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Explicit solutions for partial differential equations of Lord-Shulman thermoelasticity

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Abstract

We consider the Lord-Shulman model of thermoelasticity with one relaxation constant. The corresponding system of four linear partial differential equations is solved by means of holomorphic expansions. We prove the convergence of expansions and study the possibility to convert them in finite sums.

Keywords: Lord-Shulman thermoelasticity, Systems of Linear Partial Differential Equations, Holomorphic expansions.

1 Introduction

The finiteness of the velocity of heat propagation can be taken in account in the dynamic problems of thermoelasticity due to the presence of the inertial terms in the corresponding equations. In the Lord-Shulman model it is done considering a generalized heat transfer equation

$$q(x,t) + \tau_t \frac{\partial q(x,t)}{\partial t} = -k \,\mathrm{grad}(T)$$

instead of the classic one

$$q(x,t) = -k \,\mathrm{grad}(T).$$

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Here $q(x,t)$ is the heat flow vector, T is a temperature variation and $\tau_t > 0$ is the time relaxation constant. It leads to the more complicated system than in the classic case. We solve it by means of holomorphic expansions which were successfully applied in [1] to the system of elastic equilibrium.

This approach to a system of PDE with k unknown functions defined in the domain $D \subset R^{m+1}$ consists in

1. Extend the D to some domain $G = \{(x,y)|x \in D, y \in D_1 \subset R^{m-1}\} \subset R^{2m}$.

2. Introduce complex variables $z_1 = x_1 + \iota x_2, z_2 = x_3 + \iota y_1, \ldots, z_m = x_{m+1} + \iota y_{m-1}$ and consider G to be a domain in the multidimensional complex space C^m.

3. Change the partial differentiation in the initial system for the Cauchy operators $d_z^k = \frac{1}{2^k}\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)^k$; $d_{\bar{z}}^k = \frac{1}{2^k}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)^k$ if $z = x + \iota y$; $k \in N$ and impose additional conditions $(d_{z_j} - d_{\bar{z}_j})W(z) = 0$, $j = 2, ..., m$ which makes the solution $W(z)$ independent of y. We call this new system a complex analog for the initial one.

4. Construct solution for the complex analog as a sum of the series
$$W = \sum_{|n|=0}^{\infty} \bar{z}^n W_n(z),$$
with the functions (coefficients) $W_n(z)$ to be found. We call this series holomorphic expansion (HE) of their sum. Here $n = (n_1, \ldots, n_m)$, $z = (z_1, \ldots, z_m) \in G$, $\bar{z}^n = z_1^{n_1}, \cdots, z_m^{n_m}$. Function $W(z)$ takes values in C^k. Exactly $W(z) = (W^1, W^2, \ldots, W^k)$, $W^j : G \longrightarrow C$. The unknown functions $W_n : G \longrightarrow C^k$ have the form $W_n(z) = (W_n^1(z), W_n^2(z), \ldots, W_n^k(z))$ with $W_n^j : G \longrightarrow C$ holomorphic in G.

5. Substitute HE in the complex analog and find $W_n(z)$.

The presented process bases on the following properties of HE proved in [2].

I. A holomorphic expansion is term by term differentiable and integrable.

II. The sum holomorphic expansion is zero if and only if all the coefficients are zero.

Our interest towards Lord-Shulman model is due to its possible practical importance but it also makes a contribution to our study of systems of partial differential equations. It is the first attempt to apply holomorphic expansions to the partial differential operator of the order greater than two.

Following the ideas of [1],[2] we show that under certain conditions only finite number of terms in the HE solution differ from zero. Thus we enjoy the possibility to express the solutions explicitly by means of elementary functions. We also prove convergence theorem for formal solutions.

We do not consider boundary value problems but just system of PDE.

2 Complex form of the Lord-Shulman system

The Lord-Shulman dynamic system of generalized thermoelasticity [3] is

$$\rho\frac{\partial^2 u_j}{\partial t^2} - (\lambda+\mu)\frac{\partial\,\mathrm{div}U}{\partial x_j} - \mu\Delta u_j + \gamma\frac{\partial T}{\partial x_j} = f_j,\ j=1,2,3,$$

$$C_\epsilon\frac{\partial T}{\partial t} - k\Delta T + \tau_t C_\epsilon\frac{\partial^2 T}{\partial t^2} + \gamma\theta_0\frac{\partial\,\mathrm{div}U}{\partial t} + \gamma\theta_0\tau_t\frac{\partial^2\,\mathrm{div}U}{\partial t^2} = f_4. \quad (1)$$

Here $U=(u_1,u_2,u_3)$ is a displacement vector, T stands for the temperature deviation and $\gamma=\alpha(3\lambda+2\mu)$. The symbols $\rho,\lambda,\mu,\ \alpha,k,C_\epsilon,\ \theta_0$ and τ_t denote the density, Lame moduli, thermal expansion coefficient, conductivity, specific heat capacity for zero deformation, reference temperature and relaxation time constant respectively. The displacements u_j and the temperature deviation T depend on $x=(x_1,x_2,x_3)$ and t. The variable x belongs to some simply connected domain $D\subset R^3$ and $t\in R$. The symbol Δ stands for Laplace operator.

We introduce complex variables $z_1=x_1+\iota x_2, z_2=x_3+\iota x_4, z_3=t+\iota t_1$ and constants $a_1=\rho$, $a_2=\lambda+\mu$, $a_3=\mu$, $a_4=\gamma$, $a_5=k/C_\epsilon$, $a_6=\tau_t$, $a_7=\gamma\theta_0/C_\epsilon$ and $a_8=\gamma\theta_0\tau_t/C_\epsilon$.

The substitutions $\frac{1}{2}(V^1+V^2)$ for u_1, $\frac{1}{2\iota}(V^1-V^2)$ for u_2, $V^3/2$ for u_3, V^4 for u_4, F^j for f_j and replacement of real differentiation for Cauchy operators ($d_z^k = \frac{1}{2^k}\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)^k$; $d_{\bar z}^k = \frac{1}{2^k}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)^k$ if $z = x + \iota y$ and $k \in N$) convert the system (1) in

$$a_2 d_{\bar z_1}^2 V^2 - a_4 d_{\bar z_1} V^4 + a_2 d_{\bar z_1} d_{z_1}(V^1+V^2) + 2a_3 d_{\bar z_1} d_{z_1}(V^1+V^2) + a_2 d_{\bar z_1} d_{z_2} V^3$$
$$+a_2 d_{z_1}^2 V^1 + a_2 d_{z_1} d_{z_2} V^3 - 2a_1 d_{z_3}^2 (V^1+V^2) + 2a_3 d_{z_2}^2 (V^1+V^2) - a_4 d_{z_1} V^4 = F^1, \tag{2}$$

$$a_2 d_{\bar z_1}^2 V^2 - a_4 d_{\bar z_1} V^4 + a_2 d_{\bar z_1} d_{z_1}(V^1-V^2) + 2a_3 d_{\bar z_1} d_{z_1}(V^1-V^2) + a_2 d_{\bar z_1} d_{z_2} V^3$$
$$-a_2 d_{z_1}^2 V^1 - a_2 d_{z_1} d_{z_2} V^3 - 2a_1 d_{z_3}^2 (V^1-V^2) + 2a_3 d_{z_2}^2 (V^1-V^2) + a_4 d_{z_1} V^4 = F^2, \tag{3}$$

$$a_3 d_{\bar z_1} d_{z_1} V^3 + a_2 d_{\bar z_1} d_{z_2} V^2 + a_2 d_{z_1} d_{z_2} V^1 + (a_2+a_3) d_{z_2}^2 V^3$$
$$-a_1 d_{z_3}^2 V^3 - a_4 d_{z_2} V^4 = F^3, \tag{4}$$

$$-2a_8 d_{\bar z_1} d_{z_3}^2 V^2 + 2a_5 d_{\bar z_1} d_{z_1} V^4 - a_7 d_{\bar z_1} d_{z_3} V^2 + 2a_5 d_{z_2}^2 V^4 - 2a_6 d_{z_3}^2 V^4 - d_{z_3} V^4$$
$$-a_7 d_{z_2} d_{z_3} V^3 - 2a_8 d_{z_1} d_{z_3}^2 V^1 - a_7 d_{z_1} d_{z_3} V^1 - 2a_8 d_{z_2} d_{z_3}^2 V^3 = F^4, \tag{5}$$

$$d_{\bar z_2} V^k = d_{z_2} V^k \quad , \quad d_{\bar z_3} V^k = d_{z_3} V^k, \quad k = 1,2,3,4. \tag{6}$$

The complex valued functions V^k depend on $z = (z_1, z_2, z_3) \in G = \{(x, x_4, t, t_1) | x \in D, (t, x_4, t_1) \in R^3, \} \subset C^3$. The system (2)-(6) is a complex analog for (1) because the functions u_1, u_2, u_3 and T derived from the solutions of (2)-(6) satisfy (1). Equations (6) provide independence of unknown functions from x_4 and t_1.

Remark 1. Complex analog for system of equations is not unique.

Remark 2. In the sections 2-4 we set F^j equal to zero.

We subtract (3) from (2) and consider $J\phi(z) = \int_0^{z_1} \phi(\xi, z_2, z_3) d\xi$ to get

$$d_{\bar z_1} V^2 = \frac{-a_2 d_{z_1} V^1 + 2J(a_1 d_{z_3}^2 - a_3 d_{z_2}^2) V^2 - a_2 d_{z_2} V^3 + a_4 V^4}{a_2 + 2a_3}. \tag{7}$$

The equation (4) gives

$$d_{\bar{z}_1}V^3 = \frac{-1}{a_2+2a_3}\Bigg(2a_2 d_{z_2}V^1 - 2a_2 J^2 d_{z_2}(d_{z_2}^2 - \frac{a_1}{a_3}d_{z_3}^2)V^2$$

$$+J((3a_2+2a_3)d_{z_2}^2 - \frac{a_1}{a_3}(a_2+2a_3)d_{z_3}^2)V^3 - 2a_4 J d_{z_2}V^4\Bigg) \qquad (8)$$

and from (5) we obtain

$$d_{\bar{z}_1}V^4 = \frac{1}{2(a_2+2a_3)a_5}\left[2a_3\left(a_7 d_{z_3} + 2a_8 d_{z_3}^2\right)V^1 + \right.$$

$$J\left(2a_3 d_{z_2}d_{z_3}\left(2a_8 d_{z_3}+a_7\right)V^3\right.$$

$$+\left(-2(a_2+2a_3)a_5 d_{z_2}^2 + 2(a_2a_6+2a_3a_6+a_4a_8)d_{z_3}^2 + \right.$$

$$\left.\left.(a_2+a_4a_7+2a_3)d_{z_3}\right)V^4\right)$$

$$\left.-2J^2\left(2a_3a_8 d_{z_2}^2 d_{z_3}^2 + a_3a_7 d_{z_2}^2 d_{z_3} - 2a_1a_8 d_{z_3}^4 - a_1a_7 d_{z_2}^3\right)V^2\right]. \qquad (9)$$

The substitution of (7),(8),(9) to (2) yields

$$d_{\bar{z}_1}V^1 = J\left[\frac{a_3^2a_4a_7 d_{z_3} + 2(a_1a_2^2a_5 + a_3^2a_4a_8)d_{z_3}^2}{2a_3a_5(a_2+2a_3)(a_2+a_3)} - 2\frac{a_3 d_{z_2}^2 - a_1 d_{z_3}^2}{a_2+2a_3}\right]V^1$$

$$+J^2\left[\left(\frac{a_3a_4a_7 d_{z_2}d_{z_3} + 2(a_3a_4a_8 - a_1a_2a_5)d_{z_2}d_{z_3}^2}{2a_5(a_2+a_3)(a_2+2a_3)} + \frac{a_2}{a_2+2a_3}d_{z_2}^3\right)V^3\right.$$

$$+\left(\frac{a_3a_4(a_2+2a_3+a_4a_7)d_{z_3} + 2a_4(a_3a_4a_8 - a_1a_2a_5)d_{z_3}^2}{4a_3a_5(a_2+2a_3)(a_2+a_3)}\right.$$

$$\left.\left. - \frac{a_4 d_{z_2}^2}{a_2+2a_3} + \frac{a_4a_6 d_{z_3}^2}{2a_5(a_2+a_3)}\right)V^4\right]$$

$$+J^3\left[\frac{a_1a_3a_4a_7 d_{z_3}^3 - 2(a_1^2a_2a_5 - a_1a_3a_4a_8)d_{z_3}^4 - a_3^2a_4a_7 d_{z_2}^2 d_{z_3}}{2a_3a_5(a_2+a_3)(a_2+2a_3)}\right.$$

$$\left.+\frac{(a_1a_2^2a_5 + 2a_1a_2a_3a_5 - a_3^2a_4a_8)d_{z_2}^2 d_{z_3}^2}{a_3a_5(a_2+a_3)(a_2+2a_3)} - \frac{a_2}{a_2+2a_3}d_{z_2}^4\right]V^2. \qquad (10)$$

The last equality holds under restriction $V^1(0, z_2, z_3) = 0$. Following the technique of holomorphic expansions we consider unknown functions to be sums of the series

$$V^j = \sum_{|n|=0}^{\infty} \bar{z}_1^{n_1} \bar{z}_2^{n_2} \bar{z}_3^{n_3} V^j_{n_1,n_2,n_3}(z_1, z_2, z_3). \tag{11}$$

The functions $V^j_{n_1,n_2,n_3}(z_1, z_2, z_3)$ are holomorphic in G and $|n| = n_1 + n_2 + n_3$. All the necessary information on HE can be found in [2].

The expressions (6)-(10) provide

$$(n_1+1)V^1_{n_1+1} = J\left[\frac{a_3^2a_4a_7d_{z_3} + 2(a_1a_2^2a_5 + a_3^2a_4a_8)d^2_{z_3}}{2a_3a_5(a_2+2a_3)(a_2+a_3)} - 2\frac{a_3d^2_{z_2} - a_1d^2_{z_3}}{a_2+2a_3}\right]V^1_n$$
$$+J^2\left[\left(\frac{a_3a_4a_7d_{z_2}d_{z_3} + 2(a_3a_4a_8 - a_1a_2a_5)d_{z_2}d^2_{z_3}}{2a_5(a_2+a_3)(a_2+2a_3)} + \frac{a_2}{a_2+2a_3}d^3_{z_2}\right)V^3_n\right.$$
$$+\left(\frac{a_3a_4(a_2+2a_3+a_4a_7)d_{z_3} + 2a_4(a_3a_4a_8 - a_1a_2a_5)d^2_{z_3}}{4a_3a_5(a_2+2a_3)(a_2+a_3)}\right.$$
$$\left.\left.-\frac{a_4d^2_{z_2}}{a_2+2a_3} + \frac{a_4a_6d^2_{z_3}}{2a_5(a_2+a_3)}\right)V^4_n\right]$$
$$+J^3\left[\frac{a_1a_3a_4a_7d^3_{z_3} - 2(a_1^2a_2a_5 - a_1a_3a_4a_8)d^4_{z_3} - a_3^2a_4a_7d^2_{z_2}d_{z_3}}{2a_3a_5(a_2+a_3)(a_2+2a_3)}\right.$$
$$\left.+\frac{(a_1a_2^2a_5 + 2a_1a_2a_3a_5 - a_3^2a_4a_8)d^2_{z_2}d^2_{z_3}}{a_3a_5(a_2+a_3)(a_2+2a_3)} - \frac{a_2}{a_2+2a_3}d^4_{z_2}\right]V^2_n \equiv L_1(V_n), \tag{12}$$

$$(n_1+1)V^2_{n_1+1} =$$
$$\frac{-a_2d_{z_1}V^1_n + 2J(a_1d^2_{z_3} - a_3d^2_{z_2})V^2_n - a_2d_{z_2}V^3_n + a_4V^4_n}{a_2+2a_3} \equiv L_2(V_n), \tag{13}$$

$$(n_1+1)V^3_{n_1+1} = \frac{-1}{a_2+2a_3}\left(2a_2d_{z_2}V^1_n - 2a_2J^2d_{z_2}(d^2_{z_2} - \frac{a_1}{a_3}d^2_{z_3})V^2_n\right.$$
$$\left.+J\left((3a_2+2a_3)d^2_{z_2} - \frac{a_1}{a_3}(a_2+2a_3)d^2_{z_3}\right)V^3_n - 2a_4Jd_{z_2}V^4_n\right) \equiv L_3(V_n), \tag{14}$$

$$(n_1+1)V^4_{n_1+1} = \frac{1}{2(a_2+2a_3)a_5}\left[2a_3\left(a_7 d_{z_3} + 2a_8 d^2_{z_3}\right)V^1_n\right.$$
$$+J\left(2a_3 d_{z_2} d_{z_3}\left(2a_8 d_{z_3} + a_7\right)V^3_n\right.$$
$$+\left(-2(a_2+2a_3)a_5 d^2_{z_2} + 2(a_2a_6 + 2a_3a_6 + a_4a_8)d^2_{z_3} +\right.$$
$$\left.\left.(a_2 + a_4a_7 + 2a_3)d_{z_3}\right)V^4_n\right)$$
$$\left.-2J^2\left(2a_3a_8 d^2_{z_2} d^2_{z_3} + a_3a_7 d^2_{z_2} d_{z_3} - 2a_1a_8 d^4_{z_3} - a_1a_7 d^3_{z_2}\right)V^2_n\right] \equiv L_4(V_n), \tag{15}$$

$$(n_j+1)V^k_{n_j+1} = d_{z_j} V^k_n,\ j = 1,2,\ k = 1,2,3,4. \tag{16}$$

We put $V = (V^1, V^2, V^3, V^4)$, and $V_n = (V^1_n, V^2_n, V^3_n, V^4_n)$. The index n stands for (n_1, n_2, n_3) and $n_1 + m$ stands for $(n_1 + m, n_2, n_3)$. The meanings of the indices $n_2 + m$ and $n_3 + m$ are obvious.

The expressions (12)-(16) enable us to find all the coefficients of HE solution for (2)-(6) starting with arbitrary holomorphic functions $V^k_{0,0,0}$ (z_1, z_2, z_3), $k = 1, 2, 3, 4$. The only restriction to impose is $V^1_{0,0,0}(0, z_2, z_3) \equiv 0$.

The matrix notation transforms (11) in

$$V = \sum_{|n|=0}^{\infty} V_n(z)\bar{z}^n. \tag{17}$$

The function V is a formal solution for (2)-(6) if (12)-(16) hold, that is

$$V_n = \frac{1}{n!} d^{n_2}_{z_2} d^{n_3}_{z_3} L^{n_1}\, V_0. \tag{18}$$

Here L is a matrix operator of dimension 4×4. Its elements are:

$$L_{1,1} = J\left(\frac{a_3^2 a_4 a_7 d_{z_3} + 2(a_1 a_2^2 a_5 + a_3^2 a_4 a_8)d^2_{z_3}}{2a_3 a_5 (a_2+2a_3)(a_2+a_3)} - 2\frac{a_3 d^2_{z_2} - a_1 d^2_{z_3}}{a_2 + 2a_3}\right),$$

$$L_{1,2} = J^3\left(\frac{a_1a_3a_4a_7 d^3_{z_3} - 2(a_1^2 a_2 a_5 - a_1a_3a_4a_8)d^4_{z_3} - a_3^2 a_4 a_7 d^2_{z_2} d_{z_3}}{2a_3a_5(a_2+a_3)(a_2+2a_3)}\right.$$
$$\left.+\frac{(a_1a_2^2a_5 + 2a_1a_2a_3a_5 - a_3^2a_4a_8)d^2_{z_2}d^2_{z_3}}{a_3a_5(a_2+a_3)(a_2+2a_3)} - \frac{a_2}{a_2+2a_3}d^4_{z_2}\right),$$

$$L_{1,3} = J^2 \left(\frac{a_3a_4a_7d_{z_2}d_{z_3} + 2(a_3a_4a_8 - a_1a_2a_5)d_{z_2}d_{z_3}^2}{2a_5(a_2+a_3)(a_2+2a_3)} + \frac{a_2}{a_2+2a_3}d_{z_2}^3 \right),$$

$$L_{1,4} = J^2 \left(\frac{a_3a_4(a_2+2a_3+a_4a_7)d_{z_3} + 2a_4(a_3a_4a_8 - a_1a_2a_5)d_{z_3}^2}{4a_3a_5(a_2+2a_3)(a_2+a_3)} - \frac{a_4d_{z_2}^2}{a_2+2a_3} + \frac{a_4a_6d_{z_3}^2}{2a_5(a_2+a_3)} \right), \tag{19}$$

$$L_{2,1} = \frac{-a_2d_{z_1}}{a_2+2a_3}, \qquad L_{2,2} = \frac{2J(a_1d_{z_3}^2 - a_3d_{z_2}^2)}{a_2+2a_3},$$

$$L_{2,3} = \frac{-a_2d_{z_2}}{a_2+2a_3}, \qquad L_{2,4} = \frac{a_4}{a_2+2a_3},$$

$$L_{3,1} = \frac{-2a_2d_{z_2}}{a_2+2a_3}, \qquad L_{3,2} = \frac{2a_2J^2d_{z_2}(a_3d_{z_2}^2 - a_1d_{z_3}^2)}{a_3(a_2+2a_3)},$$

$$L_{3,3} = -\frac{J\left(a_3(3a_2+2a_3)d_{z_2}^2 - a_1(a_2+2a_3)d_{z_3}^2\right)}{a_3(a_2+2a_3)}, \qquad L_{3,4} = \frac{2a_4Jd_{z_2}}{a_2+2a_3},$$

$$L_{4,1} = \frac{a_3\left(a_7d_{z_3} + 2a_8d_{z_3}^2\right)}{(a_2+2a_3)a_5},$$

$$L_{4,2} = \frac{-J^2\left(2a_3a_8d_{z_2}^2d_{z_3}^2 + a_3a_7d_{z_2}^2d_{z_3} - 2a_1a_8d_{z_3}^4 - a_1a_7d_{z_2}^3\right)}{(a_2+2a_3)a_5}, \tag{20}$$

$$L_{4,3} = \frac{a_3J\left(2a_8d_{z_2}d_{z_3}^2 + a_7d_{z_2}d_{z_3}\right)}{(a_2+2a_3)a_5},$$

$$L_{4,4} = \frac{J\left(2(a_2a_6+2a_3a_6+a_4a_8)d_{z_3}^2 + (a_2+a_4a_7+2a_3)d_{z_3} - 2(a_2+2a_3)a_5d_{z_2}^2\right)}{2(a_2+2a_3)a_5}.$$

3 Convergence of formal solution

In this section we study convergence of the series (17) with the operator L defined by (19) and (20). The denomination $H(\Omega)$ stands for the set of complex valued vector functions holomorphic in the domain Ω. The dimension of the vector can be equal to one or four. The context makes the things clear.

Let us denote the length of the curve l by $|l|$ and state some results from the theory of functions of complex variable.

Lemma 1. *Let Ω be a domain in C and $0 \in \Omega$. Let L_ξ be a set of rectifiable curves in Ω with extremes in 0 and $\xi \in \Omega$. Then the function $l(\xi) = \inf_{l \in L_\xi} |l|$ is continuous in Ω.*

Corollary 1. *Function $l(\xi)$ reaches its global maximum M on any compact $\Lambda \subset \Omega$.*

Lemma 2. *For all points ξ of the compact Λ and for all positive ϵ there exists a curve l in Ω with extremes in 0 and ξ and with the length less than $M + \epsilon$.*

Lemma 3. *Let Ω be a simple connected domain, Λ - compact in Ω, $f(\eta) \in H(\Omega)$, $|f(\eta)| \leq R \quad \forall \eta \in \Lambda$. Then $\forall \xi \in \Lambda \quad |\int_0^\xi f(\eta) d\eta| \leq MR$.*

We follow with the study of formal solution for (2)-(6). Let G_1 be a simple connected domain and $\bar{G}_1 \subset G$. The curves are considered in G_1 and the function norms $\|V\| = \max_{k \leq 4, z \in \bar{G}_1} |V^k(z)|$ on the compact $\bar{G}_1$.

Lemma 4. *Let $r = \sum_{k=1}^m i(k) + j(k) < \infty$, where $i(k)$ and $j(k)$ take non negative integer values. For the operator $T_r = d_{z_1}^{i(1)} J^{j(1)} \cdots d_{z_1}^{i(m)} J^{j(m)} = \prod_{k=1}^m d_{z_1}^{i(k)} J^{j(k)}$ there exist i and j such that $i + j \leq r$ and one of the following equalities holds: $T_r = d_{z_1}^{i-j}$, $T_r = J^{j-i}$ or $T_r = J^j d_{z_1}^i$.*

For the proof of above lemma see [1].

Theorem 1. *Let $V_0 \in H(G)$ satisfy the inequality $\|d_z^n V_0\| < \|V_0\| C^{|n|}$ for some fixed constant $C \geq M > 1$. Then, the estimate $\|V_n\| \leq \frac{1}{n!}(20\alpha)^{n_1} C^{11n_1+n_2+n_3} \|V_0\|$ holds for α equal to the maximum absolute value of the constant coefficients in the operators $L_{i,j}$.*

Proof. The components of operator L^{n_1} consist of 4^{n_1} additive terms and each one is composed of n_1 operators $L_{i,j}$. Every operator $L_{i,j}$ is a sum of expressions $aJ^{k_1} d_{z_1}^{k_2} d_{z_2}^{k_3} d_{z_3}^{k_4}$ where a is a constant with the maximum absolute value equal to α. The number of these expressions is equal to five in $L_{1,2}$ and it is less than or equal to five in the other operators. The maximum values of k_1, k_2, k_3 and k_4 are 3,1,4 and 4, respectively. They appear in $L_{1,2}$, $L_{2,1}$, $L_{1,2}$ and $L_{1,2}$. That yields the estimation

$$\|V_n\| \leq \frac{1}{n!}(20\alpha)^{n_1} \|T_{r(n_1)} d_{z_2}^{4n_1+n_2} d_{z_3}^{4n_1+n_3} V_0\|,$$

where $r(n_1) \leq 4n_1$.

Considering the forms of $T_{r(n_1)}$ allowed in the Lemma 4 we get the desired result.

Corollary 2 *The formal solution of (2)-(6) converges uniformly on Λ if V_0 satisfies the condition of the Theorem 1.*

We have at least two possibilities to strengthen the convergence theorem. The first one is to change the condition of the theorem 1 for $\|d_z^n V_0\| < \|V_0\|(\alpha|n|)^{\beta|n|}$ where α and β are some fixed constants. This also yields to the uniform convergence of the formal solution but generally speaking it will not take place on the whole Λ.

Another possibility appears if we consider operator $P = (L^1_{i,j})_{4\times 4}$, $L^1_{2,1} = L_{2,1}$, $L^1_{2,4} = L_{2,4}$, $L^1_{i,j} = 0$ for all other combinations of i and j. Let us put $Q = L - P$. The operator P is nilpotent $(PP = 0)$ and this fact permits to reduce considerably the number of additive terms in the expression $L^{n_1} = (P+Q)^{n_1}$.

4 Finite solutions of homogeneous system

Let us rewrite (18) as

$$V_n = \frac{1}{n!} d^{n_2}_{z_2} d^{n_3}_{z_3} (P+Q)^{n_1} V_0, \tag{21}$$

where P and Q are defined at the end of the previous section.

Theorem 2. *Let $V_0 = z_2^{N_2} z_3^{N_3} \Phi(z_1)$ for some holomorphic $\Phi(z_1)$. Then $V_n \equiv 0$ for $|n|$ sufficiently big.*

Proof Let us expand the expression

$$V_{n_1,0,0} = \frac{1}{n_1!}(P+Q)^{n_1} V_0 = \frac{1}{n_1!} \sum_{i=0}^{n_1} M_i V_0.$$

Here $M_i = \sum_k N_{i,k}$ and $N_{i,k}$ is a composition of i operators P and $n_1 - i$ operators Q. The order of operators in the compositions is different for every k in the sum.

Every non-zero entrance in Q has differentiation with respect to z_2 or (and) z_3. Therefore $N_{i,k} V_0 \equiv 0$ if $i < n_1 - (N_2 + N_3)$.

Let us consider $N_{i,k}$ for $i \geq n_1 - (N_2 + N_3)$ and $n_1 \geq 2(N_2 + N_3) + 2$. These operators are composed of $N_2 + N_3$ or less operators Q and $N_2 + N_3 + 2$ or more operators P. Such a composition is zero because it necessarily has two operators P staying one after another. We proved that $V_{n_1,0,0} \equiv 0$ for $n_1 \geq 2(N_2 + N_3) + 2$.

For every n_1 (fixed), $V_{n_1,0,0}$ is a polynomial with respect to z_2 and z_3 and the corresponding degrees are less than or equal to N_2 and N_3. Therefore $V_{n_1,n_2,n_3} \equiv 0$ whenever $n_1 \geq 2(N_2+N_3)+2$, $n_2 > N_2$ or $n_3 > N_3$.

We call finite the formal solution for (2)-(6) defined by the initial function $V_0 = z_2^{N_2} z_3^{N_3} \Phi(z_1)$ because it has only a finite number of non zero terms. Corresponding solution for the system (1) can be expressed in terms of elementary functions.

Here we give some examples.

Example 1. The polynomial solution corresponding to the initial functions $V_0^1 = V_0^2 = V_0^4 = 0, \quad V_0^3 = z_1 z_2^2 z_3^2$ is

$$u_1 = x_3 r^2 \left(a_{1,1} t^2 + a_{1,2} t (4x_1^2 + r^2) + a_{1,3} r^2 (6x_1^2 + r^2) + a_{1,4} x_1^2 + a_{1,5} x_2^2\right),$$

$$u_2 = x_1 x_2 x_3 r^2 \left(a_{2,1} t + a_{2,2} r^2 + a_{2,3}\right),$$

$$u_3 = x_1 \left(t^2 \left(a_{3,1} r^2 + 8x_3^2\right) + a_{3,2} t r^4 + r^2 \left(a_{3,3} r^4 + a_{3,4} r^2 + a_{3,5} x_3^2\right)\right),$$

$$T = x_1 x_3 r^2 \left(a_{4,1} t + a_{4,2} r^2 + a_{4,3}\right),$$

where $r = |z_1| = (x_1^2 + x_2^2)^{1/2}$. The constants are

$$a_{1,1} = \frac{-8(\lambda + \mu)}{\lambda + 3\mu},$$
$$a_{1,2} = \frac{\mu \gamma^2 \theta_0}{3k(\lambda + 2\mu)(\lambda + 3\mu)},$$
$$a_{1,3} = \frac{\mu \gamma^2 \theta_0 (2\mu C_\epsilon + \gamma^2 \theta_0 + \lambda C_\epsilon)}{144 k^2 (\lambda + 2\mu)^2 (\lambda + 3\mu)},$$

$$a_{1,4} = \frac{5\mu^2 \gamma^2 \theta_0 \tau_t - 14\rho\mu\lambda k - 11\rho\mu^2 k - 3\rho k \lambda^2}{3\mu k (\lambda + 2\mu)(\lambda + 3\mu)},$$
$$a_{1,5} = \frac{\mu^2 \gamma^2 \theta_0 \tau_t - 10\rho\mu\lambda k - 7\rho\mu^2 k - 3\rho k \lambda^2}{3\mu k (\lambda + 2\mu)(\lambda + 3\mu)},$$

$$a_{2,1} = \frac{4\mu\gamma^2\theta_0}{3k(\lambda+2\mu)(\lambda+3\mu)},$$
$$a_{2,2} = \frac{\mu\gamma^2\theta_0(2\mu C_\epsilon + \gamma^2\theta_0 + \lambda C_\epsilon)}{24k^2(\lambda+2\mu)^2(\lambda+3\mu)},$$
$$a_{2,3} = -\tfrac{4}{3}\frac{\rho k(\lambda+\mu) + \mu^2\gamma^2\theta_0\tau_t}{k(\lambda+2\mu)(\lambda+3\mu)},$$

$$a_{3,1} = \frac{-2(3\lambda+5\mu)}{\lambda+3\mu},$$
$$a_{3,2} = \frac{\mu\gamma^2\theta_0}{3k(\lambda+2\mu)(\lambda+3\mu)},$$
$$a_{3,3} = \frac{\mu\gamma^2\theta_0(2\mu C_\epsilon + \gamma^2\theta_0 + \lambda C_\epsilon)}{144k^2(\lambda+2\mu)^2(\lambda+3\mu)},$$
$$a_{3,4} = \frac{-9\rho\mu^2 k - 9\rho\mu k\lambda + \mu^2\gamma^2\theta_0\tau_t - 2\rho k\lambda^2}{3\mu k(\lambda+2\mu)(\lambda+3\mu)},$$
$$a_{3,5} = \frac{2\rho}{\mu},$$

$$a_{4,1} = \frac{8\mu\gamma\theta_0}{k(\lambda+3\mu)},$$
$$a_{4,2} = \frac{\mu\gamma\theta_0(2\mu C_\epsilon + \gamma^2\theta_0 + \lambda C_\epsilon)}{3k^2(\lambda+2\mu)(\lambda+3\mu)},$$
$$a_{4,3} = \frac{8\mu\gamma\theta_0\tau_t}{k(\lambda+3\mu)},$$

Example 2. To the initial functions $V_0^1 = V_0^3 = V_0^4 = 0$, $V_0^2 = z_2^2 z_3^2 (z_1 - 1)^{-4}$ corresponds the solution

$$u_1 = x_3\left(A(2y_{1,1} + y_{1,2} + y_{1,3}) + Bty_{1,4} + 8t^3 y_{1,5}\right),$$
$$u_2 = x_3\left(-A(2y_{2,1} + y_{2,2} + y_{2,3}) + Bty_{2,4} + 8t^3 y_{2,5}\right),$$
$$u_3 = A(y_{3,1} + y_{3,2}) + Cty_{3,3},$$
$$T = Dx_3\, y_{4,1},$$

were A,B,C and D are constants given by

$$A = \frac{3\rho\gamma^2\theta_0}{3k(\lambda+2\mu)(\lambda+3\mu)}, \quad B = \frac{12\rho}{(\lambda+3\mu)},$$

$$C = \frac{-6\rho(\lambda+\mu)}{\mu(\lambda+3\mu)}, \quad D = \frac{6\rho\gamma\theta_0}{k(\lambda+3\mu)};$$

$r_1 = |z_1 - 1| = ((x_1-1)^2 + x_2^2)^{1/2}$, and $y_{i,j}$ are functions of x_1 and x_2 defined as :

$$y_{1,1} = x_1 \ln(r_1) + x_2 \arctan\frac{x_2}{x_1-1}, \qquad y_{1,2} = \frac{r^4(x_1-1)}{r_1^2},$$

$$y_{1,3} = r^2(x_1+2) - 2\pi x_2, \qquad y_{1,4} = \frac{r^2(x_1 r_1^2 - 2(x_1-1)^2)}{r_1^4},$$

$$y_{1,5} = \frac{(x_1-1)(r_1^2 - 4x_2^2)}{r_1^6},$$

$$y_{2,1} = x_2 \ln(r_1) - x_1 \arctan\frac{x_2}{x_1-1}, \qquad y_{2,2} = -\frac{r^4 x_2}{r_1^2},$$

$$y_{2,3} = -x_2 r^2 + 2\pi x_1, \qquad y_{2,4} = \frac{r^2 x_2(r_1^2 - 2(x_1-1))}{r_1^4},$$

$$y_{2,5} = \frac{x_2(3r_1^2 - 4x_2^2)}{r_1^6},$$

$$y_{3,1} = (x_1^2 - x_2^2)\ln(r_1) + 2x_1x_2 \arctan\frac{x_2}{x_1-1},$$

$$y_{3,2} = \frac{1}{2}(r^4 + 2r^2x_1 - 4\pi x_1 x_2),$$

$$y_{3,3} = \frac{r^2(r^2-x_1)}{r_1^2}, \qquad y_{4,1} = \frac{r^2(r^2-x_1)}{r_1^2}.$$

Example 3. We follow with the solution defined by the initial functions $V_0^1 = \tan(z_1)z_2z_3^2$, $V_0^2 = V_0^3 = V_0^4 = 0$. For $1 \le j \le 3$ $u_j = \Re(v_j)$ and $T = \Re(v_4)$. Here $v_j = y_{j,0} + t\, y_{j,1} + t^2\, y_{j,2}$. The functions $y_{j,m}$ are

$$y_{1,0} = x_3\left(k_{1,1}\bar{z}_1^2 \ln(\cos(z_1)) + \bar{z}_1\left(\tan(z_1)\left(k_{1,2}\,\bar{z}_1^2 + k_{1,3}\right) + z_1\left(k_{1,4}\bar{z}_1^2 + k_{1,5}\right)\right.\right.$$
$$\left.\left. + k_{1,6}\bar{z}_1 z_1^2 + k_{1,7}\bar{z}_1 \tan^2(z_1)\right)\right),$$

$$y_{2,0} = x_3\left(k_{2,1}\bar{z}_1^2 \ln(\cos(z_1)) + \bar{z}_1\left(\tan(z_1)\left(k_{2,2}\bar{z}_1^2 + k_{2,3}\right) + z_1\left(k_{2,4}\bar{z}_1^2 + k_{2,5}\right)\right.\right.$$

$$+k_{2,6}\bar{z}_1 z_1^2 + k_{2,7}\bar{z}_1 \tan^2(z_1)\right),$$

$$y_{3,0} = \bar{z}_1^2 \left(k_{3,1}\tan(z_1) + k_{3,2}\bar{z}_1 \ln(\cos(z_1)) + z_1\left(k_{3,3}\bar{z}_1 z_1 + k_{3,4}\right)\right),$$

$$y_{4,0} = x_3\bar{z}_1\left(k_{4,1}\tan^2(z_1) + \bar{z}_1 k_{4,2}\left(\tan(z_1) - z_1\right)\right),$$

$$y_{1,1} = k_{1,8}x_3\bar{z}_1\left(2z_1 - 2\tan(z_1) - \bar{z}_1\tan^2(z_1)\right),$$

$$y_{2,1} = k_{2,8}x_3\bar{z}_1\left(2z_1 - 2\tan(z_1) + \bar{z}_1\tan^2(z_1)\right),$$

$$y_{3,1} = k_{3,5}\bar{z}_1^2\left(z_1 - \tan(z_1)\right),$$

$$y_{4,1} = k_{4,3}x_3\bar{z}_1\tan^2(z_1),$$

$$y_{1,2} = x_3\tan(z_1)\left(k_{1,9}\bar{z}_1\left(1+\tan^2(z_1)\right) + 4\tan(z_1)\right),$$

$$y_{2,2} = x_3\tan(z_1)\left(k_{2,9}\bar{z}_1\left(1+\tan^2(z_1)\right) - 4\iota\tan(z_1)\right),$$

$$y_{3,2} = k_{3,6}\bar{z}_1\tan^2(z_1), \quad y_{4,2} = 0.$$

The constants $k_{j,m}$ are proportional to

$$A = \frac{\mu\gamma^2\theta_0(2\mu C_\epsilon + \gamma^2\theta_0 + \lambda C_\epsilon)}{24k^2(\lambda+2\mu)^2(\lambda+3\mu)},$$

$$B = 2\frac{\mu^2\gamma^2\theta_0\tau_t + 4\rho\mu\lambda k + 5\rho\mu^2 k + \rho k\lambda^2}{\mu k(\lambda+2\mu)(\lambda+3\mu)},$$

$$C = \frac{-\mu^2\gamma^2\theta_0\tau_t + 4\rho\mu\lambda k + 3\rho\mu^2 k + \rho k\lambda^2}{\mu k(\lambda+2\mu)(\lambda+3\mu)},$$

$$D = -\frac{\mu\gamma^2\theta_0}{k(\lambda+2\mu)(\lambda+3\mu)}, \quad E = -4\frac{\lambda+\mu}{\lambda+3\mu},$$

$$F = 8\frac{\mu\gamma\theta_0}{k(\lambda+3\mu)}, \qquad G = 8\frac{\mu\gamma\theta_0\tau_t}{k(\lambda+3\mu)},$$

$$H = \frac{\mu\gamma\theta_0(2\mu C_\epsilon + \gamma^2\theta_0 + \lambda C_\epsilon)}{k^2(\lambda+2\mu)(\lambda+3\mu)},$$

so that

$k_{1,1} = 6A,\quad k_{1,2} = -2A,\quad k_{1,3} = B,\quad k_{1,4} = 2A,\quad k_{1,5} = -B,$

$k_{1,6} = 3A,\quad k_{1,7} = C,\quad k_{1,8} = D,\quad k_{1,9} = 2E,$

$k_{2,1} = -6\iota A,\quad k_{2,2} = -2\iota A,\quad k_{2,3} = -\iota B,\quad k_{2,4} = 2\iota A,\quad k_{2,5} = \iota B,$

$k_{1,6} = 3A,\quad k_{2,7} = \iota C,\quad k_{2,8} = -\iota D,\quad k_{2,9} = 2\iota E,$

$k_{3,1} = C,\quad k_{3,2} = 2A,\quad k_{3,3} = A,\quad k_{3,4} = -C,\quad k_{3,5} = D,$

$k_{3,6} = E,\quad k_{4,1} = G,\quad k_{4,2} = H,\quad k_{4,3} = F,$

Example 4. We finally give the solution defined by the initial functions

$$V_0^1 = V_0^2 = V_0^3 = 0,\ V_0^4 = z_2^2 z_3 (z_1 - 1)^{-1/2}.$$

For $1 \leq j \leq 3 \quad u_j = \Re(v_j)$ and $T = \Re(v_4)$.
Here

$$v_j = k_{j,1}\, \bar{z}_1 \left(k_{j,2}\, y_{j,1}\, t + k_{j,3}\, y_{j,2} + (k_{j,4}\, y_{j,3}\, t + k_{j,5}\, y_{j,4})(z_1 - 1)^{-1/2}\right).$$

The functions $y_{j,m}$ are

$$y_{1,1} = 3(z_1 + \bar{z}_1) - 2,\qquad y_{2,1} = 2 - 3(z_1 + \bar{z}_1),\qquad y_{3,1} = 1,\qquad y_{4,1} = 1,$$

$$y_{1,2} = 20\bar{z}_1^2 - 30\bar{z}_1^2 z_1 - 80x_3^2 + 60\bar{z}_1 z_1 + 60\bar{z}_1 x_3^2 - 24\bar{z}_1 - 45\bar{z}_1 z_1^2 + 120x_3^2 z_1,$$

$$y_{2,2} = 20\bar{z}_1^2 - 30\bar{z}_1^2 z_1 + 80x_3^2 - 60\bar{z}_1 z_1 + 60\bar{z}_1 x_3^2 + 24\bar{z}_1 + 45\bar{z}_1 z_1^2 - 120x_3^2 z_1,$$

$$y_{3,2} = \bar{z}_1(3z_1 - 2),\qquad y_{4,2} = 6x_3^2 - 3\bar{z}_1 z_1 + 2\bar{z}_1,$$

$$y_{1,3} = 4z_1 - 2 + x_3^2 - 2z_1^2 - 3\bar{z}_1 z_1 + 3\bar{z}_1,$$

$$y_{2,3} = -4z_1 + 2 + x_3^2 + 2z_1^2 - 3\bar{z}_1 z_1 + 3\bar{z}_1,$$

$$y_{3,3} = z_1 - 1,\qquad y_{4,3} = x_3^2/\bar{z}_1 - z_1 + 1,$$

$$y_{1,4} = (z_1 - 1)(-6\bar{z}_1 z_1^2 + 12\bar{z}_1 z_1 - 5\bar{z}_1^2 z_1 + 20x_3^2 z_1 - 6\bar{z}_1 + 15\bar{z}_1 x_3^2 + 5\bar{z}_1^2 - 20x_3^2,$$

$$y_{2,4} = (z_1 - 1)(6\bar{z}_1 z_1^2 - 12\bar{z}_1 z_1 - 5\bar{z}_1^2 z_1 - 20x_3^2 z_1 + 6\bar{z}_1 + 15\bar{z}_1 x_3^2 + 5\bar{z}_1^2 + 20x_3^2,$$

$$y_{3,4} = \bar{z}_1(z_1 - 1)^2, \qquad y_{4,4} = (z_1 - 1)(3x_3^2 - \bar{z}_1 z_1 + \bar{z}_1).$$

The constants $k_{j,m}$ are proportional to

$$A = -\frac{\gamma C_\epsilon}{120k(\lambda + 2\mu)(\lambda + 3\mu)}, \quad B = -\frac{160k(\lambda + 2\mu)}{C_\epsilon},$$
$$C = -\frac{3\mu C_\epsilon + \gamma^2\theta_0 + \lambda C_\epsilon}{C_\epsilon}, \qquad D = \frac{2C_\epsilon}{3k(\lambda + 3\mu)}$$

$$k_{1,1} = A, \quad k_{1,2} = \iota B, \quad k_{1,3} = -\iota C, \quad k_{1,4} = B, \quad k_{1,5} = 4C,$$
$$k_{2,1} = A, \quad k_{2,2} = -B, \quad k_{2,3} = C, \quad k_{2,4} = \iota B, \quad k_{2,5} = 4\iota C,$$
$$k_{3,1} = 40A, \quad k_{3,2} = -\tfrac{3}{10}\iota B, \quad k_{3,3} = -\iota C, \quad k_{3,4} = \tfrac{3}{10}B, \quad k_{3,5} = -2C,$$
$$k_{4,1} = D, \quad k_{4,2} = \tfrac{8\iota}{D}, \quad k_{4,3} = -\iota C, \quad k_{4,4} = \tfrac{8}{D}, \quad k_{4,5} = -2C.$$

In the examples given above we consider u_1, u_2, u_3, and T to be real parts of $\frac{1}{2}(V^1 + V^2)$, $\frac{1}{2\iota}(V^1 - V^2)$, $V^3/2$, and V^4 respectively. The imaginary parts of these functions solve the system (1) also. Our results could be compared with closed form solutions for Lord-Shulman partial differential equations obtained in [4].

HE solutions can help in numerical treatment of initial-boundary value problems for the system (1). For instance they form a "benchmark" that would serve as the standard for judging the accuracy of approximate techniques. They also are useful in modeling of singularities.

Let us suppose that for some reason a singularity of some known type is expected in the solution of a initial-boundary value problem for (1). It is convenient to represent the solution as a sum of two functions. One has desired singularity, solve (1) but does not satisfy any fixed boundary condition. We can construct it by HE method because HE solutions inherit singularities of initial functions (see examples 2,3,4). The other function is a smooth solution of boundary value problem and it should be constructed numerically. The smoothness is an advantage. Another advantage is that the right hand side of (1) stays unchanged and is equal to zero.

Finite solutions could be used as a basis for Treftz type numerical methods [5]. Provided with the set of solutions for (1) we form a linear combination of these functions and determine the coefficients approximating initial-boundary conditions. This combination is considered as an approximate solution for the problem. It solves (1) exactly and this is an advantage for the error estimation.

5 Finite solutions of non homogeneous system

Consider non homogeneous system (1) and the corresponding non homogeneous system (2)-(6). The coefficients in the equations are all real so that there is no need to have $F^j \equiv f_j$. The real parts of V^j satisfy (1) if $\Re(F^j) = f_j$. It means that for an analytic right hand side of (1) we can construct complex analog and set F^j to be sums of holomorphic expansions just as we state it in (11) for V^j.

$$F^j = \sum_{|n|=0}^{\infty} \bar{z}_1^{n_1} \bar{z}_2^{n_2} \bar{z}_3^{n_3} F^j_{n_1,n_2,n_3}(z_1, z_2, z_3). \tag{22}$$

We skip the procedure for solving non homogeneous (2)-(6) because it is quite similar to the schedule described in the section 2. The expressions (12)-(16) transform into

$$(n_1+1)V^1_{n_1+1} = L_1(V_n) - \frac{J(a_2+2a_3)}{8a_3(a_2+a_3)}(F^1_{n_1} + \iota F^2_{n_1})$$

$$-\frac{J^3\left(2a_2a_5(a_2+a_3)d^2_{z_2} + 2(a_3a_4a_8 - a_1a_2a_5)d^2_{z_3} + a_3a_4a_7d_{z_3}\right)}{8a_3a_5(a_2+a_3)(a_2+2a_3)}(F^1_{n_1} - \iota F^2_{n_1})$$

$$+\frac{J^2a2(n_1+1)}{8a_3(a_2+a_3)}(F^1_{n_1+1} - \iota F^2_{n_1+1}) + \frac{J^2a_2d_{z_2}}{4a_3(a_2+a_3)}F^3_{n_1} - \frac{J^2a_4}{8a_5(a_2+a_3)}F^4_{n_1}, \tag{23}$$

$$(n_1+1)V^2_{n_1+1} = L_2(V_n) - \frac{J}{2(a_2+2a_3)}(F^1_{n_1} - \iota F^2_{n_1}), \tag{24}$$

$$(n_1+1)V^3_{n_1+1} = L_3(V_n) + \frac{J^2 a_2 d_{z_2}}{2a_3(a_2+2a_3)}(F^1_{n_1} - \iota F^2_{n_1}) - \frac{J}{2a_3}F^3_{n_3}, \tag{25}$$

$$(n_1+1)V^4_{n_1+1} = L_4(V_n) - \frac{J^2 d_{z_3}(a_7+2a_8)d_{z_3}}{4a_5(a_2+2a_3)}(F^1_{n_1} - \iota F^2_{n_1}) - \frac{j}{4a_5}F^4_{n_1}, \tag{26}$$

$$(n_j+1)V^k_{n_j+1} = d_{z_j}V^k_n,\ j=1,2,\ k=1,2,3,4. \tag{27}$$

The sum of the holomorphic expansion (17) solve (2)-(6) and generate solution for non homogeneous system (1). The solution is finite if the sums in (22) are finite and all F^j are polynomials with respect to z_2 and z_3. The proof is similar to the theorem 2.

Example 5. To solve (1) with $f_1 \equiv f_2 \equiv f_3 \equiv 0$ and $f_4 = 8(x_1 - 1)x_3 t^2 \left((x_1-1)^2 - 3x_2^2\right)\left((x_1-1)^2 - x_2^2\right)^{-3}$ we consider $F^1 \equiv F^2 \equiv F^3 \equiv 0$, $F^4 = (z_3+\bar z_3)^2(z_2+\bar z_2)(z_1-1)^{-3}$ and $V_0 \equiv 0$ in (23) - (27). The corresponding solution for (1) is $u_j = \Re(v_j), j=1,2,3$ and $T = \Re(v_4)$. The functions v_j are:

$$v_1 = \bar z_1 x_3 \left[6\bar z_1 ln(z_1-1)\left(A\bar z_1(\bar z_1 + 4z_1 - 4) + Bt + C\right)\right.$$
$$+Dz_1 t^2(2\bar z_1 - \bar z_1 z_1 + 2z_1 - 2z_1^2)(z_1-1)^{-2}$$
$$+B\bar z_1 t\left(z_1(3z_1^2 + 2z_1\bar z_1 + 3z_1 - 6)(z_1-1)^{-1} - 6\pi\iota\right)$$
$$+3A\bar z_1^3\left(z_1(z_1+2) - 2\pi\iota\right) + 4A\bar z_1^2\left(z_1(z_1^3 + 2z_1^2 + 6)(z_1-1)^{-1} - 6\pi\iota(z_1-1)\right)$$
$$\left.+(2C - 36A)\bar z_1^2 z_1^2(z_1-1)^{-1} + 3C\bar z_1\left(z_1(z_1+2) - 2\pi\iota\right)\right],$$

$$v_2 = \iota\bar z_1 x_3 \left[6\bar z_1 ln(z_1-1)\left(A\bar z_1(\bar z_1 - 4z_1 + 4) - Bt - C\right)\right.$$
$$+Dz_1 t^2(2\bar z_1 - \bar z_1 z_1 - 2z_1 + 2z_1^2)(z_1-1)^{-2}$$
$$-B\bar z_1 t\left(z_1(3z_1^2 - 2z_1\bar z_1 + 3z_1 - 6)(z_1-1)^{-1} - 6\pi\iota\right)$$
$$+3A\bar z_1^3\left(z_1(z_1+2) - 2\pi\iota\right) - 4A\bar z_1^2\left(z_1(z_1^3 + 2z_1^2 + 6)(z_1-1)^{-1} - 6\pi\iota(z_1-1)\right)$$

$$+(2C+36A)\bar{z}_1^2 z_1^2 (z_1-1)^{-1} - 3C\bar{z}_1\left(z_1(z_1+2)-2\pi\iota\right)\big],$$

$$\begin{aligned} v_3 = \bar{z}_1^3\,[2ln(z_1-1)\,(Bt+3A(z_1-1)\bar{z}_1+C) \\ -Dz_1^2 t^2 (z_1-1)^{-1}\bar{z}_1^{-1} + Bt\,(z_1(z_1+2)-2\pi\iota) \\ +A\bar{z}_1\left(z_1(z_1^2+3z_1-6)+6\pi\iota(z_1-1)-2\pi\iota\right) + C\left(z_1(z_1+2)\right)\big], \end{aligned}$$

$$v_4 = \bar{z}_1^2 x_3\left[E\bar{z}_1 ln(z_1-1) + Fz_1 t^2 (z_1-2)(z_1-1)^{-2}\bar{z}_1^{-1} + Gz_1^2 t(z_1-1)^{-1}\right.$$

$$\left. +H\bar{z}_1\left(z_1(z_1+2)-2\pi\iota\right) + Iz_1^2(z_1-1)^{-1}\right].$$

The constants are:

$$A = -\frac{\gamma C_\epsilon\left(C_\epsilon(\lambda+2\mu)+\gamma^2\theta_0\right)^2}{4608(\lambda+2\mu)^3 k^3}, \quad B = -\frac{\gamma C_\epsilon\left(C_\epsilon(\lambda+2\mu)+\gamma^2\theta_0\right)}{96(\lambda+2\mu)^2 k^2},$$

$$C = -\frac{\gamma C_\epsilon\left(C_\epsilon \tau_t(\lambda+2\mu)+\rho k+\gamma^2\theta_0\tau_t\right)}{96(\lambda+2\mu)^2 k^2}, \quad D = \frac{\gamma C_\epsilon}{8(\lambda+2\mu)k},$$

$$E = 96A(\lambda+2\mu)\gamma^{-1}, \qquad F = 8D(\lambda+2\mu)\gamma^{-1},$$

$$G = 24B\lambda+2\mu)\gamma^{-1}, \qquad H = E/2,$$

$$I = G\tau_t.$$

The possibility to construct exact particular solution is attractive because one can reduce a boundary value problem with singular right hand side to an homogeneous system.

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Eksplicitna rešenja parcijalnih diferencijalnih jednačina Lord-Shulman termoelastičnosti

Razmatra se Lord-Shulman model termoelastičnosti sa jednom relaksacionom konstantom. Odgovarajući sistem od četiri linearne parcijalne diferencijalne jednačine je rešen pomoću holomorfnih razvoja. Konvergencija razvoja je dokazana i proučena je moguńost njihovog prelaska u konačne sume.

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