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A contribution to the theory of the extended Lagrangian formalism for rheonomic systems

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Abstract

In this paper the generalization of the notion of variation in the extended Lagrangian formalism for the rheonomic mechanical systems (Dj. Mušicki, 2004) is formulated and analyzed in details. This formalism is based on the extension of a set of generalized coordinates by new quantities, which determine the position of the frame of reference to which the chosen generalized coordinates refer. In the process of varying, the notion of variation is extended so that these introduced quantities, being additional generalized coordinates, must also to be varied, since the position of each particle of this system is completely determined only by all these generalized coordinates.

With the consistent utilization of this notion of variation, the main results of this extended Lagrangian formalism are systematically presented, with the emphasis on the corresponding energy laws, first examined by V. Vujičić (1987), where there are two types of the energy change laws $d\mathcal{E}/dt$ and the corresponding conservation laws. Furthermore, the generalized Noether's theorem for the nonconservative systems with the associated Killing's equations (B. Vujanović, 1978) is extended to this formulation of mechanics, and applied for obtaining the corresponding energy laws. It is demonstrated that these energy laws, which are more general and

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more natural than the usual ones, are in full accordance with the corresponding ones in the vector formulation of mechanics, if they are expressed in terms of quantities introduced in this extended Lagrangian formalism.

Finally, the obtained results are illustrated by an example: the motion of a damped linear harmonious oscillator on an inclined plane, which moves along a horizontal axis, where it is demonstrated that there is valid an energy–like conservation law of Vujanović's type.

Keywords: extended Lagrangian formalism, rheonomic systems, rheonomic potential, Noether's theorem for nonconservative systems, energy–like conservation law.

1 Introduction

The modification of the analytical mechanics of rheonomic systems began with the papers by V. Vujičić [1–4], with the goal to study their energy relations more thoroughly, including the influence of nonstationary constraints. By introducing an additional generalized coordinate, which was suggested by the form of the nonstationary constraints, the main general principles of mechanics were formulated, an extended system of the Lagrangian equations was obtained, and the corresponding energy relations for such systems have been studied. Under certain conditions, the obtained energy conservation law had an unexpected form, containing an additional term arising from the nonstationary constraints, which differs essentially from its standard form.

Regarding conservation laws in general, B. Vujanović with coauthors [5–7] gave a general method for finding the integrals of motion in the usual Lagrangian formulation, which is applicable to the nonconservative systems as well. It is based on the generalization of the Noether's theorem via d'Alambert-Lagrange's principle, and the associated generalized Killing's equations. Applying it, obtaining the integrals of motion reduces to finding the particular solutions of the Killing's equations, which gives not only well-known integrals of motion, but also new ones for the nonconservative systems. Another approach to this problem was given by the author (Dj. Mušicki, [8–10]) in the form of a parametric formulation of mechanics. It is based on the separation of the double role of time for such systems (an independent variable and a parameter), and on the introduction of a new parameter, later taken as an additional generalized coordinate. Based on this, the main general principles of mechanics and the corresponding Lagrangian and Hamiltonian equations are formulated, as well as the energy conservation laws, and the corresponding Noether's theorem. The obtained results are in accordance with the corresponding ones obtained by V. Vujičić, because of the formal similarity of the roles of the time and the introduced parameter, but they have different basic ideas and interpretation.

Recently, also by the author (Dj.Mušicki, [11, 12]), this formulation of mechanics was extended in an essentially different way to the systems with more general form of the nonstationary constraints, so called extended Lagrangian formalism. This new formulation is based on the extension of the chosen set of generalized coordinates by new quantities, which determine the position of the corresponding moving frame of reference, with respect to which these generalized coordinates refer. Although formally similar to the previous parametric formulation of mechanics, it is more logically consistent, with a clear meaning of the introduced quantities as well as of this formulation itself. In the analysis of the corresponding energy relations, it is demonstrated that in this formulation there are two different types of the energy change law $d\mathcal{E}/dt$ and the corresponding conservation laws, and that they are more general and more natural than the corresponding ones in the usual Lagrangian formulation.

In this paper we shall define and analyse this extended notion of variation, which is applied to the position vectors of the particles and to the generalized coordinates. Based on this, we shall systematically present this extended Lagrangian formalism, with emphasis on the corresponding energy relations, and following Vujanović's method (1989), extend the generalized Noether's theorem to this formulation of mechanics. Finally, we shall demonstrate the equivalence of the obtained energy laws with the corresponding ones in the usual vector formulation of mechanics.

2 Basic ideas of extended Lagrangian formalism

Let us consider the motion of a mechanical system of N particles under the influence of arbitrary active forces, bounded by k nonstationary holonomic constraints, in which time appears through one or several functions $\varphi_a(t)$,

$$f_\mu[\vec{r}_\nu, \varphi_a(t)] = 0 \quad (\mu = 1, 2, \ldots, k;\ \nu = 1, 2, \ldots, N), \tag{2.1}$$

which is affirmed in all real examples. Let us determine the position of this mechanical system with respect to some system of reference by a set of generalized coordinates q^i $(i = 1, 2, \ldots, n)$, where $n = 3N - k$. We should point that in this case the system of reference is always moving in the course of time.

The fundamental idea of this formulation of mechanics (Dj.Mušicki, [11]) is based on the introduction of new quantities suggested by the form of the constraints, which change in the course of time according to the law $\tau_a = \varphi_a(t)$, and on the extension of the chosen set of generalized coordinates by these quantities. It has been demonstrated that these quantities determine the position of the frame of reference, to which the chosen generalized coordinates refer, with respect to some immobile frame of reference.

In order to determine the position of the considered mechanical system completely, let these introduced quantities $\tau_a = \varphi_a(t)$ be additional generalized coordinates, denoted by q^a, whose dependence on time is given in advance. Then, the complete set of generalized coordinates will be:

$$q^\alpha = \left\{ q^i\ (i = 1, 2, \ldots, n);\ q^a\ (a = n+1, , n+A) \right\}, \tag{2.2}$$

where the first n generalized coordinates q^i determine the position of the mechanical system with respect to the corresponding associated frame of reference, and additional ones q^a the position of this reference frame with respect to the immobile one.

Let us illustrate this idea by a simple example, the motion of a particle under the influence of arbitrary active forces on a sphere of radius R, whose center moves along a horizontal line uniformly with the velocity V (Fig.1). Assuming that this line is x-axis, and that the initial position of the center is the origin O, implies that the equation of this moving sphere represents the corresponding nonstationary constraint

$$(x - Vt)^2 + y^2 + z^2 = R^2 \tag{2.3}$$

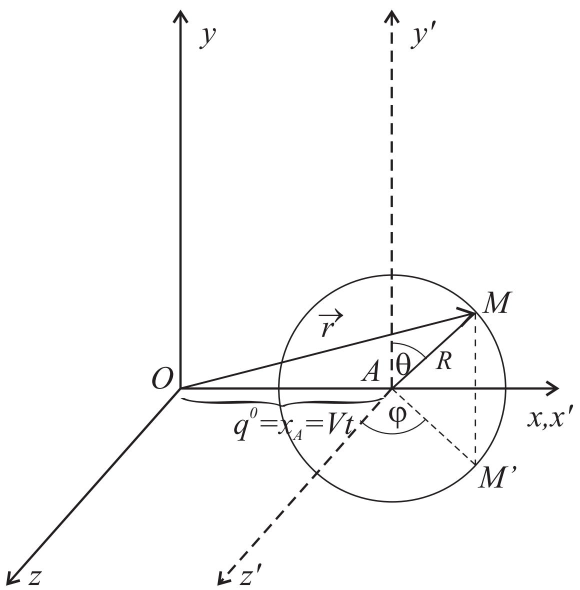

Figure 1: The particle on a moving sphere

The position of the particle on the sphere can be determined by the spherical coordinates $q^1 = \theta$ and $q^2 = \varphi$. Here, the characteristic quantity, which changes according to the law $\tau = Vt$, is the abscissa of the center A of the sphere, and needs to be taken as an additional generalized coordinate

$$\tau = x_A = q^3 \equiv q^0, \quad q^0(t) = Vt \tag{2.4}$$

This quantity determines the position of the moving reference frame with respect to the immobile one.

If we express the position vectors of particles as functions of the generalized coordinates, $\vec{r}_\nu = \vec{r}_\nu[q^i, \varphi_a(t)]$ and insert them into the nonstationary constraints (2.1), they become

$$f_\mu\{\vec{r}_\nu[q^i, \varphi_a(t)], \varphi_a(t)\} = f_\mu(x^\alpha) = 0 \quad (\mu = 1, 2, \ldots, k) \tag{2.5}$$

Therefore, in this formulation of mechanics, the considered rheonomic systems are formally reduced to the equivalent scleronomic systems, but with greater number of generalized coordinates $n' = n + A$.

As an immediate consequence, the velocity of any particle can be presented as

$$\vec{v}_\nu = \frac{d\vec{r}_\nu}{dt} = \frac{\partial \vec{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha \quad (\nu = 1, 2, \ldots, N), \tag{2.6}$$

where it is understood that the summation is over the repeated index. Then, the kinetic energy of the system is

$$T = \frac{1}{2} m_\nu \vec{v}_\nu^2 = \frac{1}{2} a_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta, \tag{2.7}$$

where

$$a_{\alpha\beta} = m_\nu \frac{\partial \vec{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \vec{r}_\nu}{\partial q^\beta}, \tag{2.8}$$

i.e. in this formulation of mechanics the kinetic energy is a homogeneous quadratic function of the generalized velocities, and so defined coefficients $a_{\alpha\beta}$ represent the components of the metric tensor of the corresponding extended configuration space.

3 Extension of the notion of variation

Since the introduced quantities $q^a = \tau_a = \varphi_a(t)$ together with the chosen generalized coordinates completely determine the position of the considered mechanical system of particles, the usual notion of variation (see e.g. Dobronravov ([13], p.75–81)) must be extended, so that besides the position vectors the quantities τ_a are varied as well. These variations of the additional generalized coordinates cause very small changes of the position of the associated frame of reference, compatible with the nonstationary constraints (2.1). Since this compatibility must remain valid in the course of time, and these constraints in each instant must be satisfied, each such change implies some change of the form of these constraints. Therefore, so extended variations δq^α $(\alpha = 1, 2, \ldots, n + A)$ are always accompanied by some change of the associated frame of reference, as well as of the corresponding constraints.

The family of varied paths of the particles of the system, together with their real trajectories starting from their positions in some instant $t = t_0$

(Fig.2), can be defined in the following way. Let us consider a family of curves

$$\vec{r}_\nu = \vec{r}_\nu(t, \gamma, \tau_a) \quad (\nu = 1, 2, \ldots, N), \tag{3.1}$$

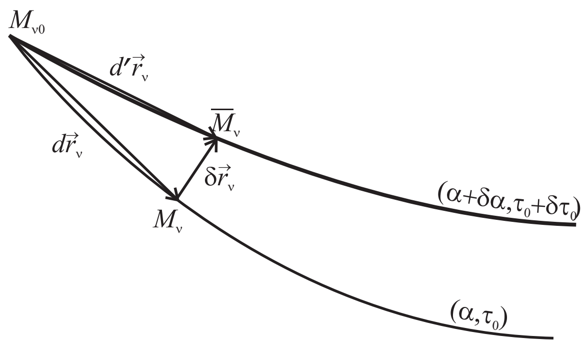

Figure 2: The family of varied paths and the variation $\delta\vec{r}_\nu$

in which, besides usual parameter here denoted by γ, we introduced the quantities τ_a $(a = 1, 2, \ldots, A)$ as the additional parameters, and suppose that for $\gamma = \gamma_0$ these equations represent the real motion of these particles. Then, the varied paths of any particle can be defined similarly as in the usual case, namely as such arbitrary paths whose values of parameters γ and τ_a differ very little from those on the real trajectory of this particle, but in accordance with the constraints.

We shall illustrate it with the simple example represented on the Fig.1, where a particle moves along the surface of a sphere, whose center moves with the velocity V along a horizontal axis. In the process of varying the particle's position, we must vary not only its polar coordinates θ and φ, but also the additional coordinate $q^0 = Vt$. This means that we must take, besides the changed functions $\theta = \theta(t)$ and $\varphi = \varphi(t)$, another function $q^0(t) = Vt + \varepsilon(t)$, which differs very little from the linear function $q^0 = Vt$, i.e. instead of the uniform motion of the center of this sphere, we will have a motion, which is close to the uniform one.

In order to formulate the variation of the position vector in so extended sense, let us consider two very close varied paths of a particle

$$\vec{r}_\nu = \vec{r}_\nu(t, \gamma, \tau_a), \quad \vec{r}_\nu = \vec{r}_\nu(t, \gamma + \delta\gamma, \tau_a + \delta\tau_a) \quad (\nu = 1, 2, ..., N) \tag{3.2}$$

Then, the simultaneous variation of the position vector will be defined as

$$\delta\vec{r}_\nu \stackrel{def}{=} \overline{\vec{r}_\nu}(t) - \vec{r}_\nu(t) = \vec{r}_\nu(t,\gamma+\delta\gamma,\tau_a+\delta\tau_a) - \vec{r}_\nu(t,\gamma,\tau_a), \tag{3.3}$$

where, $\vec{r}_\nu(t)$ and $\overline{\vec{r}_\nu}(t)$ are the functions which represent the position vectors on these varied paths taken in the same instant t.

If we expand the first function in Taylor's series, and neglect the higher terms

$$\vec{r}_\nu(t,\gamma+\delta\gamma,\tau_a+\delta\tau_a) \approx \vec{r}_\nu(t,\gamma,\tau_a) + \delta\gamma\left(\frac{\partial\vec{r}_\nu}{\partial\gamma}\right)_0 + \delta\tau_a\left(\frac{\partial\vec{r}_\nu}{\partial\tau_a}\right)_0,$$

the previous relation obtains the form

$$\delta\vec{r}_\nu = \left(\frac{\partial\vec{r}_\nu}{\partial\gamma}\right)_0\delta\gamma + \left(\frac{\partial\vec{r}_\nu}{\partial\tau_a}\right)_0\delta\tau_a \equiv \delta_0\vec{r} + \delta_\tau\vec{r}, \tag{3.4}$$

The first term $\delta_a\vec{r}_\nu = \left(\frac{\partial\vec{r}_\nu}{\partial\gamma}\right)_0\delta\gamma$ represents the variation of the position vector in the usual sense, denoted here by index zero, and the second term is specific for this formulation of mechanics.

In order to see a geometrical sense of this variation, let us imagine two very small displacements of this particle: $d\vec{r}_\nu$ along a varied path and $d'\vec{r}_\nu$ along another varied path close to the first one in the same small time interval $(t-dt,t)$, and form their difference (see Fig.2)

$$d'\vec{r}_\nu - d\vec{r}_\nu = \overrightarrow{M_\nu M_\nu} = \vec{r}_\nu(t,\gamma+\delta\gamma,\tau_a+\delta\tau_a) - \vec{r}_\nu(t,\gamma,\tau_a)$$

According to the definition (3.3), this expression represents the variation of the position vector $\delta\vec{r}_\nu$ in the instant t,

$$\delta\vec{r}_\nu = d'\vec{r}_\nu - d\vec{r}_\nu \quad (\nu=1,2,\ldots,N), \tag{3.5}$$

i.e. the variation $\delta\vec{r}_\nu$ can be treated as difference of two possible very small displacements of a particle in the same time interval dt and this is the virtual displacement of the particle.

So extended notion of variation can be formulated for the extended generalized coordinates q^α $(\alpha=1,2,\ldots,n+A)$ as well, where the family of varied paths can be described by

$$q^\alpha = q^\alpha(t,\gamma,\tau_\alpha) \quad (\alpha=1,2,\ldots,n+A), \tag{3.6}$$

with the quantities τ_a having double role: they are the additional parameters and also the additional generalized coordinates $q^a = \tau_a$. The simultaneous variation of the generalized coordinate q^α will be defined by

$$\delta q^\alpha \stackrel{def}{=} \overline{q^\alpha}(t) - q^\alpha(t) = q^\alpha(t, \gamma + \delta\gamma, \tau_a + \delta\tau_a) - q^\alpha(t, \gamma, \tau_a), \tag{3.7}$$

where $q^\alpha(t)$ and $\overline{q^\alpha}(t)$ are the functions which represent the generalized coordinates corresponding to these very close varied paths, and the first ones $q^\alpha(t)$ can coincide with the real motion of the system.

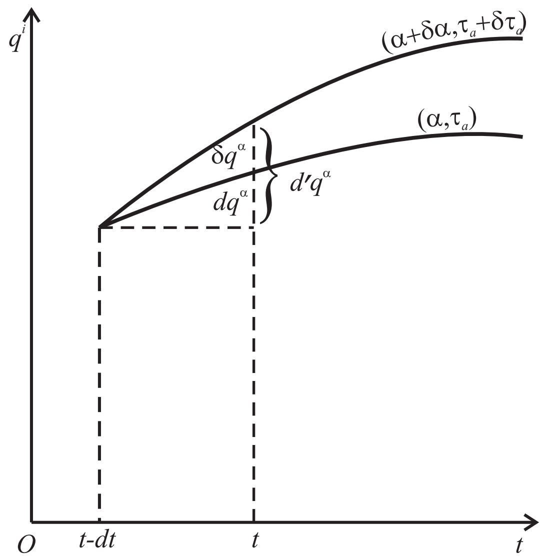

Figure 3: The variation δq^α as $\delta q^\alpha = d'q^\alpha - dq^\alpha$

If we expand the first function in (3.7) in Taylor's series, in a analogous way as in the previous case, we obtain

$$\delta q^\alpha = \left(\frac{\partial q^\alpha}{\partial \gamma}\right)_0 \delta\gamma + \left(\frac{\partial q^\alpha}{\partial \tau_a}\right)_0 \delta\tau_a \equiv \delta_0 q^\alpha + \delta_\tau q^\alpha. \tag{3.8}$$

Similarly to (3.5), this variation of generalized coordinate can be represented as the difference of its two possible very small changes in the same time interval dt (Fig.3).

$$\delta q^\alpha = d'q^\alpha - dq^\alpha \quad (\alpha = 1, 2, \ldots, n + A) \tag{3.9}$$

Finally, in this formulation of mechanics, it is possible to introduce the total or non-simultaneous variation of the generalized coordinate by generalization of the definition (3.7)

$$\Delta q^\alpha \equiv (\delta q^a)_{tot} \stackrel{def}{=} \overline{q^\alpha}(t+\Delta t) - q^\alpha(t) \quad (\alpha = 1, 2, \ldots, n+A), \tag{3.10}$$

where the functions $q^\alpha(t)$ and $\overline{q^\alpha}(t)$ correspond to two close varied paths, the first taken in the instant t, and the second in the instant $t + \Delta t$. If we expand the first term $\overline{q^\alpha}(t + \Delta t)$ in Taylor's series and insert it in the previous relation, it obtains the form

$$\Delta q^\alpha = \overline{q^\alpha}(t) - q^\alpha(t) + \dot{q}^\alpha \Delta t = \delta q^\alpha + \dot{q}^\alpha \Delta t \tag{3.11}$$

as in the usual Lagrangian formulation.

In this formulation of mechanics all the usual relations for the variations retain the same form as in the usual Lagrangian formulation, but all the variations must be understood in the extended sense. So, for example if we consider two infinitel changes of a function $\vec{r}_\nu = \vec{r}_\nu(q^\alpha, t)$ in the some time interval dt

$$d\vec{r}_\nu = \frac{\partial \vec{r}_\nu}{\partial q^\alpha} dq^\alpha + \frac{\partial \vec{r}_\nu}{\partial t} dt, \quad d'\vec{r}_\nu = \frac{\partial \vec{r}_\nu}{\partial q^\alpha} d'q^\alpha + \frac{\partial \vec{r}_\nu}{\partial t} dt,$$

their difference gives, on the base of (3.5) and (3.8)

$$\delta \vec{r}_\nu = d'\vec{r}_\nu - d\vec{r}_\nu = \frac{\partial \vec{r}_\nu}{\partial q^\alpha} \delta q^\alpha \tag{3.12}$$

For an integral of the forme $F = \int_{t_0}^{t_1} f(q^\alpha, \dot{q}^\alpha, t) dt$ its total variation will be, according its definition and after possible approximations

$$\Delta F = \int_{\overline{t_0}}^{\overline{t_1}} f(\overline{q}^\alpha, \dot{\overline{q}}^\alpha, \overline{t}) d\overline{t} - \int_{t_0}^{t_1} f(q^\alpha, \dot{q^\alpha}, t) dt = \delta F + |f \Delta t|_{t_0}^{t_1}, \tag{3.13}$$

where, for example $\overline{q}^\alpha = q^\alpha + \delta q^\alpha$, with q^α given by (3.7).

4 Work of the ideal forces of constraints

An immediate consequence of so extended notion of variation concerns the work of all the ideal forces of constraints along the arbitrary virtual displacements of particles. In order to obtain the necessary conditions for the quantities $\delta\vec{r}_\nu$, let us vary the nonstationary constraints (2.1) in the extended sense, meaning that the quantities $\tau_a = \varphi_a(t)$ are varied as well. The change in the form of these constraints arises not only from the variation of the position vectors $\vec{r}_\nu$, but also from the variation of the quantities τ_a, directly as well as through the quantities $\vec{r}_\nu$. In this way, we obtain

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu}\cdot\delta_0\vec{r}_\nu+\frac{\partial f_\mu}{\partial \tau_a}\delta\tau_a+\frac{\partial f_\mu}{\partial \vec{r}_\nu}\cdot\frac{\partial \vec{r}_\nu}{\partial \tau_a}\delta\tau_a=\frac{\partial f_\mu}{\partial \vec{r}_\nu}\cdot\left(\delta_0\vec{r}_\nu+\frac{\partial \vec{r}_\nu}{\partial \tau_a}\delta\tau_a\right)+\frac{\partial f_\mu}{\partial \tau_a}\delta\tau_a=0,$$

which, according to (3.4) can be written as

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu}\cdot\delta\vec{r}_\nu+\frac{\partial f_\mu}{\partial \tau_a}\delta\tau_a=0 \quad (\mu=1,2,\ldots,k) \tag{4.1}$$

and these are the conditions, which must be satisfied by the virtual displacements $\delta\vec{r}_\nu$. Then, the elementary work of all the ideal forces of constraints defined in the usual way by $\vec{R}_\nu=\lambda_\mu\frac{\partial f_\mu}{\partial \vec{r}_\nu}$ will be

$$\delta A=\vec{R}^{id}_\nu\cdot\delta\vec{r}_\nu=\lambda_\mu\frac{\partial f_\mu}{\partial \vec{r}_\nu}\cdot\delta\vec{r}_\nu=-\lambda_\mu\frac{\partial f_\mu}{\partial \tau_a}\delta\tau_a,$$

i.e. it has the form

$$\delta A=\vec{R}^{id}_\nu\cdot\delta\vec{r}_\nu=R^0_a\delta\tau_a, \tag{4.2}$$

where

$$R^0_a=-\lambda_\mu\frac{\partial f_\mu}{\partial \tau_a}=\vec{R}^{id}_\nu\cdot\frac{\partial \vec{r}_\mu}{\partial q^a} \tag{4.3}$$

So defined quantities $\vec{R}^0_\nu$ arise from the nonstationary constraints and represent the generalized forces which correspond to the additional generalized coordinates q^a. The property that work $\vec{R}^{id}_\nu\cdot\delta\vec{r}_\nu$ in this extended Lagrangian formalism is different from zero is one of the characteristics of this formalism, which differs essentially from the usual Lagrangian formulation, where this work is equal to zero.

As an appropriate example, consider the motion of a particle along a straight line in an inclined plane, which moves uniformly in the direction OO' under the influence of arbitrary forces. In this case, the associated frame of reference is attached to this particle, and possible elementary displacements of this pole $\Delta\vec{r}_A$ and $\Delta'\vec{r}_A$ coincide with the corresponding displacements of this particle $d\vec{r}$ and $d'\vec{r}$. In the standard analytical mechanics all these elementary displacements along the real and the varied paths end up in the same position of this moving inclined plane OC in the instant $t+\Delta t$ (Fig.4a). Therefore, the virtual displacement $\delta\vec{r}$ for any varied path is parallel to this inclined plane, and the ideal force of the constraints $\vec{R}^{id}$ is normal to it, thus $\vec{R}^{id}\cdot\delta\vec{r}=0$, what corresponds to the general definition of the ideal forces of constraints.

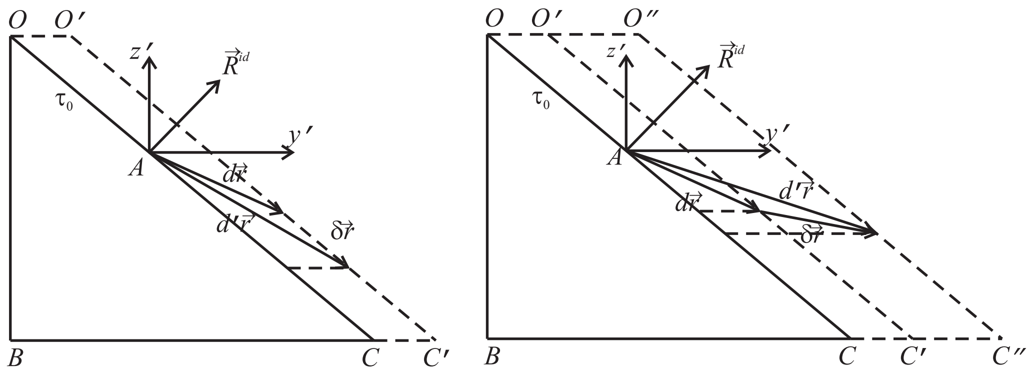

Figure 4: The variation $\delta\vec{r}$ a) in this usual formulation, b) in this formulation of mechanics

However, in this extended Lagrangian formalism, because of the additional variation of the quantity τ_a, i.e., of the pole of the associated frame of reference $AX'Y'Z'$, this pole will be displaced by an additional displacement $d_\tau\vec{r}$ (Fig.4b). This implies that the considered particle will also be displaced for the same amount, as well as this inclined plane, on which this particle must be in each instant, and therefore this inclined plane is now passing over to the final position $O'C'$. For that reason, the vector $\delta\vec{r}_\nu$ will not be parallel to this inclined plane anymore, and therefore $\vec{R}^{id}\delta\vec{r}\neq 0$. This means that the elementary work of the ideal force of constraint along arbitrary virtual displacement of this particle will be different from zero, what is in accordance with the obtained result (4.2).

5 D'Alambert-Lagrange's principle and Lagrangian equations

In order to obtain the corresponding d'Alambert-Lagrange's principle in this extended Lagrangian formalism, let us start from the fundamental equation of dynamics applied to each particle of the system

$$m_\nu \vec{a}_\nu = \vec{F}_\nu + \vec{R}_\nu^{id} + \vec{R}_\nu^* \quad (\nu = 1, 2, \ldots, N), \tag{5.1}$$

where the forces of constraints are decomposed into the ideal and the nonideal ones, the latter denoted by $\vec{R}_\nu^*$. If we multiply these equations by $\delta\vec{r}_\nu$ understood in the extended sense (3.3), sum over the repeated index, and substitute the elementary work of the ideal forces of constraints $\vec{R}_\nu^{id} \cdot \delta\vec{r}_\nu$ by the expression (4.2), we obtain

$$(\vec{F}_\nu + \vec{R}_\nu^* - m_\nu \vec{a}_\nu) \cdot \delta\vec{r}_\nu = -R_a^0 \delta\tau_a \tag{5.2}$$

This relation represents the corresponding d'Alambert-Lagrange's principle in this formulation of mechanics, which differs from the corresponding one in the usual formulation (see e.g. Goldstein, [14], p.16–21) only by the term on the right-hand side, which is specific for this extended Lagrangian formalism.

This principle can be expresses in terms of these generalized coordinates as well, putting $\delta\vec{r}_\nu = \frac{\partial \vec{r}_\nu}{\partial q^\alpha}\delta q^\alpha$ according (3.12) and having in mind that $q^a = \tau_a$. In this way, applying the analogous transformations as in the usual Lagrangian formulation, this principle passes to (see Mušicki, [11])

$$\left(Q_\alpha + R_\alpha^* + \delta_\alpha^a R_a^0 - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\alpha} + \frac{\partial T}{\partial q^\alpha} \right) \delta q^\alpha = 0, \tag{5.3}$$

where

$$Q_\alpha = \vec{F}_\nu \cdot \frac{\partial \vec{r}_\nu}{\partial q^\alpha}, \quad R_a^* = \vec{R}_\nu^* \cdot \frac{\partial \vec{r}_\nu}{\partial q^a}, \tag{5.4}$$

and δ_α^a is the Kronecker's symbol. If we decompose the generalized forces Q_α into the potential and the nonpotential ones, the latter deigned by Q_α^*, the principle (5.3) obtains the form

$$\left(\tilde{Q}_\alpha^* + \delta_\alpha^a R_a^0 - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} + \frac{\partial L}{\partial q^\alpha} \right) \delta q^\alpha = 0, \tag{5.5}$$

where the Lagrangian L and $\tilde{Q}^*_\alpha$ are given by

$$\begin{gathered} L(q^\alpha,\dot{q}^\alpha,t)=\frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta-V,\quad V=b_\alpha\dot{q}^\alpha+U(q^\alpha,t)\\ \tilde{Q}^*_\alpha=\vec{F}^*_\nu\frac{\partial\vec{r}_\nu}{\partial q^\alpha}+\vec{R}^*_\nu\frac{\partial\vec{r}_\nu}{\partial q^\alpha}=Q^*_\alpha+R^*_\alpha \end{gathered}\tag{5.6}$$

with U being the potential energy of the system.

From (5.5), because of the independence of the variations δq^α, it follows that every coefficient near to δq^α must be equal to zero, what can be decomposed into two groups of equations

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^i}-\frac{\partial L}{\partial q^i}=\tilde{Q}^*_i\quad(i=1,2,...,n)\\ &\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^a}-\frac{\partial L}{\partial q^a}=\tilde{Q}^*_a+R^0_a\quad(a=n+1,...,n+A) \end{aligned}\tag{5.7}$$

and these are the corresponding Lagrangian equations in this formulation of mechanics, which have also been obtained by V. Vujičić [2], but in less general form and in another way. We should point out, that here, the explicit dependence on time of the Lagrangian appears only in the case when the active forces are explicitly dependent on time.

The first n of these equations, coincide with the Lagrange's equations in the usual Lagrangian formulation, and the additional ones are specific for this formulation. Due to the cited property of the first group of Lagrangian equations, in this formulation of mechanics we have the same equations of motion $q^I=q^i(t)$ $(i=1,2,\ldots,n)$ as in the usual Lagrangian formulation, but the corresponding energy relations will be changed essentially.

Of special interest for this analysis is that in certain cases the quantities R^0_a can be presented as the partial derivatives of some function with respect to q^a

$$R^0_a=-\frac{\partial P}{\partial q^a}\quad\Rightarrow\quad P=-\int R^0_a dq^a\tag{5.8}$$

Then, by grouping the similar terms, Lagrangian equations (5.7) can be written in the form

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^\alpha}-\frac{\partial\mathcal{L}}{\partial q^\alpha}=\widetilde{Q}^*_\alpha\quad(\alpha=1,2,\ldots,n+A),\tag{5.9}$$

where

$$\mathcal{L}(q^\alpha, \dot{q}^\alpha, t) = L - P = T - (V + P). \tag{5.10}$$

So defined function P has the meaning of some potential energy, which arises from the nonstationary constraints.

6 Energy relations in this formulation of mechanics

In so extended Lagrangian formalism let us present the corresponding energy relations for the considered mechanical systems, what can be effectuated starting either directly from the corresponding d'Alambert-Lagrange's principle, or from the associated Lagrangian equations. In the first case, let us start from the principle (5.5) in the form

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^\alpha} - \widetilde{Q}^*_\alpha - \delta^a_\alpha R^0_a\right)\delta q^\alpha = 0 \tag{6.1}$$

and, according to (3.8), present variation δq^a as $\delta q^\alpha = d'q^\alpha - dq^\alpha$. Now, choose $d'q^\alpha$ to present the elementary change of q^α in the course of the real motion of system in the time interval $(t, t + dt)$, and let dq^α be zero, so that the variation δq^α will be

$$\delta q^\alpha = (dq^\alpha)_{real} = \dot{q}^\alpha dt \quad (\alpha = 1, 2, \ldots, n + A) \tag{6.2}$$

Then, after transforming the first term in (6.1), the d'Alambert-Lagrange's principle becomes

$$d\left(\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha\right) - \frac{\partial L}{\partial \dot{q}^\alpha}d\dot{q}^\alpha - \frac{\partial L}{\partial q^\alpha}dq^\alpha = (\widetilde{Q}^*_\alpha + \delta^a_\alpha R^0_a)dq^\alpha \tag{6.3}$$

and since, in general case,

$$dL = \frac{\partial L}{\partial q^\alpha}dq^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}d\dot{q}^\alpha + \frac{\partial L}{dt}dt,$$

the previous relation by grouping the similar terms, obtains the form:

$$d\left(\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L\right) = -\frac{\partial L}{\partial t}dt + \widetilde{Q}^*_\alpha dq^\alpha + R_a dq^a \tag{6.4}$$

In this formalism, depending on the structure of the last term, there are two possible cases, which give two different types of the energy change law $\frac{d\mathcal{E}}{dt}$ (see Mušicki [11])

(a) If $R^0_a dq^a$ can be presented as a total differential of some function

$$R^0_a dq^a = -dP \quad \Leftrightarrow \quad P - \int R^0_a dq^a, \tag{6.5}$$

the relation (6.4) after division by dt, can be written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L + P\right) = -\frac{\partial L}{\partial t} + \widetilde{Q}^*_\alpha \dot{q}^\alpha$$

or more concisely, bearing in mind that P is dependant only on q^a

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{q}^\alpha - \mathcal{L}\right) = -\frac{\partial \mathcal{L}}{\partial t} + \widetilde{Q}^*_\alpha \dot{q}^\alpha, \tag{6.6}$$

where $\mathcal{L}$ is given by (5.10). This is the corresponding energy change law, and so defined function P is identical to the quantity defined by (5.8), which is named rheonomic potential by V.Vujičić [2]. The quantities R^0_a can be obtained either directly using its definition (4.3), or from the corresponding Lagrangian equations, and then P by integration (5.8).

(b) If $R^0_a dq^a$ cannot be presented as a total differential of some function, it is necessary to start from the general relation (6.4), which can be written in the form

$$\frac{\partial \mathcal{E}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L\right) = -\frac{\partial L}{\partial t} + \widetilde{Q}^*_\alpha \dot{q}^\alpha + R^0_a \dot{q}^a \tag{6.7}$$

This is the corresponding energy change law, but in this case the rheonomic potential does not exist.

The physical meaning of the corresponding generalized energies, given in the parenthesis of (6.6) and (6.7) can be find in the following way. So, in the first case, according to (5.10) and (5.6), this generalized energy will be

$$\mathcal{E}^{ext} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{q}^\alpha - \mathcal{L} = (a_{\alpha\beta}\dot{q}^\beta - b_\alpha)\dot{q}^\alpha - \left(\frac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - b_\alpha\dot{q}^\alpha - U - P\right),$$

i.e., it is equal to

$$\mathcal{E}^{ext} = \frac{\partial \mathcal{L}}{\partial \dot{q}^{\alpha}} \dot{q}^{\alpha} - \mathcal{L} = T + U + P \tag{6.8}$$

and in the second case

$$\mathcal{E} = \frac{\partial L}{\partial \dot{q}^{\alpha}} \dot{q}^{\alpha} - L = T + U \tag{6.9}$$

Therefore, the generalized energy in the first case represents the total mechanical energy of the system extended by the rheonomic potential, which expresses the influence of the nonstationary constraints, but in the second case it coincides with the usual total mechanical energy.

Then, from the energy change laws we can obtain immediately the corresponding conservation laws. Namely, in the first case, from the relation (6.6) we can conclude: if the expression $R_a^0 dq^a$ is a total differential of some function, if the Lagrangian does not explicitly depend on time, and if the nonpotential active forces as well as the nonideal forces of constraints are absent or at least gyroscopic, then the energy conservative law is valid in the form

$$\mathcal{E}^{ext} = \frac{\partial \mathcal{L}}{\partial \dot{q}^{\alpha}} \dot{q}^{\alpha} - \mathcal{L} = T + U + P = const \tag{6.10}$$

This conservative law was obtained by V.Vujičić [2] in his modification of the dynamics of rheonomic systems, but it had less general form and a different starting point.

In the second case, for the validity of the energy conservation law, it is necessary to add one more condition $R_a^0 \dot{q}^a = 0$, and then it has the form

$$\mathcal{E} = \frac{\partial L}{\partial \dot{q}^{\alpha}} \dot{q}^{\alpha} - L = T + U = const, \tag{6.11}$$

but this rigid condition is rarely satisfied.

Let us remark that these energy change laws and the associated conservation laws differ essentially from the corresponding ones in the usual Lagrangian formulation, where the influence of the nonstationary constraints is absent and the kinetic energy is not complete, and where the corresponding energy conservation law is so–called Jacobi–Painlevé's energy integral (see e.g. Lur'e, [15], p.285-287)

$$\mathcal{E} = \frac{\partial L}{\partial \dot{q}^{\alpha}} \dot{q}^{\alpha} - L = T_2 - T_0 + U = const \tag{6.12}$$

7 Generalized Emmy Noether's theorem

The integrals of motion for the considered systems in this formulation of mechanics can be obtained in even more general form using the generalized Noether's theorem with the associated generalized Killing's equations, which is valid for the nonconservative systems as well (Vujanović and Jones, [7], p. 80–91, 111–121). This can be obtained also directly (Dj.Mušicki, [16]), starting from the total variation of action and the general Lagrangian equations. In order to adapt it for this formulation of mechanics, let us start from the total (nonsimultaneous) variation of Hamilton's action (see e.g. Dobronravov([13], p.142-146)), understood in so presented extended sense as defined by (3.13), where the variables $q^a = \tau_a(t)$ are varied as well

$$\Delta W = \Delta \int_{t_0}^{t_1} L(q^\alpha, \dot{q}^\alpha, t)dt = \int_{t_0}^{t_1} \left\{ \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}^a} (\Delta q^\alpha - \dot{q}^\alpha \Delta t) + L \Delta t \right] + \right.$$
$$\left. + \left(\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} \right) (\Delta q^\alpha - \dot{q}^\alpha \Delta t) \right\} dt, \tag{7.1}$$

but here there are $n + A$ generalized coordinates. Further, let us choose these total variations Δq^α and Δt, which represent the transformation of the generalized coordinates and time from (q^α, t) to $(\overline{q}^\alpha, \overline{t})$, in the form of a r-parametric group with r infinitesimal parameters ε^m

$$\Delta q^\alpha = \overline{q}^\alpha - q^\alpha = \varepsilon^m \xi^\alpha_m (q^\beta, \dot{q}^\beta, t), \quad \Delta t = \overline{t} - t = \varepsilon^m \xi^0_m (q^\beta, \dot{q}^\beta, t),$$
$$(\alpha = 1, 2, \ldots, n + A; \ m = 1, 2, \ldots, r) \tag{7.2}$$

Then, by inserting (7.2) into (7.1), the total variation of action can be presented as a function of ξ^a_m and ξ^0_m, i.e.,

$$\Delta W = \int_{t_1}^{t_0} \varepsilon^m \left\{ \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}^\alpha} (\xi^\alpha_m - \dot{q}^\alpha \xi^0_m) + L \xi^0_m \right] - \right.$$
$$\left. - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} \right) (\xi^\alpha_m - \dot{q}^\alpha \xi^0_m) \right\} dt \tag{7.3}$$

Let us transform the expression (7.1) substituting the variational derivative according to the Lagrangian equations (5.7) by $\widetilde{Q}^*_\alpha + \delta^a_\alpha R^0_a$. In addition, let us add and subtract the term $\frac{d\Lambda}{dt}$, where Λ can be an arbitrary function of q^α, $\dot{q}^\alpha$ and t, which defines the invariance of the action up to so called gauge term, whereby the total variation of action obtains the form

$$\Delta W = \int_{t_0}^{t_1} \left\{ \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}^\alpha} (\Delta q^\alpha - \dot{q}^\alpha \Delta t) + L\Delta t - \Lambda \right] - \right.$$
$$\left. - (\widetilde{Q}^*_\alpha + \delta^a_\alpha R^0_a)(\Delta q^\alpha - \dot{q}^\alpha \Delta t) - \frac{d\Lambda}{dt} \right\} dt \tag{7.4}$$

On the other hand, the total variation of action can be transformed in the following way

$$\Delta W = \Delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \left[\Delta L + L \frac{d}{dt}(\Delta t) \right] dt \tag{7.5}$$

and, after inserting (7.5) into (7.4), this relation can be written as

$$\int_{t_0}^{t_1} \left\{ \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}^\alpha} (\Delta q^\alpha - \dot{q}^\alpha \Delta t) + L\Delta t - \Lambda \right] - \left[\Delta L + L \frac{d}{dt}(\Delta t) + \right. \right.$$
$$\left. \left. + \widetilde{Q}^*_\alpha (\Delta q^\alpha - \dot{q}^\alpha \Delta t) + R^0_a (\Delta q^a - \dot{q}^a \Delta t) - \frac{d\Lambda}{dt} \right] \right\} dt = 0 \tag{7.6}$$

Therefore, if the following condition is satisfied

$$\Delta L + L \frac{d}{dt}(\Delta t) + Q^*_\alpha (\Delta q^\alpha - \dot{q}^\alpha \Delta t) + R^0_a (\Delta q^a - \dot{q}^a \Delta t) - \frac{d\Lambda}{dt} = 0, \tag{7.7}$$

since the time interval (t_0, t_1) is arbitrary, the integrand in (7.6) must be zero, from where follows

$$I = \frac{\partial L}{\partial \dot{q}^\alpha} \Delta q^\alpha + \left(L - \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha \right) \Delta t - \Lambda = const \tag{7.8}$$

After substituting Δq^α and Δt by the expressions (7.2), and with $\Lambda = \varepsilon^m \Lambda_m$, we obtain

$$I = \left[\frac{\partial L}{\partial \dot{q}^\alpha} \xi^\alpha_m + \left(L - \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha \right) \xi^0_m - \Lambda_m \right] \varepsilon^m = const$$

and, because of the independence of parameters ε^m, this conservation law will be decomposed into r independent ones

$$I_m = \frac{\partial L}{\partial \dot{q}^\alpha} \xi^\alpha_m + \left(L - \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha \right) \xi^0_m - \Lambda_m = const \quad (m = 1, 2, \ldots, r) \tag{7.9}$$

If $R_a^0 dq^a$ can be presented as a total differential of some function

$$R_a^0 dq^a = -dP \quad \Rightarrow \quad R_a^0 = -\frac{\partial P}{\partial q^\alpha}, \tag{7.10}$$

in accordance with (6.5), let us start from the relation (7.6) and transform the specific term in the following way

$$\begin{aligned} R_a^0(\Delta q^a - \dot{q}^a \Delta t) &= R_a^0 \Delta q^a - \frac{R_a^0 dq^a}{dt}\Delta t = -\frac{\partial P}{\partial q^a}\Delta q^a + \frac{dP}{dt}\Delta t = \\ &= -\Delta P + \frac{d}{dt}(P\Delta t) - P\frac{d}{dt}(\Delta t), \end{aligned}$$

since P is dependant only on the variables q^a. Then the relation (7.6) by grouping the similar terms, can be written in the form

$$\begin{aligned} \int_{t_0}^{t_1} \Big\{ \frac{d}{dt}\Big[\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\Delta q^\alpha + \Big(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{q}^\alpha\Big)\Delta t - \Lambda\Big] - \Big[\Delta\mathcal{L} + \\ + \mathcal{L}\frac{d}{dt}(\Delta t) + \widetilde{Q}^*_\alpha(\Delta q^\alpha - \dot{q}^\alpha \Delta t) - \frac{d\Lambda}{dt}\Big]\Big\}\, dt = 0, \end{aligned} \tag{7.11}$$

where $\mathcal{L}$ is given by (5.10). From here we conclude: if

$$\Delta\mathcal{L} + \mathcal{L}\frac{d}{dt}(\Delta t) + \widetilde{Q}^*_\alpha(\Delta q^\alpha - \dot{q}^\alpha \Delta t) - \frac{d\Lambda}{dt} = 0, \tag{7.12}$$

the following quantity remains constant in the course of time

$$\begin{aligned} I' &= \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\Delta q^\alpha + \Big(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{q}^\alpha\Big)\Delta t - \Lambda = \\ &= \Big[\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\xi^\alpha_m + \Big(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{q}^\alpha\Big)\xi^0_m - \Lambda_m\Big]\varepsilon^m = const, \end{aligned}$$

which can be also decomposed into r independent integrals of motion

$$I'_m = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\xi^\alpha_m + \Big(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{q}^\alpha\Big)\xi^0_m - \Lambda_m = const \quad (m = 1, 2, \dots, r) \tag{7.13}$$

Consequently, for any r-parametric transformation of generalized coordinates and time which satisfies the condition (7.7) or (7.12), there are r-mutually independent integrals of motion (7.9) or (7.13) respectively. This statement represents some generalization of our formulation of Noether's theorem in a parametric formulation of mechanics (Mušicki, [9]) and is valid for any mechanical system which can be nonconservative as well. In the special case when $\Delta t = 0$, $\widetilde{Q}^*_\alpha = 0$, and $\frac{d\Lambda}{dt} = 0$, the condition (7.12) for the validity of the integrals of motion is reduced to $\Delta\mathcal{L} = \delta\mathcal{L} = 0$ (because of $\Delta t = 0$) and the variation of action will be

$$\delta W = \int_{t_0}^{t_1} \delta\mathcal{L} dt = 0, \tag{7.14}$$

which corresponds to the usual formulation of the Noether's theorem, but with $\mathcal{L}$ insted of L.

Now, analogously to the procedure given by B.Vujanović [7] in the usual formulation of mechanics, we shall find the equations which must be satisfied by the functions ξ^α_m and ξ^0_m. If we start from the condition (7.7), write the total variation of Lagrangian ΔL explicitly, and apply the relation between the total variation of $\dot{q}^\alpha$ and time derivative of Δq^α (see e.g. Dobronravov, [13], p.145)

$$\Delta q^\alpha = \Delta \frac{dq^\alpha}{dt} = \frac{d}{dt}(\Delta q^\alpha) - \dot{q}^\alpha \frac{d}{dt}(\Delta t), \tag{7.15}$$

the condition (7.7) obtains the form

$$\begin{aligned} &\frac{\partial L}{\partial q^\alpha}\Delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}\left[\frac{d}{dt}(\Delta q^\alpha) - \dot{q}^\alpha \frac{d}{dt}(\Delta t)\right] + \frac{\partial L}{\partial t}\Delta t + L\frac{d}{dt}(\Delta t) + \\ &\qquad + \widetilde{Q}^*_\alpha(\Delta q^\alpha - \dot{q}^\alpha \Delta r) + R^0_a(\Delta q^a - \dot{q}^a \Delta t) - \frac{d\Lambda}{dt} = 0 \end{aligned} \tag{7.16}$$

After substituting Δq^α and Δt by the expression (7.2), and with $\Lambda = \varepsilon^m \Lambda_m$, this relation becomes a linear combination of the parameters ε^m, and because of their independence, each coefficient near to ε^m must be equal to zero

$$\begin{aligned} &\frac{\partial L}{\partial q^\alpha}\xi^\alpha_m + \frac{\partial L}{\partial \dot{q}^\alpha}\dot{\xi}^\alpha_m + \left(L - \frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha\right)\dot{\xi}^0_m + \frac{\partial L}{\partial t}\xi^0_m + \\ &\widetilde{Q}^*_\alpha(\xi^\alpha_m - \dot{q}^\alpha \xi^0_m) + R^0_a(\xi^a_m - \dot{q}^a \xi^0_m) - \dot{\Lambda}_m = 0 \end{aligned} \tag{7.17}$$

and this relation can be named, by analogy with corresponding Vujanović's relation ([7], p.118) in the usual Lagrangian formulation, the basic Noether's identity. If we expand all time derivatives of the functions ξ_m^α, ξ_m^0 and Λ_m, after grouping the similar terms we obtain a relation of the form $A_m + \ddot{q}^\beta B_{m\beta} = 0$, in which functions A_m and $B_{m\beta}$ do not depend on $\ddot{q}^\beta$. This relation can be satisfied only if $A_m = 0$, and $B_{m\beta} = 0$, what gives the following two systems of equations

$$\frac{\partial L}{\partial q^\alpha}\xi_m^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}\left[\frac{\partial \xi_m^\alpha}{\partial q^\beta}\dot{q}^\beta + \frac{\partial \xi_m^\alpha}{\partial t} - \dot{q}^\alpha\left(\frac{\partial \xi_m^0}{\partial q^\beta}\dot{q}^\beta + \frac{\partial \xi_m^0}{\partial t}\right)\right] + \frac{\partial L}{\partial t}\xi_m^0 + L\left(\frac{\partial \xi_m^0}{\partial q^\beta}\dot{q}^\beta + \right.$$
$$\left. + \frac{\partial \xi_m^0}{\partial t}\right) + \widetilde{Q}_\alpha^*(\xi_m^\alpha - \dot{q}^\alpha\xi_m^0) + R_a^0(\xi_m^a - \dot{q}^a\xi_m^0) - \frac{\partial \Lambda_m}{\partial q^\beta}\dot{q}^\beta - \frac{\partial \Lambda_m}{\partial t} = 0, \tag{7.18}$$

and
$$\frac{\partial L}{\partial \dot{q}^\alpha}\left(\frac{\partial \xi_m^\alpha}{\partial \dot{q}^\beta} - \dot{q}^\alpha\frac{\partial \xi_m^0}{\partial \dot{q}^\beta}\right) + L\frac{\partial \xi_m^0}{\partial \dot{q}^\beta} - \frac{\partial \Lambda_m}{\partial \dot{q}^\beta} = 0. \tag{7.19}$$

If $R_a^0 dq^a$ is a total differential of some function, by comparing the condition (7.7) with (7.12) we conclude that the latter relation (7.12) differs from the previous one only by $\mathcal{L}$ instead of L and by absence of the term $R_a^0(\Delta q^a - \dot{q}^a\Delta t)$. Thus, for example, the basic Noether's identity, instead of (7.17) in this case will have the form

$$\frac{\partial \mathcal{L}}{\partial q^\alpha}\xi_m^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{\xi}_m^\alpha + \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{q}^\alpha\right)\dot{\xi}_m^0 + \frac{\partial \mathcal{L}}{\partial t}\xi_m^0 + \widetilde{Q}_\alpha^*(\xi_m^\alpha - \dot{q}^\alpha\xi_m^0) - \dot{\Lambda}_m = 0. \tag{7.20}$$

These equations (7.18) and (7.19) are the corresponding generalized Killing's equations for the functions ξ_m^α and ξ_m^0, which differ from the corresponding ones in the usual Lagrangian formulation (see Vujanović and Jones, [7], p. 118–121) only by the specific term $R_a^0(\xi_m^a - \dot{q}^a\xi_m^0)$, and their meaning consists in the following. If there exists at least one particular solution of this system of equations, the condition for the existence of integrals of motion (7.7) will be satisfied, implying that for each such transformation of generalized coordinates and time (7.2), there are r-mutually independent integrals of motion (7.9). In this way, finding the integrals of motion is reduced to finding the particular solutions of this system of equations, what in ceratin cases can give some energy–like integrals of motion for the nonconservative systems as well.

Now, we shall demonstrate that from so formulated generalized Noether's theorem, we can obtain the cited energy conservation laws. As it is known, in the usual formulation of mechanics, the energy conservation laws are a consequence of the homogeneity of time, which, under certain conditions, implies the invariance of the Lagrangian with respect to the translation of time. The latter can be demonstrated either directly, or from the Noether's theorem (see e.g. Dobronravov, [13], p.156).

Following this idea, for the transformation of the generalized coordinates and time (7.2), here we shall also take the translation of time,

$$\Delta t = \bar{t} - t = const, \quad \Delta q^\alpha = \overline{q}^\alpha - q^\alpha = 0, \tag{7.21}$$

to which, according to (7.2), the following functions correspond

$$\xi_m^0 = 1, \quad \xi_m^\alpha = 0 \tag{7.22}$$

If we insert these expressions into the generalized Killing's equations (7.18) and (7.19), and take $\Lambda_m = 0$, then, the first equation gives

$$\frac{\partial L}{\partial t} - \widetilde{Q}_\alpha^* \dot{q}^\alpha = 0 \tag{7.23}$$

and the second one is identically satisfied. Therefore, the Killing's equations are satisfied under condition (7.23), which implies that the condition (7.7) for the existence of the integrals of motion is satisfied as well. The corresponding integral of motion, according to (7.9) and (7.22) will be

$$I_m = \frac{\partial L}{\partial \dot{q}^\alpha} \xi_m^\alpha + \left(L - \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha \right) \xi_m^0 - \Lambda_m = L - \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha$$

and on the basis of (6.11) we obtain

$$I_m = - \left(\frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha - L \right) = -(T + U) = const \tag{7.24}$$

If $R_a^0 dq^0$ is a total differential of some function, according to (7.13) we have

$$I_m' = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \xi_m^\alpha + \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \dot{q}^\alpha \right) \xi_m^0 - \Lambda_m = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \dot{q}^\alpha$$

and the corresponding integral of motion by using (6.10) will be

$$I'_m = -\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\dot{q}^\alpha - \mathcal{L}\right) = -(T + U + P) = const, \tag{7.25}$$

These results coincide with the energy conservation law (6.11) and (6.10), what confirms the correctness of this method for obtaining the energy integrals in this formulation of mechanics.

8 Connection with the vector formulation of mechanics

Let us examine how these energy change laws refer to the corresponding laws in the standard vector formulation of mechanics. In this aim, suppose that all the potential active forces are the usual ones ($\vec{F}_\nu = -\frac{\partial U}{\partial \vec{r}_\nu}$) and start from the kinetic energy law in the vector form

$$dT = \vec{F}_\nu \cdot d\vec{r}_\nu + \vec{R}_\nu \cdot d\vec{r}_\nu = \vec{F}^{pot}_\nu \cdot d\vec{r}_\nu + \vec{F}^*_\nu \cdot d\vec{r}_\nu + \vec{R}^{id}_\nu \cdot d\vec{r}_\nu + \vec{R}^*_\nu \cdot d\vec{r}_\nu, \tag{8.1}$$

where the active forces are decomposed into the potential and nonpotential ones, and the forces of constraints into the ideal and nonideal ones. Because of the cited assumption, the first term can be transformed in the following way

$$\vec{F}^{pot}_\nu \cdot d\vec{r}_\nu = -\frac{\partial U}{\partial \vec{r}_\nu} \cdot d\vec{r}_\nu = -dU + \frac{\partial U}{\partial t}dt \tag{8.2}$$

If we put $q^a = t$, then, after introducing the notation (5.4), the sum of the second and the fourth term can be presented as

$$\begin{aligned} \left(\vec{F}^*_\nu + \vec{R}^*_\nu\right) \cdot d\vec{r}_\nu &= \left(\vec{F}^*_\nu + \vec{R}^*_\nu\right) \cdot \left(\frac{\partial \vec{r}_\nu}{\partial q^i}dq^i + \frac{\partial \vec{r}_\nu}{\partial t}dt\right) = \\ &= \left(\vec{F}^*_\nu + \vec{R}^*_\nu\right) \cdot \frac{\partial \vec{r}_\nu}{\partial q^a}dq^\alpha = (Q^*_\alpha + R^*_\alpha)dq^\alpha \end{aligned} \tag{8.3}$$

The third term, the elementary work of the ideal forces of constraints along the arbitrary possible displacements of particles, can be transformed starting from the fact that these possible displacements are bounded by the following conditions

$$\frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot d\vec{r}_\nu + \frac{\partial f_\mu}{\partial t} dt = 0 \quad (\mu = 1, 2, \dots, k), \tag{8.4}$$

which are obtained by differentiating the constraints (2.1). Then, the work of the ideal forces of constraints, defined by $\vec{R}^{id}_\nu = \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu}$, can be presented in the form

$$\vec{R}^{id}_\nu \cdot d\vec{r}_\nu = \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot d\vec{r}_\nu = -\lambda_\mu \frac{\partial f_\mu}{\partial t} dt,$$

in which, according to (4.3), the coefficient next to dt, expressed in terminology of this formalism, represents the quantity R^0_a for $\tau = q^a = t$,

$$\vec{R}^{id}_\nu \cdot d\vec{r}_\nu = R^0_{(\tau = t)} dt \tag{8.5}$$

Then the kinetic energy law (8.1), divided by dt gives the energy change law

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}(T + U) = \frac{\partial U}{\partial t} + \widetilde{Q}^*_\alpha \cdot \dot{q}^\alpha + R^0_{a(\tau_a = t)}, \tag{8.6}$$

where $\widetilde{Q}^*_\alpha$ is given by (5.4).

If $R^0 dt$ can be presented as a total differential of some function, the term $R^0_{(\tau = t)} dt$ in (8.5) according to (6.5), can be interpreted as the negative value of the quantity P for $q^a = t$, i.e.

$$\vec{R}^{id}_\nu \cdot d\vec{r}_\nu = R^0_{(q^a = t)} dt = -dP_{(q^a = t)} \tag{8.7}$$

By inserting these expressions into the kinetic energy law (8.1), we obtain

$$dT = -dU + \frac{\partial U}{\partial t} dt + (Q^*_\alpha + R^*_\alpha) dq^\alpha - dP_{(q^a = t)},$$

which, after grouping the similar terms, and dividing by dt, can be presented in the form

$$\frac{d\mathcal{E}^{ext}}{dt} = \frac{d}{dt}[T + U + P_{(q^a = t)}] = \frac{\partial U}{\partial t} + \widetilde{Q}^*_\alpha \dot{q}^\alpha \tag{8.8}$$

The analogous results can be obtained from the equations with multipliers, which in the general case have the form

$$m_\nu \frac{dv_k}{dt} = \vec{F}_\nu^{pot} + \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu} + \vec{F}_\nu^* + \vec{R}_\nu^* \quad (\nu = 1, 2, \ldots, N) \tag{8.9}$$

If we multiply this equation by $d\vec{r}_\nu = \vec{v}_\nu dt$ and sum over the repeated index, we obtain

$$m_\nu \vec{v}_\nu \cdot d\vec{v}_\nu = d\left(\frac{1}{2} m_\nu \vec{v}_\nu^2\right) = \vec{F}^{pot} \cdot d\vec{r}_\nu + \lambda_\mu \frac{\partial f_\mu}{\partial \vec{r}_\nu} \cdot d\vec{r}_\nu + (\vec{F}_\nu^* + \vec{R}_\nu^*) \cdot d\vec{r}_\nu \tag{8.10}$$

With the same supposition about the potential forces, the first term on the right-hand side can be written in the form (8.2), and the second one represents the elementary work of the ideal forces of constraints and is presented by (8.5).

Then the relation (8.10), by grouping the similar terms and divided by dt, gives the energy change law in the form

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(\frac{1}{2} m_\nu \vec{v}_\nu^2 + U\right) = \frac{\partial U}{\partial t} + (\vec{F}_\nu^* + \vec{R}_\nu^*) \cdot \vec{v}_\nu + R^0_{(q^a = t)} \tag{8.11}$$

If $R^0 dt$ is a total differential of some function, in a similar way, putting $R^0 dt = -dP_{(\tau = t)}$ in the previous relation, we obtain

$$\frac{d\mathcal{E}^{ext}}{dt} = \frac{d}{dt}\left(\frac{1}{2} m_\nu \vec{v}_\nu^2 + U + P_{(\tau_a = t)}\right) = \frac{\partial U}{\partial t} + (\vec{F}_\nu^* + \vec{R}_\nu^*) \cdot \vec{v}_\nu \tag{8.12}$$

These results are equivalent with the previous ones (8.6) and (8.7), but the previous results are expressed in the extended generalized coordinates as in this formulation of mechanics, and the latter ones in the initial vector form.

If we compare these energy change laws with the corresponding laws (6.7) and (6.6) in this formulation of mechanics, bearing in mind that here the explicit dependence of the Lagrangian on time arises only from such dependence of the active forces, so that $\partial L/\partial t = \partial \mathcal{L}/dt = -\partial U/\partial t$, we come to the following conclusion:

The energy change laws, which is obtained in the vector formulation of mechanics and expressed in terms of here introduced quantities, are completely equivalent to the corresponding laws in this formulation of mechanics. This confirms the correctness of this extended Lagrangian formalism, demonstrates its importance, and its advantage with respect to the usual Lagrangian formulation.

9 An example

Consider the motion of a linear harmonious oscillator with the attraction center in the point A_0 (Fig.5), moving along the straight line on an inclined plane, in a damping medium with a resistance force proportional to the velocity of this oscillator. Further, assume that this inclined plane moves along a horizontal axis according to the given law $x_A = x_A(t)$ and choose it to be x-axis, with the origin in the point in which the pole A was in the instant $t = 0$. Then, since in any instant it is necessary that

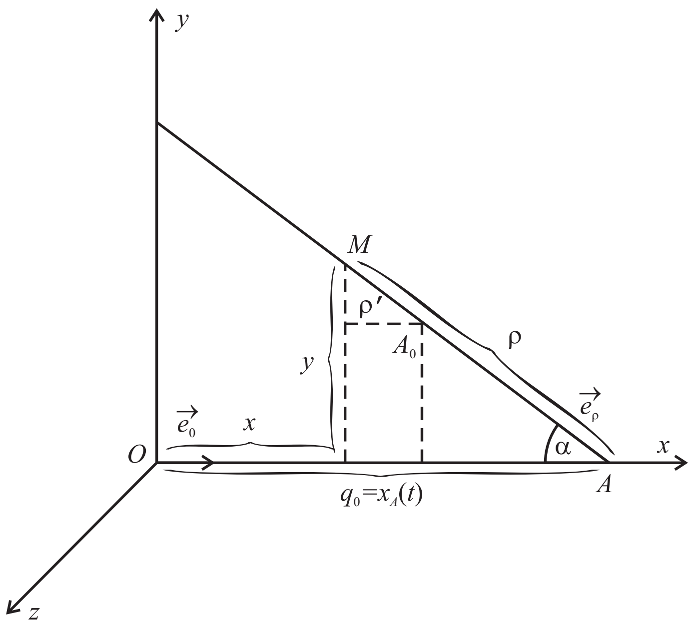

Figure 5: Damped linear oscillator on a moving inclined plane

$$\tan\alpha = \frac{y}{x_A(t) - x},$$

this motion is restricted by the nonstationary holonomic constraints

$$\begin{aligned} f_1(\vec{r}, t) &\equiv [x_A(t) - x]\sin\alpha - y\cos\alpha = 0, \\ f_2(\vec{r}, t) &\equiv z = 0. \end{aligned} \tag{9.1}$$

This motion has only one degree of freedom and the position of the oscilator can be determined by the generalized coordinate $q^1 = \rho$, presented in the figure. As an additional generalized coordinate here must be taken the quantity $\tau = x_{A(t)}$, since it is the quantity, suggested by the form of the constraint (9.1), which changes according to the law $\tau = x_A(t)$. So, in this case the complete extended set of generalized coordinates will be

$$q^\alpha = \{q^1 = \rho,\ q^a \equiv q_0 = x_A(t)\} \tag{9.2}$$

The relations between the coordinates (x, y, z) and (ρ, q_0) are

$$x = q_0 - \rho\cos\alpha, \quad y = \rho\sin\alpha, \quad z = 0 \tag{9.3}$$

and the position vector of the point M can be presented as

$$\vec{r} = q_0\vec{e}_x + \rho\vec{e}_\rho, \tag{9.4}$$

where $\vec{e}_x$ and $\vec{e}_\rho$ are the orts of x-axis and ρ-axis respectively.

The kinetic energy of this oscillator, according to (9.2) is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left[(\dot{q}_0 - \dot{\rho}\cos\alpha)^2 + (\dot{\rho}\sin\alpha)^2\right] = \\ &= \frac{1}{2}m(\dot{\rho}^2 - 2\dot{\rho}\dot{q}_0\cos\alpha + \dot{q}_0^2) \end{aligned} \tag{9.5}$$

and since its potential energy, arising from the attraction force $\vec{F}^* = -k\rho'\vec{e}_\rho$ is $U = \frac{1}{2}k\rho'^2$ where $\rho' = \rho - \rho_0$, the Lagrangian of this linear oscillator will be

$$L = T - U = \frac{1}{2}m(\dot{\rho}^2 - 2\dot{\rho}\dot{q}_0\cos\alpha + \dot{q}_0^2) - \frac{1}{2}k(\rho - \rho_0)^2 \tag{9.6}$$

The generalized forces which correspond to the reststence force $\vec{F}^* = -\beta\dot{\rho}\vec{e}_\rho$, according to their general definition and (9.4) are

$$\begin{aligned} Q^*_\rho &= \vec{F}^* \cdot \frac{\partial\vec{r}}{\partial\rho} = -\beta\dot{\rho}(\vec{e}_\rho \cdot \vec{e}_\rho) = -\beta\dot{\rho} \\ Q^*_{q_0} &= \vec{F}^* \cdot \frac{\partial\vec{r}}{\partial q_0} = -\beta\dot{\rho}(\vec{e}_\rho \cdot \vec{e}_x) = \beta\dot{\rho}\cos\alpha \end{aligned} \tag{9.7}$$

Then the corresponding extended set of the Lagrangian equations (5.7) for ρ and q_0 in this case obtains the form

$$\begin{aligned} &\frac{d}{dt}(m\dot{\rho} - m\dot{q}_0\cos\alpha) + k(\rho - \rho_0) = -\beta\dot{\rho} \\ &\frac{d}{dt}(m\dot{q}_0 - m\dot{\rho}\cos\alpha) = \beta\dot{\rho}\cos\alpha + R_0 \end{aligned} \tag{9.8}$$

In order to examine whether the rheonomic potential exists, for which it is necessary that $R_0 dq_0$ can be presented as a total differential of some function, let us express it using the second Lagrangian equation (9.8)

$$R_0 dq_0 = \dot{q}_0 R_0 dt = \dot{q}_0 d(m\dot{q}_0 - m\dot{\rho}\cos\alpha - \beta\rho\cos\alpha) \tag{9.9}$$

From this relation we can conclude that this case will occurs only if $\dot{q}_0 = const \equiv V$, i.e. $q_0 = Vt$. The quantity R_0 can be found by elimination of the variable q_0 from these Lagrangian equations, resolving the first equation with respect to $\frac{d}{dt}(m\dot{\rho})$ and inserting it in the second one,

$$m\ddot{q}_0 - \cos\alpha[m\ddot{q}_0\cos\alpha - k(\rho - \rho_0) - \beta\dot{\rho}] = \beta\dot{\rho}\cos\alpha + R_0,$$

from which it follows

$$R_0 = k\cos\alpha(\rho - \rho_0) + m\ddot{q}_0\sin^2\alpha \tag{9.10}$$

If $q_0 = Vt$, the last term disappears, and we obtain the rheonomic potential, according to (5.10), in the form

$$P = -\int R_0 dq_0 = -k\cos\alpha(\rho - \rho_0)q_0, \tag{9.11}$$

since in this process of integration all the variables except q_0 must be treated as the constants, in accordance with the relation $\partial P/\partial q_0 = -R_0$. Therefore, the rheonomic potential in this example exists only in the case $q_0 = Vt$, and then it is given by the expression (9.11) and in the following we shall suppose that always $q_0 = Vt$.

The motion of this linear oscillator is determined by the first Lagangian equation (9.8), which has the same form as in the usual Lagrangian formulation

$$\frac{d^2\rho}{dt^2} + \frac{\beta}{m}\frac{d\rho}{dt} + \frac{k}{m}(\rho - \rho_0) = \ddot{q}_0\cos\alpha$$

and in the case $q_0 = Vt$, this differential equation is reduced to the equation with constant coefficients and can be written as

$$\frac{d^2\rho'}{dt^2} + 2\mu\frac{d\rho'}{dt} + \omega^2\rho' = 0 \quad (\rho' = \rho - \rho_0), \tag{9.12}$$

where

$$2\mu = \frac{\beta}{m}, \quad \omega^2 = \frac{k}{m} \tag{9.13}$$

As it is known, the solution of this equation depends on the roots of the characteristics equation and there are three cases (see e.g. Landau and Lipschitz (1973, p.97-99).

1. If $\mu < \omega$, the solution represents the amortized oscillations

$$\rho' = Ae^{-\mu t}\sin(\omega' t + \alpha), \quad \omega' = \sqrt{\omega^2 - \mu^2} \tag{9.14}$$

2. If $\mu > \omega$, the solution represents the aperiodic motion

$$\rho' = C_1 e^{-mt} + C_2 e^{-nt} \quad (m, n > 0) \tag{9.15}$$

3. If $\mu = \omega$, the solution represents the asymptotic motion

$$\rho' = e^{-\mu t}(C_1 t + C_2), \tag{9.16}$$

In order to examine whether some of the energy conservation laws is valid in this example, let us consider the conditions under which these laws (6.10) and (6.11) are valid. In this case the effect of the nonconservative, resistence force, according to (9.7) is

$$\tilde{Q}^*_\alpha \dot{q}^\alpha = Q^*_\rho \dot{\rho} + Q^*_{q_0} \dot{q}_0 = -\beta\dot{\rho}(\dot{\rho} - \cos\dot{q}_0) \tag{9.17}$$

and in the general case it is different from zero, as well as the characteristic term $R^0_a \dot{q}^a$. Therefore the necessary conditions for the validity of these laws are not satisfied, and none of these two energy conservation laws here is valid.

However, we shall demonstrate that in the considered case $q_0 = Vt$, when $R^0_a dq^a$ is a total differential of some function, there exists an energy–like conservation law, which can be obtained by the method of Vujanović (1989), adapted for this formulation of mechanics. We shall realize this procedure with some modifications in two stages: a) the transition to an equivalent quasiconservative system, which corresponds to the case as if the nonconservative force $\vec{F}^*$ is formally absent, and b) the finding of a particular solution of the basic Noether's identity i.e. of the corresponding generalized Killing's equations.

In this aim, let us start from the corresponding Lagrangian equations (5.11)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} - \frac{\partial \mathcal{L}}{\partial q^\alpha} = \tilde{Q}^*_\alpha \quad (\alpha = 1, 2, \dots, n + A) \tag{9.18}$$

and seek such new Lagrangian

$$\overline{\mathcal{L}}(q^\alpha, \dot{q}^\alpha, t) = f(t)\mathcal{L}(q^\alpha, \dot{q}^\alpha, t) \tag{9.19}$$

that the Lagrangian equations (9.18) will be transformed to the Euler-Lagrange's equations

$$\frac{d}{dt}\frac{\partial \overline{\mathcal{L}}}{\partial \dot{q}^\alpha} - \frac{\partial \overline{\mathcal{L}}}{\partial q^\alpha} = 0 \quad (\alpha = 1, 2, \dots, n + A) \tag{9.20}$$

In this way an equivalent quasiconservative system will be defined, corresponding formally to $Q^* = 0$ (the absence of this term on the right-hand side of above equation). If we insert the expression (9.19) in the equation (9.20) we get

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\frac{df}{dt} + f\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} - \frac{\partial \mathcal{L}}{\partial q^\alpha}\right) = 0$$

and substitute the variational derivative by the corresponding expression from the Lagrangian equation (9.21), the above equations passes to

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\frac{df}{dt} + \tilde{Q}^*_\alpha \cdot f = 0 \quad (\alpha = 1, 2, \dots, n + A) \tag{9.21}$$

These equations determine the function f, and in this example we have two equations, for ρ and q_0, which have the form

$$\begin{aligned} (m\dot{\rho} - m\dot{q}_0 \cos\alpha)\frac{df}{dt} - 2\mu m\dot{\rho} f = 0 \\ (m\dot{q}_0 - m\dot{\rho}\cos\alpha)\frac{df}{dt} + 2\mu m\dot{\rho}\cos\alpha \dot{f} = 0 \end{aligned} \tag{9.22}$$

By elimination of q_0, adding the first equation to the second one multiplied by $\cos\alpha$, we obtain the following equation and its solution

$$\frac{df}{dt} + 2\mu f = 0 \quad \Rightarrow \quad f(t) = e^{2\mu t}(C = 1) \tag{9.23}$$

Then,according to (9.19), (5.12) and (9.11), the new Lagrangian will be

$$\begin{aligned}\overline{\mathcal{L}}(q^\alpha, \dot{q}^\alpha, t) = e^{2\mu t}\mathcal{L}(q^\alpha, \dot{q}^\alpha, t) = e^{2\mu t}\bigg[\frac{1}{2}m(\dot{\rho}^2 - 2\dot{\rho}\dot{q}_0\cos\alpha + \dot{q}_0^2) - \\ -\frac{1}{2}k(\rho-\rho_0)^2 + k\cos\alpha(\rho-\rho_0)q_0\bigg],\end{aligned} \tag{9.24}$$

where the influence of the resistence force is included in the exponential factor.

In order to find a particular solution of the basic Noether's identity, let us start from this relation in the form (7.20), apply it to this quasi-conservative system ($Q^*_\alpha = 0$) and choose the generates of transformation ξ^0_m, ξ^1_m and ξ^2_m in the following form

$$\xi^0_m = 0, \quad \xi^1_m = F(q^\alpha, \dot{q}^\alpha, t), \quad \xi^2_m = \lambda F(q^\alpha, \dot{q}^\alpha, t) \tag{9.25}$$

If we insert these expressions in (7.20), and put $\tilde{Q}^*_\alpha = 0$, we get

$$F\left(\frac{\partial\overline{\mathcal{L}}}{\partial\rho} + \lambda\frac{\partial\overline{\mathcal{L}}}{\partial q_0}\right) + \dot{F}\left(\frac{\partial\overline{\mathcal{L}}}{\partial\dot{\rho}} + \lambda\frac{\partial\overline{\mathcal{L}}}{\partial\dot{q}_0}\right) - \dot{\Lambda}_m = 0 \tag{9.26}$$

and this relation, after inserting these partial derivatives and multypling it by $e^{-2\mu t}$, obtains the form

$$\begin{aligned}&Fk[(\rho-\rho_0)(\lambda\cos\alpha - 1) + q_0\cos\alpha] + \\ &+ \dot{F}m\left[-\dot{\rho}(\lambda\cos\alpha - 1) - \dot{q}_0(\cos\alpha - \lambda)\right] - \dot{\Lambda}_m e^{-2\mu t} = 0\end{aligned} \tag{9.27}$$

Here is suitable to introduce a quantity, similar to the coefficient next to Fk, which connects the variables $\rho - \rho_0$ and q_0, so that its negative time derivative is equal to the coefficient next to $\dot{F}m$. In order to realize this aim, it is necessary to take $\lambda = 0$, whose immediate consequence is that the generator of transformation ξ^2_m, according to (9.25) is equal to zero.

$$\lambda = 0 \quad \Rightarrow \quad \xi^2_m = \lambda F = 0, \tag{9.28}$$

and the relation (9.27) is then simplified to

$$Fk\left[-(\rho-\rho_0) + q_0\cos\alpha\right] + \dot{F}m(\dot{\rho} - \dot{q}_0\cos\alpha) - \dot{\Lambda}_m e^{-2\mu t} = 0 \tag{9.29}$$

The form of this relation suggests us the introduction of the following quantity, which satisfies the cited condition

$$\eta = (\rho - \rho_0) - q_0 \cos \alpha, \tag{9.30}$$

and the previous relation (9.29), can be written as

$$m(\dot{F}\dot{\eta} - \omega^2 F\eta)^{\cdot} - \dot{\Lambda}_m e^{-2\mu t} = 0, \tag{9.31}$$

where ω^2 is given by (9.13).

Further, following the procedure employed by Vujanović (1989) for the similar cases, let us choose the function ξ_m^1 in the form

$$\xi_m^1 = F = A(\dot{\eta} + \mu\eta) \tag{9.32}$$

and then the previous relation (9.32) becomes

$$mA\left[(\dot{\eta}\ddot{\eta} - \omega^2\eta\dot{\eta}) + \mu(\dot{\eta}^2 - \omega^2\eta^2)\right] - \dot{\Lambda}_m e^{-2\mu t} = 0 \tag{9.33}$$

This relation determines the gauge function Λ_m, and after following transformations

$$\begin{aligned} \frac{d\Lambda_m}{dt} &= Ame^{2\mu t}\left[(\dot{\eta}\ddot{\eta} - \omega^2\eta\dot{\eta}) + \mu(\dot{\eta}^2 - \omega^2\eta^2)\right] = \\ &= \frac{Am}{2}\left[e^{2\mu t}\frac{d}{dt}(\dot{\eta}^2 - \omega^2\eta^2) + \frac{d}{dt}(e^{2\mu t})(\dot{\eta}^2 - \omega^2\eta^2)\right] \end{aligned}$$

we see that this function Λ_m has the form

$$\Lambda_m = \frac{Am}{2}e^{2\mu t}(\dot{\eta}^2 - \omega^2\eta^2) \tag{9.34}$$

In this way, with so chosen expressions for the generators of transformation and the gauge function, the basic Noether's identity is satisfied. Since this is the condition for existence of the integrals of motion, there exists a corresponding integral of motion, which is determined by (7.13)

$$I'_m = \frac{\partial\overline{\mathcal{L}}}{\partial\dot{q}^\alpha}\xi_m^\alpha + \left(\overline{\mathcal{L}} - \frac{\overline{\mathcal{L}}}{\dot{q}^\alpha}\dot{q}^\alpha\right)\xi_m^a - \Lambda_m = const \tag{9.35}$$

In this case, according to (9.24), (9.25), (9.28), (9.32) and (9.34) we have

$$I'_m = \frac{\partial \overline{\mathcal{L}}}{\partial \dot{\rho}} \dot{\xi}^1_m - \Lambda_m = e^{2\mu t} m(\dot{\rho} - \dot{q}_0 \cos\alpha) A(\dot{\eta} + \mu\eta) - \frac{Am}{2} e^{2\mu t}(\dot{\eta}^2 - \\ - \omega^2\eta^2) = Ame^{2\mu t}\left[\dot{\eta}(\dot{\eta} + \mu\eta) - \frac{1}{2}(\dot{\eta}^2 - \omega^2\eta^2)\right],$$

i.e. this energy–like integral of motion has the form

$$I'_m = Am\left(\frac{1}{2}\dot{\eta}^2 + \frac{1}{2}\omega^2\eta^2 + \mu\eta\dot{\eta}\right) e^{2\mu t} = const \tag{9.36}$$

or explicitely

$$I'_m = Am\left[\frac{1}{2}(\dot{\rho} - \dot{q}_0\cos\alpha)^2 + \frac{1}{2}\omega^2\left[[(\rho - \rho_0) - q_0\cos\alpha\right]^2 + \\ + \mu\left[(\rho - \rho_0) - q_0\cos\alpha\right](\dot{\rho} - \dot{q}_0\cos\alpha)\right] e^{2\mu t} = const \tag{9.37}$$

Consequently, although in this example none of the energy conservation laws is valid, there exists an energy–like integral of motion of Vujanović's type in the form (9.36), from which we see that in this formulation of mechanics also the integrals of motion of this type exist.

10 General conclusion

1. The main characteristic of this extended Lagrangian formalism is that extended generalized coordinates q^α $(\alpha = 1, 2, \ldots, n + A)$ determine the position of the considered mechanical system with respect to an inertial frame of reference, which is the same to which all the dynamical quantities as well as the energy laws refer. This makes the extended Lagrangian formalism essentially different from the standard Lagrangian formulation, but, because of this property, more preferable with respect to the usual Lagrangian formulation.

2. In this formalism, the position of the mechanical system is determined completely by the extended set of generalized coordinates. Because of this fact, it was necessary to introduce an extended notion of the variation, where the additional generalized coordinates

$q^a = \varphi_a(t)$ must be varied as well. Based on this extended notion of variation, main results of this extended Lagrangian formalism were presented, and the generalized Noether's theorem extended to this formulation of mechanics which enables us to obtain the energy–like conservation laws for the nonconservative systems as well.

3. The obtained energy change laws and the corresponding conservation laws differ essentially from those ones in the usual Lagrangian formulation. However, they are in full accordance with the corresponding laws in the vector formulation of mechanics, expressed in terms of quantities introduced in this extended Lagrangian formalism. Due to the cited essential characteristic, these energy laws are more general and more natural than the corresponding ones in the standard Lagrangian formulation, including the influence of the nonstationary constraints on the energy relations as will.

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Doprinos teoriji proširenog Lagranževog formalizma za reonomne mehaničke sisteme

U ovom radu formulisana je i detaljno analizirana generalizacija pojma varijacije u proširenom Lagranžovom formalizumu za reonomne sisteme (Dj. Mušicki, 2004). Na toj osnovi sistematski je prikazana ova formulacija mehanike, sa doslednim korišćenjem ovog pojma. Sam prošireni Lagranžev formalizam zasniva se na proširenju skupa izabranih generalisanih koordinata novim veličinama, koje odredjujuju položaj pridruženog sistema referencija, u odnosu na koji se odnose ove generalisane koordinate.

Iz ovog razloga ove uvedene veličine takodje moraju biti varirane, što predstavlja prošireni pojam varijacije u ovom slučaju. U tako formulisanom proširenom Lagranževom formalizmu, formulisani su odgovarajući Dalamber-Lagranžev princip i Lagranževe jednačine koje iz njega proizilaze i prikazani su odgovarajući energijski zakoni kao i Neterina teorema prilagodjena ovoj formulaciji mehanike. Pokazano je da su tako dobijeni energijski zakoni logički konzistentniji, opštiji i prirodniji od odgovarajućih zakona u standardnoj Lagranževoj formulaciji za reonomne sisteme, a u potpunoj saglasnosti sa odgovarajućim energijskim zakonima u vektorskoj formulaciji mehanike ako se oni izraze pomoću veličina uvedenih u proširenom Lagranževom formalizmu.

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