Theoret. Appl. Mech., Vol.35, No.4, pp. 363–379, Belgrade 2008

Influence of rotatory inertia on stochastic stability of a viscoelastic rotating shaft

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Abstract

The stochastic stability problem of a viscoelastic Voigt-Kelvin balanced rotating shaft subjected to action of axial forces at the ends is studied. The shaft is of circular cross-section, it rotates at a constant rate about its longitudinal axis of symmetry. The effect of rotatory inertia of the shaft cross-section and external viscous damping are included into account. The force consists of a constant part and a time-dependent stochastic function. Closed form analytical solutions are obtained for simply supported boundary conditions. By using the direct Liapunov method almost sure asymptotic stability conditions are obtained as the function of stochastic process variance, external damping coefficient, retardation time, angular velocity, and geometric and physical parameters of the shaft. Numerical calculations are performed for the Gaussian process with a zero mean and variance σ^2 as well as for harmonic process with amplitude H.

Keywords: Random loading, Liapunov functional, Almost sure stability, Rotatory inertia, Gaussian and harmonic process

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List of notations

A	cross-section area
I	inertia moment of shaft cross-section with respect to the neutral axis
E	Young's modulus
f_{cr}	dimensionless Euler's critical force
f_o	dimensionless constant component of axial force
$f(t)$	dimensionless stochastic component of axial force
$\overline{F}$	axial force
ℓ	length of the shaft
r	radius of gyration
p	probability density function
P	probability
t	dimensionless time
T	time
X, Y, Z	shaft coordinates
z	dimensionless axial shaft coordinate
u, v	flexural displacements in X and Y direction, respectively
$\mathbf{V}$	Liapunov's functional
$\overline{\beta}$	external damping coefficient
$\overline{\varsigma}$	retardation time
β	dimensionless external damping coefficient
ς	dimensionless retardation time
$\overline{\Omega}$	angular velocity
Ω	dimensionless angular velocity
ρ	density
σ^2	variance of stochastic loading
$E\{.\}$	mathematical expectation
$\parallel . \parallel$	distance of solution from the trivial solution

1 Introduction

One of the most fundamental components of a mechanical system is a rotating shaft. It is not surprising that through the years considerable effort has been directed at obtaining a better understanding of such mechanisms. Rotating shafts, as elements of construction, often can take position to lose stability. The stability problem of rotating shafts arises when shafts are required to run smoothly at high speed. Destabilizing factors can be compressive force, the normal inertia force as well as certain types of damping.

When a slightly bent shaft is exposed to axial loads in the presence of creep, lateral deflections will increase with time. Under certain conditions and after a certain elapsed time during which deflections continually increase, collapse of shaft can occur. This buckling is called "creep buckling". The time to colapse is called critical time. In this case the creep buckling problem involves the evaluation of critical times rather than critical loads.

The dynamic stability of rotating shafts, with omission of the compressive force, was first analyzed by Bishop [1] using a modal approach. The same problem using the direct Liapunov method was examined by Parks and Pritchard [2].

J. Shaw and S. W. Shaw [3] considered instabilities and bifurcations in non-linear rotating shaft made of viscoelastic Voigt-Kelvin material without compressive force.

Uniform stochastic stability of the rotating shafts, when the axial force is a wide-band Gaussian process with zero mean was studied by Tylikowski [4].

Tylikowski and Hetnarski [5] examined the influence of the activation through the change of the temperature on dynamic stability of the shape memory alloy hybrid rotating shaft.

Young and Gau [6], [7] investigated dynamic stability of a pretwisted cantilever beam with constant and non-constant spin rates, subjected to axial random forces. By using stochastic averaging method, they determined mean-square stability condition in [6] and first and second moment stability conditions in [7].

In the present paper we investigate almost sure stability of the rotating viscoelastic Voigt-Kelvin shaft, where is external damping included. This is extension of the problem considered erlier in [8]. The axial force is stochastic process with known density function. Problem is solved by direct Liapunov method, and stability regions are given as function of geometric and physical parameters of the shaft.

2 Problem formulation

Let us consider a shaft rotating about its longitudinal axis with angular velocity $\overline{\Omega}$, (Fig. 1).

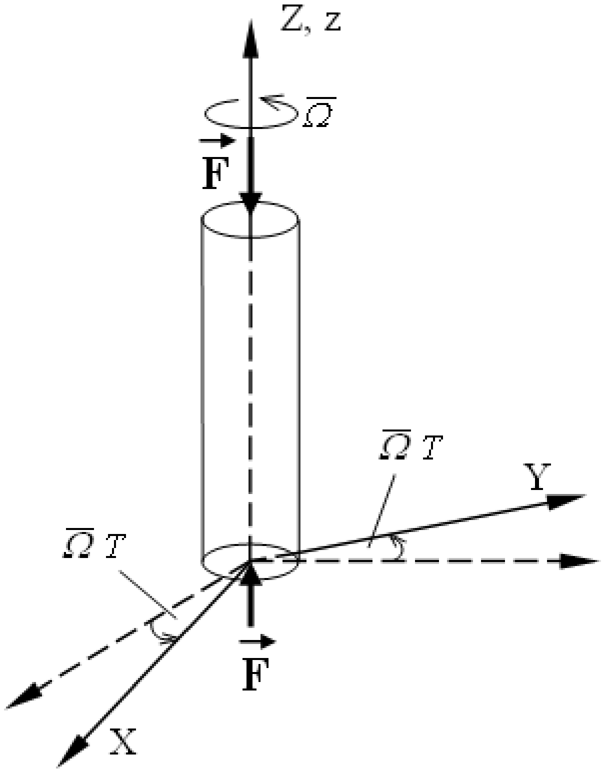

Figure 1: The rotating shaft and co-ordinate systems

The governing differential equations can be written as:

$$\rho A\left(\frac{\partial^2 u}{\partial T^2}-2\overline{\Omega}\frac{\partial v}{\partial T}-\overline{\Omega}^2 u\right)-\rho I\frac{\partial^4 u}{\partial T^2\partial Z^2}+\overline{\beta}\left(\frac{\partial\, u}{\partial T}-\overline{\Omega}\, v\right)+$$

$$+EI\overline{\varsigma}\frac{\partial^5 u}{\partial T\partial\, Z^4}+EI\frac{\partial^4 u}{\partial\, Z^4}+\bar{F}(T)\frac{\partial^2 u}{\partial\, Z^2}=0, \tag{1}$$

$$\rho A\left(\frac{\partial^2 v}{\partial T^2}+2\overline{\Omega}\frac{\partial u}{\partial T}-\overline{\Omega}^2 v\right)-\rho I\frac{\partial^4 v}{\partial T^2\partial\, Z^2}+\overline{\beta}\left(\frac{\partial\, v}{\partial T}+\overline{\Omega}\, u\right)+$$

$$+EI\bar{\varsigma}\frac{\partial^5 v}{\partial T\partial Z^4}+EI\frac{\partial^4 v}{\partial Z^4}+\bar{F}(T)\frac{\partial^2 v}{\partial Z^2}=0, \qquad (2)$$

where u, v are - flexural displacements in the X and Y direction, ρ - mass density, A - area of the cross-section of shaft, I - axial moment of inertia, E - Young modulus of elasticity, $\overline{\beta}$ - external damping coefficient, $\overline{\varsigma}$ - retardation time, T - time and Z - axial coordinate.

Using the following transformations:

$$Z=z\ell\ ,\quad e^2=\frac{I}{Al^2},\quad k_t=\sqrt{\frac{\rho A l^4}{EI}}\ ,$$

$$T=k_t t,\quad f_o+f(t)=\frac{\overline{F}(t)\ell^2}{EI}\ ,$$

$$2\varsigma=\frac{\overline{\varsigma}\,k_t}{\rho A},\quad 2\beta=\frac{\overline{\beta}\,k_t}{\rho A},\quad \Omega=\overline{\Omega}\,k_t, \qquad (3)$$

where ℓ is length of the shaft, β is reduced damping coefficient, ς is reduced retardation time, we get governing equations as:

$$\frac{\partial^2 u}{\partial t^2}-2\Omega\frac{\partial v}{\partial t}-\Omega^2 u-e^2\frac{\partial^4 u}{\partial t^2\partial z^2}+2\varsigma\frac{\partial^5 u}{\partial t\partial z^4}+$$

$$+2\beta\left(\frac{\partial u}{\partial t}-\Omega\,v\right)+\frac{\partial^4 u}{\partial z^4}+(f_o+f(t))\,\frac{\partial^2 u}{\partial z^2}=0, \qquad (4)$$

$$\frac{\partial^2 v}{\partial t^2}+2\Omega\frac{\partial u}{\partial t}-\Omega^2 v-e^2\frac{\partial^4 v}{\partial t^2\partial z^2}+2\varsigma\frac{\partial^5 v}{\partial t\partial z^4}+$$

$$2\beta\left(\frac{\partial v}{\partial t}+\Omega\,u\right)+\frac{\partial^4 v}{\partial z^4}+(f_o+f(t))\,\frac{\partial^2 v}{\partial z^2}=0,\quad z\in(0,\ 1). \qquad (5)$$

Boundary conditions for the simply supported shaft are:

$$u(t,0)=u(t,1)=\frac{\partial^2 u}{\partial z^2}(t,0)=\frac{\partial^2 u}{\partial z^2}(t,1)=0, \qquad (6)$$

$$v(t,0)=v(t,1)=\frac{\partial^2 v}{\partial z^2}(t,0)=\frac{\partial^2 v}{\partial z^2}(t,1)=0\cdot$$

The purpose of the present paper is the investigation of almost sure asymptotic stability of the rotating shaft subjected to stochastic time-dependent axial loads. To estimate perturbated solutions it is necessary to introduce a measure of distance $\|.\|$ of solutions of Eqs. (4) and (5) with nontrivial initial conditions and the trivial one. Following Kozin [9], the equilibrium state of Eqs. (4) and (5) is said to be almost sure stochastically stable, if:

$$P\left\{\lim_{t\to\infty}\|\boldsymbol{w}\,(.,t)\|=0\right\}=1, \tag{7}$$

where $\boldsymbol{w}=col(u,v)$, matrix column.

3 Stability analyses

With the purpose of applying the Liapunov method, we can construct the functional by means of the Parks - Pritchard's method [2]. Thus, let us write Eqs. (4) and (5) in the formal form $\mathsf{L}\boldsymbol{w}=0$, where L is the matrix:

$$\mathsf{L}=\begin{bmatrix}\ell_{11} & \ell_{12}\\ \ell_{21} & \ell_{22}\end{bmatrix} \tag{8}$$

with elements

$$\ell_{11}=\ell_{22}=\frac{\partial^2}{\partial t^2}-\Omega^2-e^2\frac{\partial^4}{\partial t^2\partial z^2}+2\varsigma\frac{\partial^5}{\partial t\partial z^4}+2\beta\frac{\partial}{\partial t}+\frac{\partial^4}{\partial z^4}+(f_o+f(t))\frac{\partial^2}{\partial z^2},$$

$$\ell_{12}=-\ell_{21}=-2\Omega\left(\beta+\frac{\partial}{\partial t}\right), \tag{9}$$

and introduce the linear operator:

$$\mathsf{N}=\begin{bmatrix}2\left(\frac{\partial}{\partial t}-e^2\frac{\partial^3}{\partial t\,\partial z^2}+\varsigma\frac{\partial^4}{\partial z^4}+\beta\right) & -2\Omega\\ 2\Omega & 2\left(\frac{\partial}{\partial t}-e^2\frac{\partial^3}{\partial t\,\partial z^2}+\varsigma\frac{\partial^4}{\partial z^4}+\beta\right)\end{bmatrix}, \tag{10}$$

which is a formal derivative of the operator L with respect to $\partial/\partial t$.

Integrating the scalar product of the vectors $\mathsf{L}\boldsymbol{w}$ $\mathsf{N}\boldsymbol{w}$ on rectangular $C=[z:0\leq z\leq 1]\times\,[\tau:0\leq\tau\leq t]$ with respect to Eqs. (4) and (5), it is clear:

$$\int_0^1\int_0^\tau \mathsf{L}\boldsymbol{w}\ \mathsf{N}\ \boldsymbol{w}\ dz\ d\tau=0. \tag{11}$$

After applying partial integration to Eq. (11), the sum of two integrals may be obtained. In the first, integration is only on the spatial domain, and it is chosen to be the Liapunov functional:

$$\mathbf{V}=\int_0^1\left\{\left(\frac{\partial u}{\partial t}+\beta u+\varsigma\frac{\partial^4 u}{dz^4}-e^2\frac{\partial^4 u}{\partial t\,\partial z^2}-\Omega v\right)^2+\right.$$

$$\left(\frac{\partial v}{\partial t}+\beta v+\varsigma\frac{\partial^3 v}{dz^4}-e^2\frac{\partial^3 v}{\partial t\,\partial z^2}+\Omega u\right)^2+$$

$$-\left(f_o+\Omega^2e^2\right)\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]+\left(1-f_oe^2\right)\left[\left(\frac{\partial^2 u}{\partial z^2}\right)^2+\left(\frac{\partial^2 v}{\partial z^2}\right)^2\right]+$$

$$+e^2\left[\left(\frac{\partial^3 u}{\partial z^3}\right)^2+\left(\frac{\partial^3 v}{\partial z^3}\right)^2\right]+\beta^2\left(u^2+v^2\right)+2\beta\varsigma\left[\left(\frac{\partial^2 u}{\partial z^2}\right)^2+\left(\frac{\partial^2 v}{\partial z^2}\right)^2\right]+$$

$$\left.\varsigma^2\left[\left(\frac{\partial^4 u}{\partial z^4}\right)^2+\left(\frac{\partial^4 v}{\partial z^4}\right)^2\right]\right\}dz. \tag{12}$$

Since it is evident

$$V\,\Big|_0^t-\int_0^t\frac{dV}{dt}dt=0, \tag{13}$$

then the second integral in (11) is a time derivative of the functional (12) along equations (4) and (5):

$$\frac{d\mathbf{V}}{dt}=-2\int_0^1\left\{\beta\left[\left(\frac{\partial u}{\partial t}-\Omega v\right)^2+\left(\frac{\partial v}{\partial t}+\Omega u\right)^2\right]+\right.$$

$$\beta e^2\left[\left(\frac{\partial^2 u}{\partial t\partial z}-\Omega\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial^2 v}{\partial t\partial z}+\Omega\frac{\partial u}{\partial z}\right)^2\right]+$$

$$+\beta\left[\left(\frac{\partial^2 u}{\partial z^2}\right)^2+\left(\frac{\partial^2 v}{\partial z^2}\right)^2\right]-\beta\left(f_o+\Omega^2e^2\right)\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]+$$

$$+\varsigma\left[\left(\frac{\partial^3 u}{\partial t\partial z^2}\right)^2+\left(\frac{\partial^3 v}{\partial t\partial z^2}\right)^2+e^2\left(\frac{\partial^4 u}{\partial t\partial z^3}\right)^2+e^2\left(\frac{\partial^4 v}{\partial t\partial z^3}\right)^2+\right. \tag{14}$$

$$\left(\frac{\partial^4 u}{\partial z^4}\right)^2 + \left(\frac{\partial^4 v}{\partial z^4}\right)^2\Bigg] - \varsigma\left[\Omega^2\left(\frac{\partial^2 u}{\partial z^2}\right)^2 + \Omega^2\left(\frac{\partial^2 v}{\partial z^2}\right)^2 + \right.$$

$$\left. f_o\left(\frac{\partial^3 u}{\partial z^3}\right)^2 + f_o\left(\frac{\partial^3 v}{\partial z^3}\right)^2\right] +$$

$$+ f(t)\frac{\partial^2 u}{\partial z^2}\left(\frac{\partial u}{\partial t} + \beta u + \varsigma\frac{\partial^4 u}{\partial z^4} - e^2\frac{\partial^3 u}{\partial t\partial z^2} - \Omega v\right) +$$

$$f(t)\frac{\partial^2 v}{\partial z^2}\left(\frac{\partial v}{\partial t} + \beta v + \varsigma\frac{\partial^4 v}{\partial z^4} - e^2\frac{\partial^3 v}{\partial t\partial z^2} + \Omega u\right)\Bigg\} dz.$$

Functional **V** will be a Liapunov functional if it is a positive definite. By using well known Steklov's inequalities:

$$\int_0^1 \left[\left(\frac{\partial^{n+1} u}{\partial z^{n+1}}\right)^2 + \left(\frac{\partial^{n+1} v}{\partial z^{n+1}}\right)^2\right] dz \geq \pi^2 \int_0^1 \left[\left(\frac{\partial^n u}{\partial z^n}\right)^2 + \left(\frac{\partial^n v}{\partial z^n}\right)^2\right] dz,$$

$$n = 1, 2, 3, \qquad (15)$$

and omitting dynamical terms, we can write:

$$\mathbf{V} \geq \left[\pi^2\left(1 - f_0 e^2\right) + \pi^4 e^2 - \left(f_0 + \Omega^2 e^2\right)\right] \int_0^1 \left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] dz \qquad (16)$$

so, the positive definite condition reduces to relation:

$$f_0 \leq \pi^2 - \frac{\Omega^2 e^2}{1 + \pi^2 e^2}. \qquad (17)$$

4 Stability under constant axial force

If $f(t) = 0$, by using relations (15), we can estimate the first derivative of Liapunov functional (14)

$$\frac{d\mathbf{V}}{dt} = -2\int_0^1 \left\{\beta\left(1 - \frac{f_0 + \Omega^2 e^2}{\pi^2}\right)^2 \left[\left(\frac{\partial^2 u}{\partial z^2}\right)^2 + \left(\frac{\partial^2 v}{\partial z^2}\right)^2\right] + \right.$$

$$\zeta\left(1-\frac{f_0\pi^2+\Omega^2}{\pi^{42}}\right)\left[\left(\frac{\partial^4 u}{\partial z^4}\right)^2+\left(\frac{\partial^4 v}{\partial z^4}\right)^2\right]\Bigg\}\,dz, \tag{18}$$

and it will be negative definite when:

$$f_o \le \pi^2 - \Omega^2 e^2, \tag{19a}$$

$$f_0 \le \pi^2 - \frac{\Omega^2}{\pi^2}. \tag{19b}$$

As is $e = r/\ell < 1/\pi \approx 0.3181$, the relation (19b) is stronger than (17) and (19a). So, relation (19b) represents the dynamic stability condition of the rotating shaft under constant force. We may observe that if $f_o = f_{cr} = \pi^2$, where f_{cr} is Euler's critical force, then $\Omega = 0$.

In the absence of axial force ($f_o = 0$), we find:

$$\Omega \le \pi^2 \;\rightarrow\; \overline{\Omega} \le \frac{\pi^2}{\ell^2}\sqrt{\frac{EI}{\rho A}} = \omega_1, \tag{20}$$

where ω_1 denotes the first natural frequency of the shaft at rest. The angular velocity Ω may be larger if $f_o < 0$, (i.e., f_o is tensile force).

5 Almost-sure stability

Let a scalar function $\lambda(t)$ be defined as:

$$\frac{1}{\mathbf{V}}\frac{d\mathbf{V}}{dt} \le \lambda(t). \tag{21}$$

As a maximum point is a particular case of the stationary point, we may write:

$$\delta\left(\dot{\mathbf{V}} - \lambda\mathbf{V}\right) = 0. \tag{22}$$

By using the associated Euler's equations we obtain:

$$\left(\lambda\ell_e^{(2)} + 2\beta\ell_e + 2\varsigma\frac{\partial^4\ell_e}{\partial z^4}\right)\frac{\partial u}{\partial t} +$$

$$\left[\lambda\beta\ell_e + \lambda\varsigma\frac{\partial^4\ell_e}{\partial z^4} + f(t)\frac{\partial^2\ell_e}{\partial z^2}\right]u - \Omega\left(\lambda\ell_e + 2\beta\ell_e\right)v = 0,$$

$$\left(\lambda\ell_e^{(2)} + 2\beta\ell_e + 2\varsigma\frac{\partial^4\ell_e}{\partial z^4}\right)\ \frac{\partial v}{\partial t} - \tag{23}$$

$$\Omega\left(\lambda\ell_e + 2\beta\ell_e\right)u + \left[\lambda\beta\ell_e + \lambda\varsigma\frac{\partial^4\ell_e}{\partial z^4} + f(t)\frac{\partial^2\ell_e}{\partial z^2}\right]\ v = 0,$$

$$\left[\lambda\beta\ell_e + \lambda\varsigma\frac{\partial^4\ell_e}{\partial z^4} + f(t)\frac{\partial^2\ell_e}{\partial z^2}\right]\ \frac{\partial u}{\partial t} + \Omega\left(\lambda\ell_e + 2\beta\ell_e\right)\ \frac{\partial v}{\partial t} + \ell_{uv}u = 0,$$

$$-\Omega\left(\lambda\ell_e + 2\beta\ell_e\right)\ \frac{\partial u}{\partial t} + \left[\lambda\beta\ell_e + \lambda\varsigma\frac{\partial^4\ell_e}{\partial z^4} + f(t)\frac{\partial^2\ell_e}{\partial z^2}\right]\ \frac{\partial v}{\partial t} + \ell_{uv}v = 0,$$

where:

$$\ell_e = 1 - e^2\frac{\partial^2}{\partial z^2}, \quad \ell_e^{(2)} = \left(1 - e^2\frac{\partial^2}{\partial z^2}\right)\left(1 - e^2\frac{\partial^2}{\partial z^2}\right) = 1 - 2e^2\frac{\partial^2}{\partial z^2} + e^4\frac{\partial^4}{\partial z^4},$$

$$\ell_{uv} = \lambda\left[\Omega^2 + 2\beta\left(\beta + \varsigma\frac{\partial^4}{\partial z^4}\right) + 2\alpha\left(\beta\frac{\partial^4}{\partial z^4} + \varsigma\frac{\partial^8}{\partial z^8}\right) + \right.$$

$$\left.\left(f_o + \Omega^2 e^2\right)\frac{\partial^2}{\partial z^2} + \left(1 - f_o e^2\right)\frac{\partial^4}{\partial z^4} - e^2\frac{\partial^6}{\partial z^6}\right] +$$

$$2\beta\left[\Omega^2\ell_e + \left(f_o + \Omega^2 e^2\right)\frac{\partial^2}{\partial z^2} + \frac{\partial^4}{\partial z^4} + f(t)\frac{\partial^2}{\partial z^2}\right] + \tag{24}$$

$$+2\varsigma\left[-\Omega^2\frac{\partial^4}{\partial z^4} + f_o\frac{\partial^6}{\partial z^6} + \frac{\partial^8}{\partial z^8} + f(t)\frac{\partial^6}{\partial z^6}\right].$$

After simplifying, we get two equations:

$$\left[\left(\lambda\beta\ell_e + \lambda\varsigma\frac{\partial^4\ell_e}{\partial z^4} + f(t)\frac{\partial^2\ell_e}{\partial z^2}\right)^{(2)} + \Omega^2\left(\lambda^+ 2\beta\right)^2\ell_e^{(2)} - \right.$$

$$\left.\left(\lambda\ell_e^{(2)} + 2\beta\ell_e + 2\varsigma\frac{\partial^4\ell_e}{\partial z^4}\right)\ell_{uv}\right]\left\{\begin{array}{c} u \\ v \end{array}\right\} = 0. \tag{25}$$

According to the boundary condition (10), we may write the solution in the form:

$$u(z,t) = \sum_{m=1}^{\infty} U_m T_m(t)\sin\alpha_m z \tag{26}$$

$$v(z,t) = \sum_{m=1}^{\infty} V_m T_m(t) \sin \alpha_m z$$

where $\alpha_m = m\pi$, and from (25) we obtain algebraic equation:

$$A_m \lambda_m^2 + 2B_m \lambda_m + C_m = 0, \tag{27}$$

where:

$$\begin{aligned} A_m &= \left(1 + e^2\alpha_m^2\right) \left[\left(\beta + \varsigma\alpha_m^4\right)^2 + \alpha_m^2 \left(1 + e^2\alpha_m^2\right) \left(\alpha_m^2 - f_o\right) - e^2\Omega^2\alpha_m^2\right], \\ B_m &= 2\left(\beta + \varsigma\alpha_m^4\right) \left[\left(\beta + \varsigma\alpha_m^4\right)^2 + \alpha_m^2 \left(1 + e^2\alpha_m^2\right) \left(\alpha_m^2 - f_o\right) - e^2\Omega^2\alpha_m^2\right], \\ C_m &= -\alpha_m^4 \left(1 + e^2\alpha_m^2\right) f^2(t) - 4\beta^2\Omega^2 \left(1 + e^2\alpha_m^2\right) + 8\beta\Omega^2 \left(\beta + \varsigma\alpha_m^4\right) + \\ &\quad + 4\left(\beta + \varsigma\alpha_m^4\right)^2 \left[\alpha_m^2 \left(\alpha_m^2 - f_o\right) - \alpha_m^2 f(t) - \Omega^2\right]. \end{aligned} \tag{28}$$

Hence, from Eq. (27)

$$\lambda_m = \frac{1}{1 + e^2\alpha_m^2} \times$$

$$\left\{\sqrt{\frac{\left[2\,(\beta + \varsigma\,\alpha_m^4)^2 + (1 + e^2\alpha_m^2)\,\alpha_m^2 f(t)\right]^2 + 4\,(\beta e^2 - \varsigma\,\alpha_m^2)^2\,\Omega^2\alpha_m^4}{(\beta + \varsigma\,\alpha_m^4)^2 + \alpha_m^2\,(1 + e^2\alpha_m^2)\,(\alpha_m^2 - f_o) - e^2\Omega^2\alpha_m^2}} - \right.$$

$$\left. 2\left(\beta + \varsigma\,\alpha_m^4\right)\right\}. \tag{29}$$

By solving the differential inequality (21), we obtain estimation of the functional $\mathbf{V}$, when the process $f(t)$ is ergodic and stationary it can be concluded that the trivial solutions of equations (4) are almost sure asymptotically stable, if:

$$E\left\{\max_m \lambda_m(t)\right\} \leq 0, \tag{30}$$

where E denotes the operator of the mathematical expectation. For $\beta = 0$ relation (29) reduces to the result obtained in [8].

6 Numerical results and discussion

The relation (30), give us possibility to obtain minimal damping coefficients guaranteeing the asymptotic and almost sure asymptotic stability called critical damping coefficient. The domain where the retardation times are greater than the critical damping coefficient is called the stability region or almost sure stability region. The stability regions are given as functions of loading variance, external damping coefficient, angular velocity, dimensionless parameter $e = r/\ell$, where $r = \sqrt{I/A}$ is the radius of gyration and constant component of the axial loading for Gaussian and harmonic process.

Knowledge of the probability density function $p(f)$ for the process $f(t)$ gives us possibility to obtain more precise results, (see Kozin, [9]). The boundaries of the almost sure stability are calculated by using the corresponding Gauss-Cristofel quadratures, and presented with a full line for Gaussian process, and a dashed line for the harmonic one. For Gaussian process we take the parameters of Gauss-Hermite quadrature, and for harmonic process we set $f(t) = H\cos(\omega t + \theta)$, where H, ω are fixed amplitude and frequency, and θ is a uniformly distributed random phase on the interval $[0, 2\pi)$. In order to compare both processes the variance of harmonic process $\sigma^2 = H^2/2$ is used, and we take the Gauss-Chebyshev quadrature, (see Pavlovi *et al.* [8]). Calculations were performed for the first mode, *(m = 1),* and constant part of axial loading is absent $(f_o = 0)$. In [10] is shown that for higher modes *(m = 2, 3)* regions of instability decrease, and in [8] that stability regions are larger when constant component of axial loading changes from pressure to tension. Also, critical angular velocity and stability regions can be enlarged by applying tension axial loading.

In Fig. 2 stability regions in plane variance and external damping coefficient, are plotted as a function of the retardation time when the influence of rotatory inertia is neglected $(e = 0)$. It is evident that for fixed external damping higher values of retardation time allow higher critical variances. So, we can conclude that increasing of the retardation time leads to the noticable larger stability regions and that viscoelasticity increases the almost sure stability of the rotating shaft.

In Fig. 3 stability regions in plane variance and angular velocity are plotted as a function of the external damping coefficient, when is taken

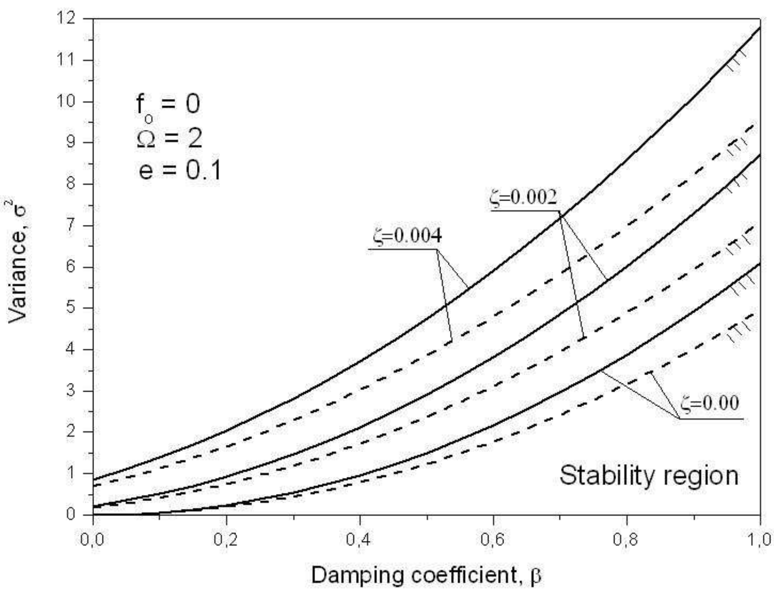

Figure 2: Influence of retardation time on stability regions

influence of rotatory inertia *(e = 0.1)*. As in previous case, increasing of the external damping coefficient leads to increasing of stability regions.

Fig. 4 illustrates effect of cross-section rotatory inertia on almost sure stability when external damping coefficient and retardation time are fixed ($\beta = 0.5, \quad \varsigma = 0.001$). When is angular velocity less than first natural frequency $\left(\overline{\Omega} \le \omega_1 \quad \Rightarrow \quad \Omega \le \pi^2\right)$, influence of rotatory inertia can be neglected, but on higher angular velocities, increasing of parameter e causes decreasing of stability regions.

7 Conclusion

By means of the direct Liapunov method, the stochastic stability of a viscoelastic rotating shaft subjected to axial time-dependent forces at the ends is analyzed. By taking into account rotatory inertia of the shaft cross-section, almost sure stability conditions are obtained. Stability regions are calculated for Gaussian and harmonic processes and shown as functions of angular velocity, external damping coefficient and retardation time.

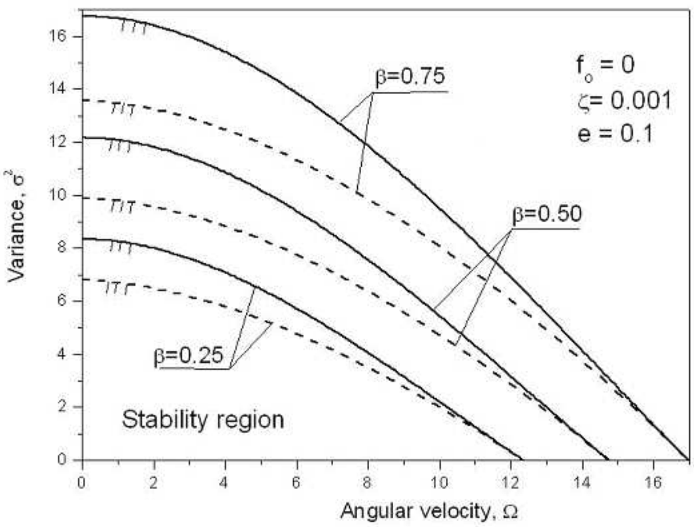

Figure 3: Influence of external damping coefficient on stability regions

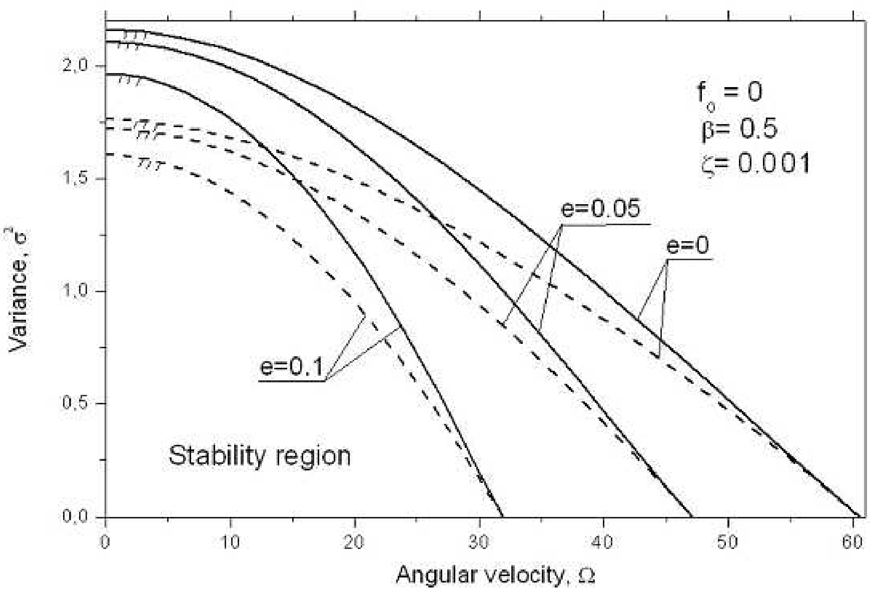

Figure 4: Influence of cross-section rotatory inertia on stability regions

According to previous, we can emphasize the following conclusions:

1. In general, stability regions for Gaussian process are larger than for harmonic one.

2. Stability regions are noticeable larger when external damping coefficient and retardation time increase.

3. Influence of cross-section rotatory inertia is remarcable when the shaft rotates by angular velocities higher than first natural frequency of the shaft.

4. Viscoelasticity increases the almost sure stability of the rotating shaft.

Acknowledgement: This work was supported by Ministry of Science and Environmental Protection of the Republic of Serbia, throw the project No 144023.

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Submitted on March 2008, revised on August 2008.

Uticaj inercije obrtanja na stohastičku stabilnost viskoelastičnog obrtnog vratila

Proučava se problem stohastičke stabilnosti viskoelastičnog Voigt-Kelvinovog uravnoteženog obrtnog vratila podvrgnutog dejstvu aksijalnih sila na njegovim krajevima. Vratilo je kružnog poprečnog preseka i obrće se konstantnom ugaonom brzinom oko svoje geometrijske ose. Uzet je u obzir uticaj inercije obrtanja poprečnog preseka vratila i spoljašnje viskozno prigušenje. Sila se sastoji od konstantnog dela i vremenski promenljive slučajne funkcije. Analitičko rešenje u zatvorenom obliku je dobijeno za slučaj prosto oslonjenog vratila. Korišćenjem direktne metode Ljapunova dobijeni su uslovi skoro sigurne asimptotske stabilnosti u funkciji varijanse procesa, koeficijenta spoljašnjeg prigušenja, vremena retardacije, ugaone brzine, i geometrijskih i fizičkih parametara vratila. Numerička sračunavanja su izvršena za Gausov proces sa nultim matematičkim očekivanjem i varijansom σ^2, kao i za harmonijski proces amplitude H.

doi:10.2298/TAM0804363P Math.Subj.Class.: 93E15, 74D05