x_1, x_2). \tag{13}$$

Let us remark here that, like in [1], we can obtain system of differential equations (10) by differentiation of $Q(w, x_1, x_2) = 0$ and substituting

$$(\frac{\partial Q}{\partial w})^2 = R(x_1)R(x_2), \quad (\frac{\partial Q}{\partial x_1})^2 = R(x_2)\varphi(w), \quad (\frac{\partial Q}{\partial x_2})^2 = R(x_1)\varphi(w),$$

where is

$$\varphi(w) = 4w^3 + 6w^2 l_1 + (c^2 - k^2)w - c^2 l^2 = (w + \frac{3l_1}{2})(4w^2 + c^2 - k^2) - c^2 l^2.$$

We can see that $\varphi(w) = S(s)$ for $s = w - \frac{l_1}{2}$. In order to apply Golubov's method, next problem was to find function $h(w)$ so that

$$Q(w, x_1, x_2)h(w) = A^2 + \varphi(w)B$$

for some expressions $A(x_1, x_2, w)$, $B(x_1, x_2, w)$. We found that:

$$h(w) = 4(w + \frac{3l_1}{2}), \quad A = [4(w + \frac{3l_1}{2})(\frac{x_1 x_2}{2} - w) + cl(x_1 + x_2)],$$

$$B = (x_1 + x_2)^2 - 4(w + \frac{3l_1}{2}).$$

If we change w into $\tilde{s} - \frac{3l_1}{2}$, $\varphi(w)$ turns into

$$\tilde{\varphi}(\tilde{s}) = \tilde{s}(4(\tilde{s} - \frac{3l_1}{2})^2 + c^2 - k^2) - c^2 l^2 = 4(\tilde{s} - \tilde{e}_1)(\tilde{s} - \tilde{e}_2)(\tilde{s} - \tilde{e}_3). \quad (14)$$

Denote $Q(w, x_1, x_2)(x_1 - x_2)^{-2}$ with $F(w)$. Expanding $F(w)$ into Taylor's polynomial we get

$$F(w) = F(u) + (w - u)F'(u) + (w - u)^2 =$$

$$\frac{1}{(x_1 - x_2)^2} \left[2\sqrt{u + \frac{3l_1}{2}}(\frac{1}{2}x_1 x_2 - u) + \frac{lc(x_1 + x_2)}{2\sqrt{u + \frac{3l_1}{2}}} \right]^2 +$$

$$\frac{\varphi(u)}{(x_1 - x_2)^2} \left[\frac{(x_1 + x_2)^2}{4(u + \frac{3l_1}{2})} - 1 \right] + (w - u)\frac{2u(x_1 - x_2)^2 - R(x_1, x_2)}{(x_1 - x_2)^2}$$

$$+ (w - u)^2 = 0. \quad (15)$$

Now we replace $w = \tilde{s} - \frac{3l_1}{2}$, $u = z - \frac{3l_1}{2}$ and roots of (15) denote with $\tilde{s}_1 - z$ and $\tilde{s}_2 - z$ so

$$(\tilde{s}_1 - z)(\tilde{s}_2 - z) = \Big[2\sqrt{z}\frac{\frac{1}{2}x_1x_2 - z + \frac{3l_1}{2}}{x_1 - x_2} + \frac{lc}{2\sqrt{z}}\frac{x_1 + x_2}{x_1 - x_2}\Big]^2 + \frac{\tilde{\varphi}(z)}{(x_1 - x_2)^2}\Big[\frac{(x_1 + x_2)^2}{4z} - 1\Big]. \tag{16}$$

If we take $z = \tilde{e}_i, \quad i = 1, 2, 3$ then

$$\sqrt{\tilde{e}_i}\frac{x_1x_2 - 2\tilde{e}_i + 3l_1}{x_1 - x_2} + \frac{lc}{2\sqrt{\tilde{e}_i}}\frac{x_1 + x_2}{x_1 - x_2} = \sqrt{(\tilde{s}_1 - \tilde{e}_i)(\tilde{s}_2 - \tilde{e}_i)}. \tag{17}$$

Next, denote with

$$X = \frac{x_1x_2 + 3l_1}{x_1 - x_2} = \frac{p^2 + q^2 + 3l_1}{2q\imath},$$

$$Y = \frac{1}{x_1 - x_2} = \frac{1}{2q\imath},$$

$$Z = \frac{x_1 + x_2}{x_1 - x_2} = \frac{p}{q\imath}$$

and

$$P_i = \sqrt{(\tilde{s}_1 - \tilde{e}_i)(\tilde{s}_2 - \tilde{e}_i)}, \quad i = 1, 2, 3.$$

Then (17) turns into:

$$\sqrt{\tilde{e}_\alpha}X - 2\tilde{e}_\alpha\sqrt{\tilde{e}_\alpha}Y + \frac{lc}{2\sqrt{\tilde{e}_\alpha}}Z = P_\alpha. \tag{18}$$

Changing $i = 1, 2, 3$ into (18) and solving that system, we get X, Y, Z. Then we obtain

$$q = \frac{1}{2\imath Y} = \frac{-1}{2\imath\sum_{i=1}^3 2\frac{P_i\sqrt{\tilde{e}_i}}{\tilde{\varphi}'(\tilde{e}_i)}} = \frac{\imath}{4\sum_{i=1}^3 \frac{P_i\sqrt{\tilde{e}_i}}{\tilde{\varphi}'(\tilde{e}_i)}},$$

$$p = \frac{Z}{2Y} = \frac{\sum_{i=1}^3 \frac{4\sqrt{\tilde{e}_j\tilde{e}_k}}{\tilde{\varphi}'(\tilde{e}_i)}P_i}{-4\sum_{i=1}^3 \frac{P_i\sqrt{\tilde{e}_i}}{\sqrt{2}\tilde{\varphi}'(\tilde{e}_i)}} = -\frac{\sum_{\alpha=1}^3 \frac{\sqrt{\tilde{e}_j\tilde{e}_k}}{\tilde{\varphi}'(\tilde{e}_i)}P_i}{\sum_{i=1}^3 \frac{P_i\sqrt{\tilde{e}_i}}{\tilde{\varphi}'(\tilde{e}_i)}},$$

$$r=\frac{2\dot{p}}{q}=\frac{\sum_{i=1}^{3}\frac{\sqrt{\tilde{e}_i}P_{jk}}{\tilde{\varphi}'(\tilde{e}_i)}}{\imath(\sum_{i=1}^{3}\frac{\sqrt{\tilde{e}_i}P_i}{\tilde{\varphi}'(\tilde{e}_i)})}$$

where (i,j,k) is cyclic permutation of $(1,2,3)$. With P_{ij} we denoted

$$P_{ij}=P_iP_j\left[\frac{\frac{d\tilde{s}_1}{dt}}{(\tilde{s}_1-\tilde{e}_i)(\tilde{s}_1-\tilde{e}_j)}+\frac{\frac{d\tilde{s}_2}{dt}}{(\tilde{s}_2-\tilde{e}_i)(\tilde{s}_2-\tilde{e}_j)}\right]$$

for $e_4=k_1+l_1$, $e_5=k_2+l_1$, $i,j=1,2,3,4,5$.

Next we obtain γ'' from equations $2\dot{p}=qr$, $2\dot{q}=-pr-c\gamma''$, so

$$qc\gamma''=-\frac{d(p^2+q^2)}{dt}=-\frac{d(\frac{X}{Y})}{dt}$$

and we get

$$\gamma''=\frac{1}{\imath c}\frac{\sum_{i=1}^{3}\frac{\sqrt{\tilde{e}_i\tilde{e}_j}}{\tilde{\varphi}'(\tilde{e}_k)}P_{ij}}{\sum_{i=1}^{3}\frac{\sqrt{\tilde{e}_i}}{\tilde{\varphi}'(\tilde{e}_i)}P_i}.$$

From constant of energy and $2(p^2+q^2)=\frac{2X}{Y}-6l_1$ we can express

$$2c\gamma=\frac{2X}{Y}+r^2-12l_1=4\frac{\sum_{i=1}^{3}\frac{\sqrt{\tilde{e}_i}(\tilde{e}_j+\tilde{e}_k)P_i}{\tilde{\varphi}'(\tilde{e}_i)}}{\sum_{i=1}^{3}\frac{P_i\sqrt{\tilde{e}_i}}{\tilde{\varphi}'(\tilde{e}_i)}}-\left(\frac{\sum_{i=1}^{3}L_iP_{jk}}{\sum_{i=1}^{3}L_iP_i}\right)^2-12l_1,$$

where with L_i we denoted $\imath(\tilde{e}_j-\tilde{e}_k)\sqrt{\tilde{e}_i}$. Finally, we obtain γ' from equation $\dot{r}=c\gamma'$. Then,

$$\begin{aligned}&-c\gamma'\,(L_1P_1+L_2P_2+L_3P_3)^2\\&=\sum_{i=1}^{3}(\tilde{e}_j-\tilde{e}_k)^2\tilde{e}_iP_{i4}P_{i5}\\&+\sum_{i=1}^{3}(\tilde{e}_i-\tilde{e}_j)(\tilde{e}_k-\tilde{e}_i)\sqrt{\tilde{e}_i\tilde{e}_j}(P_{j4}P_{k5}+P_{j5}P_{k4}+(\tilde{e}_j-\tilde{e}_k)^2P_i).\end{aligned}$$

In paper [7] S. Kovalevskaya obtained formulas for $p,q,r,\gamma,\gamma',\gamma''$ using Weierstrass $\wp$ and σ function after long and not easy calculations. With e_1,e_2,e_3 she denotes the roots of $\mathcal{S}(s)$ and uses change of variable $w=s-$

$\frac{l_1}{2}$, so $\tilde{e}_i = e_i + l_1$. With these remarks it is easy to check that expressions we got here are the same as those obtained by S. Kovalevskaya. She did, also, expressed P_i, P_{ij} for $i, j \in \{1, 2, 3, 4, 5\}$ through ϑ functions. More about these formulas and notation she used, one can see in [3], [11].

5 Conclusions

Even after more than hundred years after publishing, Kovalevskaya's paper [7] is still quite a challenge. Up to now, her method has been neither understood, nor improved. This paper offered simple alternative to Kovalevskaya's formulas without mentioning properties of $\wp$ and σ functions. New techniques are still developing. Some geometric approaches to this problem were proposed in [8], [2] and that will be the field of our future research.

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Razni pristupi čigri Kovaljevske

Proučavaju se jednačine kretanja krutog tela oko nepomične tačke u slučaju Kovaljevske. Daje se interpretacija "tajanstvene smene promenljivih" u radu Sofije Kovaljevske. Ovde predloženi rad takodje povezuje integracioni postupak Golubova sa originalnim jednačinama dobijenim Sofijom Kovaljevskom.

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