Theoret. Appl. Mech., Vol.36, No.3, pp. 167–205, Belgrade 2009

Eulerian elastoplasticity: basic issues and recent results

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Abstract

Traditional formulations of elastoplasticity in the presence of finite strain and large rotation are Eulerian type and widely used; they are based upon, among other things, the additive decomposition of the stretching or the Eulerian strain-rate into elastic and plastic parts. In such formulations, yield functions and objective rate constitutive equations are expressed in terms of objective Eulerian tensor quantities, including the stretching, the Kirchhoff stress, internal state variables, etc. Each of these quantities transforms in a corotational manner under a change of the observing frame. According to the principle of material frame-indifference or objectivity, each constitutive function should be invariant, whenever the observing frame is changed to another one by any given time-dependent rotation. In this work the general form of constitutive equations is discussed. Several frequently used objective rates are analyzed with respect to their serviceability to develop a self-consistent formulation, i.e. to be integrable to deliver an elastic in particular hyperelastic relation for vanishing plastic deformation. This would be of great importance, e.g., for so-called spring back calculations in metal forming.

Keywords: Elastoplasticity; finite deformation; Eulerian rate formulation; hypoelasticity; objective stress rates; logarithmic rate; consistency criteria; anisotropy; deformation cycles.

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1 Introduction

It seems that the first effort in modern description of elastoplasticity for finite deformation goes back to the late fifties and early sixties of the last century. This effort has led to an Eulerian rate type theory based on the true stress and the natural deformation rate, representing a simple, direct extension of the Prandtl-Reuss theory. A closely related development in this respect emerged slightly earlier when Truesdell [1] proposed the theory of hypoelasticity, relating an objective rate of Cauchy or Kirchhoff stress and the rate of deformation via a 4th-order stress-dependent tensor.

In classical theory for small deformation, the composite structure of elastoplasticity is established by introducing strain-like variables and their rates labelled as “elastic” or “plastic”. It is assumed that the infinitesimal strain $\boldsymbol{\varepsilon}$ is additively separated into a reversible elastic and an irreversible plastic part. By virtue of the incremental essence of elastoplastic behaviour, the following rate form is introduced:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} . \tag{1}$$

For finite elastoplastic deformation, the natural deformation rate for the flow-like characteristic would be the stretching $\mathbf{D}$. Then, a direct extension of the separation (1) is as follows:

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}} , \tag{2}$$

where $\mathbf{D}^{\mathrm{e}}$ and $\mathbf{D}^{\mathrm{p}}$ are called elastic and plastic stretching, respectively.

For the general case of finite deformation, differences of opinion have been voiced concerning the appropriate decomposition and the formulation of constitutive relations (cf. [2], [3]). As one result of this discussion three different approaches to describe a physically reasonable decomposition of finite elastoplasticity are known[1]:

First, the classical setting of Prandtl and Reuss is also adopted for a finite deformation description by additively splitting the stretching $\mathbf{D}$, refer to Eq. (2). The elastic part therein is then usually described by a hypoelastic relation containing an objective corotational or non-corotational rate, preferably, the Jaumann rate or the Green/Naghdi rate of Cauchy or Kirchhoff stress.

[1]The three different approaches and their respective assets and drawbacks - and several additional thoughts - are discussed the recent paper [3].

However, these hypoelastic relations were shown not to be self-consistent in the sense that they cannot be integrated to give an elastic relation ([4]). It means that even a purely elastic process will produce dissipation. Moreover, it turned out that in simulations of simple shear problems spurious oscillations were observed for some of the rates. Thus, as a consequence of these shortcomings until recently composition (2) was believed not to be adequate to describe finite elastoplasticity. Instead it was judged to be a reasonable concept only for the description of metal plasticity with finite total but infinitesimal small elastic deformations.

Recently, [5] could prove that the stretching may be integrated to give an Eulerian strain, namely the Hencky strain $\mathbf{h}$. Moreover, these authors could show that a linear hypoelastic relation is indeed integrable to deliver an elastic relation. The objective rate that has to be used in both cases is the logarithmic rate. Thus, it is believed that it would be the time to reanimate the simple idea of Prandtl and Reuss, and use composition (2) even for real finite deformation.

The second decomposition is related with a multiplicative split of the deformation gradient $\mathbf{F}$ into an elastic part $\mathbf{F}^{\mathrm{e}}$ and a plastic one $\mathbf{F}^{\mathrm{p}}$ according to

$$\mathbf{F} = \mathbf{F}^{\mathrm{e}}\mathbf{F}^{\mathrm{p}}\,. \tag{3}$$

This setting is now frequently used in almost all descriptions of finite inelastic deformations. In [6] it has been shown that the above attempt with the logarithmic rate and the Hencky strain is able to consistently combine both settings.

The third setting is based on a Lagrangean description, and an a priori introduction of a plastic strain $\mathbf{E}^{\mathrm{p}}$ as a primitive variable ([7]). It turns out then that the total Lagrangean strain $\mathbf{E}$ appears to be

$$\mathbf{E} = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{p}}\,, \tag{4}$$

where, however, $\mathbf{E}^{\mathrm{e}}$ only in the case of infinitesimal small deformations is related with the notion of elastic deformations. In general, the so-called elastic part has been introduced by defining the difference $\mathbf{E} - \mathbf{E}^{\mathrm{p}}$, refer to [2].

Here, according to the original concept of Prandtl and Reuss, for each process of elastic-plastic deformation the additive decomposition of the total stretching $\mathbf{D}$ is assumed. In a natural and direct manner, the elastic part $\mathbf{D}^{\mathrm{e}}$ may be interpreted as the deformation rate related to the elastic deformation

$\mathbf{F}^{\mathrm{e}}$. The plastic part $\mathbf{D}^{\mathrm{p}}$ may be associated with both the elastic deformation $\mathbf{F}^{\mathrm{e}}$ and the plastic deformation $\mathbf{F}^{\mathrm{p}}$. When there is no change in the microstructure responsible for plastic deformation, the deformation is purely elastic and, thus, the elastic part $\mathbf{D}^{\mathrm{e}}$ corresponds to the total stretching $\mathbf{D}$.

The concept of objectivity is important in the description of large deformations. Numerous objective rates have been presented in the past, and, thus, the question arises, if all of them are equally suitable. In particular, if relevant material line rotations occur, unreasonable phenomena, like stress oscillation in simple shear, dissipation in elastic strain cycles etc., may occur. Such unreliable results may have considerably negative effects in the description of elastoplastic deformations, since the elastic deformation part, though sometimes small compared with plastic deformations, may substantially influence the result of the total deformation.

Dienes [8] revealed the occurrence of stress oscillations in large elastic simple shear. A similar observation was made by Lehmann [9] seven years earlier for simple shear of a rigid plastic body. It is widely accepted that this unreasonable phenomenon results from the selected stress rate (both used a Jaumann rate) in the deformation description. Thus, in the last decades, a large variety of objective stress rates has been presented to avoid these oscillations, all of them generally leading to different results. The question arises if the concept of objectivity is sufficient to account for cited problems and if some of these rates will show other unreliabilities in other deformation processes.

Xiao et al. [10] presented a hypoelasticity model based on the logarithmic stress rate. It has been shown that this model is exactly integrable and, moreover, is derivable from an elastic potential, thus combining hyperelasticity and hypoelasticity. Also, this property is exclusively bound to the logarithmic rate. It was shown (Bruhns et al., [11]), that this model showed proper results in the case of simple shear.

The question remains about results in the case of more complex strain paths. Lin [12] introduced a plain strain square cycle. He could show that some non-corotational stress rates, namely Truesdell and Oldroyd rates, lead to erroneous results: stresses arrive at much too high values and residual stresses remain at the end of the cycle. In the case of three corotational stress rates, namely Jaumann rate ([13, 14]), Green/Naghdi rate ([7, 15],) and the logarithmic rate ([16, 17, 5]), the stress development is very similar; however, only in case of the logarithmic rate all stresses return back to their initial zero state at the end of the cycle.

For a simple shear deformation mode progressing monotonically, use of Jaumann rate was known to result in aberrant oscillatory shear stress response. Nevertheless, use of Naghdi rate was found to produce reasonable monotonic shear stress response. On the other hand, for a single cycle of deformation recovering the original shape only once - although the most recent study has reported residual stresses resulting from Naghdi rate and other rates - their magnitudes may be regarded acceptable for small deformations. Here a pertinent question may be: Whenever a strain cycle is constantly repeated, how will the residual stresses or errors change with the cycle number? Will their magnitudes remain within an acceptable range or, to the contrary, steadily accumulate?

Since strain cycles and cyclic loading may be frequently met in engineering problems, it seems important to investigate the foregoing questions. We shall study the stress responses of the most widely used hypoelastic model under constantly repeated strain cycles. Here the main ideas are as follows: (i) We consider smooth strain cycles; (ii) we do not restrict ourselves to a single cycle but treat constantly repeated cycles; (iii) we study how residual stresses or errors change with the cycle number.

Another issue in Eulerian formulations of finite elastoplasticity is concerned with the different objective rates that may appear in the different constitutive equations. Could these rates be chosen independently or should they rather be fixed in a uniform manner? That is, provided the hypoelastic part of the composite elastoplastic model contains a corotational Jaumann stress rate, could the rate equation of a tensor-valued internal variable, e.g. for the back-stress, contain another corotational or in particular non-corotational rate?

A similar question arises with the consistency condition $\dot{f} = 0$ for the yield function f of this elastoplastic model. If a material time derivative is applied to the scalar-valued function f, and thus to the different tensor products contained in this function, this derivative via chain rule also has to be applied to the tensors themselves. This can directly be done for tensors of Lagrangean type. The question, however, remains what should be done with Eulerian quantities? Which objective time derivative should be taken and what would be the criterion to answer this question?

Moreover, the principle of material symmetry requires that each constitutive function should fulfill invariance restrictions under the action of the initial material symmetry group. Generally, for a solid material, the latter is a proper subgroup of the full orthogonal group, except for isotropic materials. Sometimes it was thought (see, e.g., [18]) that, within the framework of Eulerian formulations, the objectivity principle implies that each constitutive function is isotropic, i.e., invariant under the full orthogonal group, and therefore that Eulerian formulations of constitutive functions may be applicable only to isotropic materials and therefore not to materials with any given type of initial anisotropy, such as transverse isotropy, orthotropy and crystallographic symmetries, etc.

The above issues and other related issues will be elaborated upon and investigated in the subsequent sections. At the end of this introduction, we explain some notations that will be used:

$$(\mathbf{XY})_{ij} = X_{ik}Y_{kj}\,, \quad \mathbf{X} : \mathbf{Y} = \mathrm{tr}(\mathbf{XY}^{\mathrm{T}}) = X_{ij}Y_{ij}\,, \quad \mathrm{tr}\,\mathbf{X} = X_{ii}\,,$$

$$(\mathbb{C} : \mathbf{X})_{ij} = \mathbb{C}_{ijkl}X_{kl}\,, \quad \mathbf{X} : \mathbb{C} : \mathbf{Y} = X_{ij}\mathbb{C}_{ijkl}Y_{kl}\,,$$

for any two second-order tensors $\mathbf{X}$ and $\mathbf{Y}$, and any fourth-order tensor $\mathbb{C}$. Moreover, we will make use of the notation

$$\mathbf{Q}\star\mathbf{X} = \mathbf{QXQ}^{\mathrm{T}}\,, \quad \mathbf{Q}\star\mathbb{C} = \mathbf{QQ}\mathbb{C}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\,,$$

wherein $\mathbf{Q}$ is an orthogonal tensor.

2 Basic relations

2.1 Kinematics

Let $\mathbf{X}$ be the position of a material particle in the reference configuration $\mathscr{B}_0$ and $\mathbf{x}$ the position in the current configuration $\mathscr{B}$. The deformation gradient $\mathbf{F}$ describes the motion of the body

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}\,, \quad \det \mathbf{F} > 0\,. \tag{5}$$

The particle velocity $\mathbf{v}$ and the velocity gradient $\mathbf{L}$ are defined by

$$\mathbf{v} = \dot{\mathbf{x}}\,, \quad \mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}\,, \tag{6}$$

where a superposed dot denotes the material time derivative. The deformation gradient **F** can be uniquely decomposed into its left and right multiplicative decompositions

$$\mathbf{F} = \mathbf{V}\mathbf{R} = \mathbf{R}\mathbf{U}\,, \quad \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}\,, \tag{7}$$

where the positive definite symmetric tensors **V** and **U** are called left and right stretch tensors, and **R** is the proper orthogonal rotation tensor. The objective left Cauchy-Green tensor **B** is computed from **V** and may be represented through its m distinct eigenvalues b_s and its eigenprojections $\mathbf{B}_s$ as

$$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \sum_s^m b_s \mathbf{B}_s\,, \quad \sum_s^m \mathbf{B}_s = \mathbf{I}\,. \tag{8}$$

Herein **I** is the second order identity tensor. The following relations hold

$$\mathbf{B}_s\mathbf{B}_k = \begin{cases} \mathbf{0}, & s \neq k\,, \\ \mathbf{B}_s\,, & s = k\,. \end{cases} \tag{9}$$

For given eigenvalues b_s the eigenprojections $\mathbf{B}_s$ can be determined from Sylvester's formula

$$\mathbf{B}_s = \delta_{1m}\mathbf{I} + \prod_{k\neq s}^m \frac{\mathbf{B} - b_k\mathbf{I}}{b_s - b_k}\,, \tag{10}$$

where δ_{sk} is the Kronecker symbol. We note that whereas **V** and likewise **B** are of Eulerian type, in a similar way we also may define a Lagrangean type right Cauchy-Green tensor **C**,

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \sum_s^m b_s \mathbf{C}_s\,, \tag{11}$$

with the eigenprojections $\mathbf{C}_s$ and the same eigenvalues b_s.

This allows us to introduce a general definition of strain measures ([19]) in terms of the left and right Cauchy-Green tensors **B** and **C**:

$$\mathbf{e} = \mathbf{g}(\mathbf{B}) = \sum_{s=1}^m g(b_s)\mathbf{B}_s\,, \quad \mathbf{E} = \mathbf{g}(\mathbf{C}) = \sum_{s=1}^m g(b_s)\mathbf{C}_s\,, \tag{12}$$

where $g(b)\,, \forall\, b > 0\,,$ is a scale function with initial conditions $g(1) = 0\,, g'(1) = \frac{1}{2}$. These definitions as a subclass also include the well-known Doyle-Ericksen or Seth-Hill strains with

$$\mathbf{e}^{(n)} = \frac{1}{2n}(\mathbf{B}^n - \mathbf{I})\,, \quad \mathbf{E}^{(n)} = \frac{1}{2n}(\mathbf{C}^n - \mathbf{I})\,, \quad g(b) = \frac{1}{2n}(b^n - 1)\,. \tag{13}$$

The Eulerian Almansi strain tensor $\mathbf{e}$ defined over $\mathscr{B}$ and the Lagrangean Green strain tensor $\mathbf{E}$ defined over $\mathscr{B}_0$ may be derived from (13) for $n = -1$ and $n = 1$, respectively,

$$\begin{aligned} \mathbf{e} &= \mathbf{e}^{(-1)} = \sum_{s=1}^{m} \frac{1}{2}(1 - b_s^{-1})\mathbf{B}_s = \tfrac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}), \\ \mathbf{E} &= \mathbf{E}^{(+1)} = \sum_{s=1}^{m} \frac{1}{2}(b_s - 1)\mathbf{C}_s = \tfrac{1}{2}(\mathbf{C} - \mathbf{I}). \end{aligned} \tag{14}$$

Moreover, the numbers $n = \pm 1$ also furnish

$$\begin{aligned} \mathbf{a} &= \mathbf{e}^{(+1)} = \sum_{s=1}^{m} \frac{1}{2}(b_s - 1)\mathbf{B}_s = \tfrac{1}{2}(\mathbf{B} - \mathbf{I}), \\ \mathbf{A} &= \mathbf{E}^{(-1)} = \sum_{s=1}^{m} \frac{1}{2}(1 - b_s^{-1})\mathbf{C}_s = \tfrac{1}{2}(\mathbf{I} - \mathbf{C}^{-1}), \end{aligned} \tag{15}$$

where the so-called Eulerian Finger tensor[2] $\mathbf{a}$ is a forward-rotated Green tensor

$$\mathbf{a} = \mathbf{R} \star \mathbf{E} = \mathbf{R}\mathbf{E}\mathbf{R}^{\mathrm{T}} \tag{16}$$

and, likewise, the Lagrangean Piola tensor $\mathbf{A}$ is a backward-rotated Almansi tensor

$$\mathbf{A} = \mathbf{R}^{\mathrm{T}} \star \mathbf{e} = \mathbf{R}^{\mathrm{T}}\mathbf{e}\mathbf{R}. \tag{17}$$

In particular, the limiting process $n \to 0$, or the logarithmic scale function $g(b) = \frac{1}{2}\ln b$ results in Hencky's logarithmic strain measure ([20], [5])

$$\begin{aligned} \mathbf{h} &= \mathbf{e}^{(0)} = \sum_{s=1}^{m} \frac{1}{2}\ln b_s \mathbf{B}_s = \tfrac{1}{2}\ln \mathbf{B}, \\ \mathbf{H} &= \mathbf{E}^{(0)} = \sum_{s=1}^{m} \frac{1}{2}\ln b_s \mathbf{C}_s = \tfrac{1}{2}\ln \mathbf{C}, \end{aligned} \tag{18}$$

with the rotational transformation property

$$\mathbf{h} = \mathbf{R} \star \mathbf{H} \tag{19}$$

[2] We here follow the notation of Haupt [21].

and vice versa. The logarithmic strains **h** and **H**, respectively, will be of particular importance for the description of finite elastoplasticity. They possess some intrinsic advantages in contrast to other measures of strain, i.e. the property of additivity, readily observed in one-dimensional loading. However, due to the transcendental form of these measures, their use was often limited to particular cases. In the succeeding sections, our focus will be on the Eulerian strain measures, and in particular on the Eulerian logarithmic strain **h**, and appropriate rates.

Apart from the above mentioned rotational correspondences (16) and (17), the Lagrangean Green tensor and the Eulerian Almansi tensor are moreover related through the following transformation:

$$\mathbf{E} = \mathbf{F}^{\mathrm{T}}\mathbf{e}\mathbf{F} \tag{20}$$

and vice versa

$$\mathbf{e} = \mathbf{F}^{\text{-T}}\mathbf{E}\mathbf{F}^{\text{-1}}\,. \tag{21}$$

The velocity gradient is decomposed into the symmetric and objective deformation rate **D** and the skew-symmetric vorticity **W**

$$\mathbf{L} = \mathbf{D} + \mathbf{W}\,, \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}})\,, \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}})\,. \tag{22}$$

We note that the material time derivative of the Green strain tensor has the following property

$$\dot{\mathbf{E}} = \frac{1}{2}\mathbf{F}^{\mathrm{T}}(\mathbf{L} + \mathbf{L}^{\mathrm{T}})\mathbf{F} = \mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F}\,. \tag{23}$$

For later purpose it will become essential to know, whether there exists a similar relation between any of the given Eulerian strains $\mathbf{e}^{(n)}$, say **e**, and the stretching **D**. As both quantities are of Eulerian type, the most simple relation would read

$$\overset{\circ}{\mathbf{e}}{}^{*} = \mathbf{D}\,, \tag{24}$$

where the asterisk underlines that the type of objective rates herein is yet undetermined.

2.2 Corotational rates

Under a change of observer or change of frame a physical quantity should not alter. The rotating frame $(\bullet)^*$ may be defined by its spin $\boldsymbol{\Omega}^*$, which determines the skew-symmetric second-order Eulerian tensor $\mathbf{Q}(t)$ to within a constant proper orthogonal tensor through the linear tensorial differential equation

$$\boldsymbol{\Omega}^* = \dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q} = -\mathbf{Q}^{\mathrm{T}}\dot{\mathbf{Q}}\,. \qquad (25)$$

In the transformed $\boldsymbol{\Omega}^*$-frame, the objective symmetric second-order Eulerian tensor $\mathbf{A}$ has the representation

$$\mathbf{A}^* = \mathbf{Q}\star\mathbf{A} = \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}}\,. \qquad (26)$$

The material time derivative of (26) then gives

$$\dot{\mathbf{A}}^* = \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{A}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{A}\dot{\mathbf{Q}}^{\mathrm{T}} = \mathbf{Q}\star\overset{\circ}{\mathbf{A}}{}^*\,, \qquad (27)$$

where the corotational rate of the Eulerian tensor $\mathbf{A}$ defined by the Eulerian spin $\boldsymbol{\Omega}^*$ is

$$\overset{\circ}{\mathbf{A}}{}^* = \dot{\mathbf{A}} + \mathbf{A}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\mathbf{A}\,. \qquad (28)$$

From (27), it is evident that the corotational rate of an objective Eulerian tensor defined by an Eulerian spin $\boldsymbol{\Omega}^*$ is a material time derivative in an $\boldsymbol{\Omega}^*$-frame

$$\mathbf{Q}\star\overset{\circ}{\mathbf{A}}{}^* = \dot{\overline{\mathbf{Q}\star\mathbf{A}}}\,. \qquad (29)$$

Corotational rates of objective tensors must be objective measures to ensure that any superimposed rigid rotation has no effect on them. This condition is essential to the notion of work-conjugacy which will be presented later. However, not every rate $\overset{\circ}{\mathbf{A}}{}^*$ of an objective Eulerian tensor $\mathbf{A}$ is objective. This generally depends on its defining spin $\boldsymbol{\Omega}^*$, which should be associated with the deformation and motion of the deforming body in an appropriate manner ([22]).

The corotational rate of a symmetric tensor $\mathbf{A}$ is defined by Eq. (28), where $\boldsymbol{\Omega}^*$ is the skew-symmetric spin tensor. Xiao et al. [23] showed that the most general form of objective corotational rates is related to the spin

$$\boldsymbol{\Omega}^* = \mathbf{W} + \mathbf{N} = \mathbf{W} + \sum_{s\neq k}^{m} h\left(\frac{b_s}{I_1}, \frac{b_k}{I_1}\right)\mathbf{B}_s\mathbf{D}\mathbf{B}_k\,. \qquad (30)$$

Herein, I_1 is the first basic invariant of **B** and the spin function h enjoys the property $h(x,y) = -h(y,x)$. Here and henceforth, the notation $\sum_{s\neq k}^{m}(\bullet)$ represents the summation for all $s,k = 1,\ldots,m$ with $s \neq k$. The sum vanishes for $m = 1$.

Following the above mentioned objectivity requirements, an explicit basis-free expression of $\mathbf{\Omega}^*$ in terms of **D**, **W** and **B** is obtained with

$$\mathbf{N} = \begin{cases} \mathbf{0}\,, & m = 1\,, \\ h_{12}/(b_1 - b_2)[\mathbf{BD}]\,, & m = 2\,, \\ \nu_1[\mathbf{BD}] + \nu_2[\mathbf{B}^2\mathbf{D}] + \nu_3[\mathbf{B}^2\mathbf{DB}]\,, & m = 3\,, \end{cases} \tag{31}$$

where the coefficients are isotropic invariants of **B**, given by

$$h_{ij} = h\left(\frac{b_i}{I_1}, \frac{b_j}{I_1}\right)\,, \tag{32}$$

$$\begin{aligned} \nu_k &= -\frac{(-1)^k}{\Delta}\left(b_1^{3-k}h_{23} + b_2^{3-k}h_{31} + b_3^{3-k}h_{12}\right)\,, \quad k = 1,2,3\,, \\ \Delta &= (b_1 - b_2)(b_2 - b_3)(b_3 - b_1)\,, \end{aligned} \tag{33}$$

and the following notation is used:

$$[\mathbf{A}] = 2\mathrm{skw}(\mathbf{A})\,. \tag{34}$$

The three eigenvalues b_1, b_2 and b_3 of **B** (possibly repeated) can be determined by the basic or the principal invariants of **B**.

From above, we see that the explicit basis-free expressions (31)-(34) enable us to determine the spin $\mathbf{\Omega}^*$ directly using the deformation gradient **F** given under any coordinate system. Particularly, the spins related to three well known objective rates considered here are:

$\overset{\circ}{(\bullet)}{}^{\mathrm{J}}$ for the Jaumann rate

$$\mathbf{\Omega}^{\mathrm{J}} = \mathbf{W}\,; \tag{35}$$

$\overset{\circ}{(\bullet)}{}^{\mathrm{R}}$ for the polar or Green/Naghdi rate

$$\mathbf{\Omega}^{\mathrm{R}} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \mathbf{W} + \sum_{s\neq k}^{m} \frac{\sqrt{b_k} - \sqrt{b_s}}{\sqrt{b_k} + \sqrt{b_s}}\mathbf{B}_s\mathbf{D}\mathbf{B}_k\,; \tag{36}$$

$\overset{\circ}{(\bullet)}{}^{\log}$ for the logarithmic rate

$$\mathbf{\Omega}^{\log} = \dot{\mathbf{R}}^{\log}(\mathbf{R}^{\log})^{\mathrm{T}} = \mathbf{W} + \sum_{s \neq k}^{m} \left(\frac{b_k + b_s}{b_k - b_s} - \frac{2}{\ln b_k - \ln b_s} \right) \mathbf{B}_s \mathbf{D} \mathbf{B}_k \,. \tag{37}$$

For long time it was believed that the stretching **D** cannot be written as a corotational rate of any strain measure. Recently, it was proved that for the logarithmic rate of the Eulerian Hencky strain **h** the relation

$$\overset{\circ}{\mathbf{h}}{}^{\log} = \dot{\mathbf{h}} + \mathbf{h}\mathbf{\Omega}^{\log} - \mathbf{\Omega}^{\log}\mathbf{h} = \mathbf{D} \tag{38}$$

holds (cf. [5, 23], see also [16], [17]): The objective corotational rate of Eulerian logarithmic strain **h** defined by the logarithmic spin $\mathbf{\Omega}^{\log}$ is identical to the Eulerian stretching tensor **D**. This result answers the question related with assertion (24).

The proper orthogonal tensor $\mathbf{R}^{\log}$ defined by the linear tensorial differential equation

$$\dot{\mathbf{R}}^{\log} = -\mathbf{R}^{\log}\mathbf{\Omega}^{\log}\,, \qquad \mathbf{R}^{\log}\big|_{t=0} = \mathbf{I}\,, \tag{39}$$

is called logarithmic rotation. Using $\mathbf{R}^{\log}$, the rotated correspondence

$$\overline{\mathbf{R}^{\log} \star \mathbf{A}}^{\boldsymbol{\cdot}} = \mathbf{R}^{\log} \star \overset{\circ}{\mathbf{A}}{}^{\log} \tag{40}$$

holds, which will be of particular interest when applied to **h** using the kinematical relation (38)

$$\overline{\mathbf{R}^{\log} \star \mathbf{h}}^{\boldsymbol{\cdot}} = \mathbf{R}^{\log} \star \mathbf{D}\,. \tag{41}$$

The left-hand side of Eq. (40) represents the material time rate of a Lagrangean tensor. This measure can be integrated with respect to time and rotated forward into the current configuration to give

$$\mathbf{A} = (\mathbf{R}^{\log})^{\mathrm{T}} \star \int_0^t \mathbf{R}^{\log} \star \overset{\circ}{\mathbf{A}}{}^{\log}\, \mathrm{d}s\,. \tag{42}$$

This technique is called corotational integration. Corotational integration of the stretching **D** gives the Eulerian Hencky strain **h**.

2.3 Non-corotational rates

In the preceding subsection, we have discussed corotational rates. Now we broaden the scope to take also non-corotational rates into consideration. Replacing the spin $\mathbf{\Omega}^*$ in the defining formula (28) by a general asymmetric 2nd order tensor $\mathbf{\Psi}^*$, we come to the following definition of a non-corotational rate of Oldroyd's type:

$$\overset{\triangledown}{\mathbf{A}}{}^* \equiv \dot{\mathbf{A}} + \mathbf{A}\mathbf{\Psi}^* + \mathbf{\Psi}^{*\mathrm{T}}\mathbf{A} \,. \tag{43}$$

The well-known Oldroyd rates are two examples of the above definition by setting $\mathbf{\Psi}^* = \mathbf{L}$ and $\mathbf{\Psi}^* = -\mathbf{L}^{\mathrm{T}}$, respectively. Accordingly, we have the lower Oldroyd rate

$$\overset{\triangledown}{\mathbf{A}}{}^{\mathrm{Ol}} \equiv \dot{\mathbf{A}} + \mathbf{A}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A} \tag{44}$$

and the upper Oldroyd rate[3]

$$\overset{\triangledown}{\mathbf{A}}{}^{\mathrm{Ou}} \equiv \dot{\mathbf{A}} - \mathbf{A}\mathbf{L}^{\mathrm{T}} - \mathbf{L}\mathbf{A} \,. \tag{45}$$

More generally, setting $\mathbf{\Psi}^* = \mathbf{W} - n\mathbf{D}$ with $n \in (-\infty, +\infty)$, we have Hill's class of objective rates ([24]):

$$\overset{\triangledown}{\mathbf{A}}{}^{\mathrm{H}} \equiv \dot{\mathbf{A}} + \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A} - n(\mathbf{A}\mathbf{D} + \mathbf{D}\mathbf{A}) \,. \tag{46}$$

This class includes the Oldroyd rates as particular cases for $n = \pm 1$. A remarkable property of the Oldroyd rates is as follows:

$$\overset{\triangledown}{\mathbf{e}}{}^{\mathrm{Ol}} = \dot{\mathbf{e}} + \mathbf{e}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{e} = \mathbf{D} \,, \tag{47}$$

$$\overset{\triangledown}{\mathbf{a}}{}^{\mathrm{Ou}} = \dot{\mathbf{a}} - \mathbf{a}\mathbf{L}^{\mathrm{T}} - \mathbf{L}\mathbf{a} = \mathbf{D} \,, \tag{48}$$

where $\mathbf{e}$ and $\mathbf{a}$ are Almansi and Finger tensor, respectively, refer to Eqs. $(14)_1$ and $(15)_1$.

Here we would like to explore whether there exist more definitions of non-corotational rates of Oldroyd's type which can also establish relationships like

[3] These rates are sometimes also named: Eq. (44) as Cotter/Rivlin rate and Eq. (45) as Oldroyd rate, respectively.

Eqs. (47) or (48). Towards this end, we would like to find out what strain $\mathbf{e}$ and what asymmetric tensor $\mathbf{\Psi}^*$ together make the relationship

$$\overset{\triangledown}{\mathbf{e}}{}^* \equiv \dot{\mathbf{e}} + \mathbf{e}\mathbf{\Psi}^* + \mathbf{\Psi}^{*\mathrm{T}}\mathbf{e} = \mathbf{D} \tag{49}$$

identically hold. Let $\mathbf{\Omega}^*$ and $\mathbf{X}^*$ be the skew-symmetric and symmetric parts of $\mathbf{\Psi}^*$. Then, we may recast Eq. (49) as

$$\mathbf{e}\mathbf{X}^* + \mathbf{X}^*\mathbf{e} = \mathbf{D} - \overset{\circ}{\mathbf{e}}{}^* . \tag{50}$$

The skew-symmetric part $\mathbf{\Omega}^*$ is a general spin, given by Eq. (30). Hence, $\overset{\circ}{\mathbf{e}}{}^*$ in Eq. (50) is the corotational rate of $\mathbf{e}$ defined by the spin $\mathbf{\Omega}^*$. Since every eigenvalue of $\mathbf{e}$, i.e., $g(b_\sigma)$, is non-vanishing and monotonically increasing with b_σ, we infer that Eq. (50) has a unique solution $\mathbf{X}^*$ for any given strain $\mathbf{e}$ and for any given spin $\mathbf{\Omega}^*$. By means of the eigenprojection method and setting

$$g(b) = \frac{1}{2n}(b^n - 1) , \quad h(x,y) = n\frac{x^n + y^n}{x^n - y^n} - \frac{x+y}{x-y} , \tag{51}$$

the following solution was attained

$$\begin{cases} \mathbf{e} = \mathbf{e}^{(n)} = \dfrac{1}{2n}(\mathbf{B}^n - \mathbf{I}) , \\ \mathbf{\Omega}^* = \mathbf{\Omega}^{(n)} = \mathbf{W} + \displaystyle\sum_{s\neq k=1}^{m} \left(n\frac{b_s^n + b_k^n}{b_s^n - b_k^n} - \frac{b_s + b_k}{b_s - b_k} \right) \mathbf{B}_s\mathbf{D}\mathbf{B}_k , \end{cases} \tag{52}$$

where

$$\mathbf{X}^* = -n\mathbf{D} . \tag{53}$$

Hence, the rate (43) is of the form

$$\overset{\triangledown}{\mathbf{A}}{}^{(n)} \equiv \dot{\mathbf{A}} + \mathbf{A}\mathbf{\Omega}^{(n)} - \mathbf{\Omega}^{(n)}\mathbf{A} - n(\mathbf{A}\mathbf{D} + \mathbf{D}\mathbf{A}) , \tag{54}$$

which is just a non-corotational rate of Hill's type (cf. Eq. (46)) with the replacement of the vorticity tensor $\mathbf{W}$ by the spin $\mathbf{\Omega}^*$ given by Eq. $(52)_2$. In summary, with Eq. (52) the relationship (49) becomes

$$\overset{\triangledown}{\mathbf{e}}{}^{(n)} = \dot{\overline{\mathbf{e}^{(n)}}} + \mathbf{e}^{(n)}\mathbf{\Omega}^{(n)} - \mathbf{\Omega}^{(n)}\mathbf{e}^{(n)} - n(\mathbf{e}^{(n)}\mathbf{D} + \mathbf{D}\mathbf{e}^{(n)}) = \mathbf{D} . \tag{55}$$

This holds for every number n. It means that the stretching $\mathbf{D}$ is expressible as a Hill's type non-corotational rate of any given Seth-Hill strain $\mathbf{e}^{(n)}$. Both equations (47) and (48) are two particular cases of this general fact where $n = \pm 1$.

A perhaps interesting observation can be made by noting that the rate defined by Eqs. (54) and (52) is corotational, if and only if $n = 0$. Then, we have

$$\lim_{n\to 0} \mathbf{e}^{(n)} = \mathbf{e}^{(0)} = \mathbf{h} = \frac{1}{2}\ln \mathbf{B}\,, \quad \lim_{n\to 0} \frac{n}{b_s^n - b_k^n} = \frac{1}{\ln b_s - \ln b_k}\,,$$

and finally

$$\lim_{n\to 0} \boldsymbol{\Omega}^{(n)} = \boldsymbol{\Omega}^{\log}\,. \tag{56}$$

2.4 Stresses and stress power

Here and in what follows, we will use the Cauchy stress $\boldsymbol{\sigma}$ as a natural Eulerian stress measure. Frequently, the weighted Cauchy stress $J\boldsymbol{\sigma}$ is referred to as Kirchhoff stress $\boldsymbol{\tau}$

$$\boldsymbol{\tau} = J\boldsymbol{\sigma}\,, \quad J = \det(\mathbf{F})\,. \tag{57}$$

As Lagrangean counterpart, the symmetric second Piola-Kirchhoff stress $\mathbf{S}$ may be derived from $\boldsymbol{\tau}$ via

$$\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}}\,. \tag{58}$$

From the different stresses acting on different configurations, we may introduce the stress power per unit volume of $\mathscr{B}_o$

$$\dot{w} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{E}}) = \mathrm{tr}(\boldsymbol{\tau}\mathbf{D})\,, \tag{59}$$

where here a simple transformation has led to Hill's work conjugacy relation ([25, 19]), stating that the second Piola-Kirchhoff stress $\mathbf{S}$ and the Green strain $\mathbf{E}$ are conjugate stress and strain tensors. An extension of (59) to Eulerian measures of stress and strain has been proposed by Xiao et al. [23]. They consider a pair of symmetric Eulerian tensors ($\mathbf{s}$, $\mathbf{e}$), where $\mathbf{s}$ is an objective stress measure and $\mathbf{e}$ an objective measure of strain. In a frame defined by the spin $\mathbf{\Omega}^*$ relative to a fixed background frame, this pair turns out to be work conjugate if the relation

$$\dot{w} = \mathrm{tr}(\mathbf{s}\,\overset{\circ}{\mathbf{e}}{}^{*}) \tag{60}$$

is satisfied. One possible solution of this requirement is that the Kirchhoff stress $\boldsymbol{\tau}$ and the Hencky strain $\mathbf{h}$ form a work conjugate pair under a logarithmic rotation $\mathbf{R}^{\log}$,

$$\dot{w} = \mathrm{tr}(\boldsymbol{\tau}\,\overset{\circ}{\mathbf{h}}{}^{\log}) = \mathrm{tr}(\boldsymbol{\tau}\mathbf{D})\,. \tag{61}$$

This surprising result has motivated the question which of the possible pairs of work conjugate stresses and strains should be preferably taken in an Eulerian description of finite elastoplasticity? And, moreover, as these formulations are closely related with the introduction of yield functions and consistency conditions, i.e. tensor functions and their time derivatives in a rotating system, the second interesting question arises, whether there is any preference to one of the above mentioned objective rates.

3 Constitutive relations

The separation (1) and its direct extension (2) are the starting-points for classical elastoplasticity theories at small and finite deformations. In the above decompositions, the different variables are intended for the formulation of elastic and plastic behaviour, respectively. For this purpose, the main idea in accord with the incremental essence of elastoplasticity is to formulate the relationship between instantaneous elastic and plastic deformation rates and the instantaneous stress rate. These can be symbolically written in

$$\text{deformation rates} = F(\text{stress, internal variables; stress rate}),$$

where here internal variables are introduced to characterize the hardening of the material induced by micro-structural rearrangement, and where each function F should be a homogeneous function of degree one in the stress rate due to the rate-independence property. On the other hand, physically the instantaneous rate of each internal variable is related to the instantaneous plastic deformation rate $\mathbf{D}^{\mathrm{p}}$. Then we have

$$\text{rate of internal variable} = H(\text{stress, internal variables; }\mathbf{D}^{\mathrm{p}}),$$

where each function H is homogeneous of degree one in the plastic deformation rate $\mathbf{D}^{\mathrm{p}}$. Here, particular attention should be paid to the objectivity requirement for instantaneous rates.

With the elastic strain $\boldsymbol{\varepsilon}^{\mathrm{e}}$, the elastic behaviour at small deformation may be described by Hooke's law. Its rate form may be given by

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \frac{\dot{\boldsymbol{\sigma}}}{2\mu} - \nu \frac{\operatorname{tr} \dot{\boldsymbol{\sigma}}}{E} \mathbf{I}, \tag{62}$$

where Poisson's ratio ν and shear modulus μ are related to give Young's modulus $E = 2\mu(1+\nu)$. A direct extension of Eq. (62) may be obtained by simply replacing the elastic strain rate $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ with the elastic stretching $\mathbf{D}^{\mathrm{e}}$. However, the resultant equation is not objective, since the rate $\dot{\boldsymbol{\sigma}}$ is known to be non-objective. From this fact arises the need to define an objective stress rate. Let, e.g., $\boldsymbol{\sigma}$ be replaced by the Kirchhoff stress $\boldsymbol{\tau}$, then a direct objective extension of Eq. (62) is:

$$\mathbf{D}^{\mathrm{e}} = \frac{\overset{\circ}{\boldsymbol{\tau}}}{2\mu} - \nu \frac{\operatorname{tr} \overset{\circ}{\boldsymbol{\tau}}}{E} \mathbf{I}. \tag{63}$$

This pertains to the simplest hypoelastic equation (of grade zero). Thus, a general Eulerian form of the objective elastic rate equation may be given by

$$\mathbf{D}^{\mathrm{e}} = \mathbb{H}(\boldsymbol{\tau}) : \overset{\circ}{\boldsymbol{\tau}}, \tag{64}$$

where the stress-dependent tensor of moduli $\mathbb{H}(\boldsymbol{\tau})$ characterises the elastic behaviour in a general sense.

For the sake of simplicity, usually one scalar variable κ and an objective Eulerian stress-like variable $\boldsymbol{\alpha}$ known as back stress are used to describe isotropic and kinematic hardening. Then the yield function is formulated by a scalar function $f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$. Hence, the yield surface is given by

$$f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) = 0. \tag{65}$$

A widely used yield function for metals is of von Mises type

$$f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) = \frac{1}{2}(\boldsymbol{\tau}' - \boldsymbol{\alpha}) : (\boldsymbol{\tau}' - \boldsymbol{\alpha}) - \tau_0^2, \quad \boldsymbol{\tau}' = \boldsymbol{\tau} - \frac{1}{3}(\operatorname{tr} \boldsymbol{\tau})\mathbf{I}, \tag{66}$$

where the yield shear stress τ_0 may rely on κ, and the back stress is deviatoric.

When the yield surface is moving and possibly expanding and the stress stays on this surface (i.e., during loading) continuing plastic flow is induced. With the plastic stretching $\mathbf{D}^{\mathrm{p}}$, a general form of the flow rule may be formulated as

$$\mathbf{D}^{\mathrm{p}} = \xi \; \mathbb{A}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) : \overset{\circ}{\boldsymbol{\tau}} \; . \tag{67}$$

In the above, $\mathbb{A}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ is a 4th-order tensor-valued constitutive function.

The scalar internal variable κ may be governed by an evolution equation of the form

$$\dot{\kappa} = \mathbf{K}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) : \mathbf{D}^{\mathrm{p}} \, , \tag{68}$$

where $\mathbf{K}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ is a symmetric 2nd-order tensor-valued constitutive function and the so-called plastic indicator ξ is either 1 (loading) or 0 (unloading or elastic process). If κ is assumed as plastic work, then the right-hand side of Eq. (68) is simply given by $\boldsymbol{\tau} : \mathbf{D}^{\mathrm{p}}$. As mentioned before, the back stress $\boldsymbol{\alpha}$ is intended for the characterization of deformation-induced anisotropy of plastic behaviour as evidenced by the Bauschinger effect. To formulate its evolution equation, an objective rate of the back stress $\boldsymbol{\alpha}$, say $\overset{\diamond}{\boldsymbol{\alpha}}$, is also necessary. Then, a general form of objective evolution equation for $\boldsymbol{\alpha}$ may be of the form:

$$\overset{\diamond}{\boldsymbol{\alpha}} = \mathbb{K}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) : \; \mathbf{D}^{\mathrm{p}} \, , \tag{69}$$

where $\mathbb{K}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ is again a 4th-order tensor-valued constitutive function. A widely-used particular rule for anisotropic hardening is as follows ([26]):

$$\overset{\diamond}{\boldsymbol{\alpha}} = c \, \mathbf{D}^{\mathrm{p}} \, , \tag{70}$$

where the scalar parameter c is known as kinematic hardening modulus. In general, the definitions of the rates $\overset{\diamond}{\boldsymbol{\alpha}}$ and $\overset{\circ}{\boldsymbol{\tau}}$ need not be the same.

Even in the case of small deformations, the flow rule (67) with constitutive function $\mathbb{A}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ might not be tractable due to the complexity of irreversible elastoplastic phenomena. Drastic simplifications have to be introduced. The notion of plastic potential initiated by von Mises represents a significant step towards reasonable simplification, which leads to the following form of the flow rule:

$$\mathbf{D}^{\mathrm{p}} = \xi \; \frac{1}{h} \left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}} \right) \frac{\partial p}{\partial \boldsymbol{\tau}} \, , \tag{71}$$

where the hardening modulus h and the plastic potential p are scalar functions of variables $(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$. The former may be determined by the consistency condition for plastic flow, i.e., $\dot{f} = 0$. A further step is to assume that the plastic potential is nothing else but the yield function f. Thus follows the associated flow rule (normality rule):

$$\mathbf{D}^{\mathrm{p}} = \xi \, \frac{1}{h} \left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}} \right) \frac{\partial f}{\partial \boldsymbol{\tau}} . \tag{72}$$

With the above assumption, the combination of the elastic rate equation (64) and the flow rule (72) yields a simple relation between total deformation rate and the stress rate as follows:

$$\mathbf{D} = \left(\mathbb{H}(\boldsymbol{\tau}) + \xi \, \frac{1}{h} \frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}} \right) : \overset{\circ}{\boldsymbol{\tau}} \, . \tag{73}$$

At this stage, there would be no compelling reason why the definitions of the two objective rates $\overset{\circ}{\boldsymbol{\tau}}$ and $\overset{\diamond}{\boldsymbol{\alpha}}$, emerging in the above equations, should be the same. There are many candidates for either of them. The four classical rates used earlier are well-known, including the Jaumann rate, the Oldroyd rates and the Truesdell rate[4]. Thus arises the issue of how to select suitable rates for the above formulations. Within the context of perfect elastoplasticity, Prager [27] was the first to realize that a consistency requirement would be necessary for the composite structure of an Eulerian elastoplastic formulation to be free of inconsistency. Observing that the yield surface should keep unchanged in any process of unloading, he introduced the basic requirement: The yield surface should be stationary with the vanishing of the stress rate. With this yielding stationarity criterion, Prager demonstrated that, of the above four classical rates, only the corotational Jaumann rate would be admissible for the consistent formulation of Eulerian finite elastoplasticity. This elementary work established the prominent role of the Jaumann rate in classical Eulerian elastoplasticity. This theory, in particular, the J_2-flow theory with the von

[4]The Truesdell rate is a specific upper Oldroyd rate with an additional volume stretching term, originally formulated for true stress

$$\overset{\triangledown}{\boldsymbol{\sigma}}{}^{\mathrm{Tr}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\mathbf{L}^{\mathrm{T}} - \mathbf{L}\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} .$$

Mises type yield function (66), has been developed by many researchers and found wide application.

However, the foundation of the classical theory was shaken by an unexpected discovery of spurious phenomena known as shear oscillations. It seems that Lehmann [9] was the first to reveal that the rigid plastic J_2-flow theory with Prager's kinematic hardening rule would predict an oscillating shear stress response to monotonically progressing simple shearing deformation. Ten years later, this phenomenon was rediscovered by Nagtegaal and de Jong [28]. On the other hand, Dienes [8] demonstrated that a similar phenomenon would emerge also for the hypoelastic rate equation (63) with $\mathbf{D}^{\mathrm{e}} = \mathbf{D}$ which was assumed to describe purely elastic behaviour. Extensive discussions and studies since then have been made towards clarification. A number of plausible alternative rates, such as Green/Naghdi rate and Lie derivatives, have been suggested to replace the Jaumann rate. Although instructive in some cases, conclusions in this respect were drawn merely from non-oscillatory shear stress responses to simple shearing.

Another unexpected finding was made by Simo and Pister [4], who demonstrated that the widely-used hypoelastic rate equation (63) fails to be exactly integrable to really define an elastic, in particular, hyperelastic relation for each of the well-known objective rates. Since this finding, a general tendency is to believe (see, e.g., [18]) that the just-mentioned non-integrability property would likely be true for all possible rates. This would imply that the classical Eulerian elastoplasticity theory might be self-inconsistent in the sense of formulating elastic behaviour via the hypoelastic equation (63). In fact, a non-integrable hypoelastic formulation is path-dependent and dissipative, and thus would deviate essentially from the recoverable elastic-like behaviour.

On the other hand, it is noted that the yield function $f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ and the stress-dependent moduli tensor $\mathbb{H}(\boldsymbol{\tau})$ in the general hypoelastic equation (64) as well as other constitutive functions should be isotropic functions of their respective variables due to the objectivity requirement, refer to Simo and Hughes [18]. Therefore, the classical Eulerian rate theory of elastoplasticity was believed to be applicable only to initially isotropic materials.

4 Consistent Eulerian formulation with logarithmic rate

The inherent inseparability of the total elastoplastic deformation as a physical entity would render us in a dilemma in the effort to attain a consistent, physically pertinent formulation of finite elastoplasticity. On the one hand, without an additional elastic or plastic deformation-like variable like $\mathbf{E}^{\mathrm{p}}$ or $\mathbf{F}^{\mathrm{e}}$, it would hardly be possible to present a realistic formulation of elastic behaviour. On the other hand, such an additional variable associated with the unstressed (intermediate) state could in principle not be separated out of the total elastoplastic deformation.

A possible way out of this dilemma might be, not to separate the total elastoplastic deformation $\mathbf{F}$. Instead, the physically pertinent quantities Kirchhoff stress $\boldsymbol{\tau}$ and natural deformation rate $\mathbf{D}$ should be employed. This means that we should return to the basic idea of the earlier Eulerian rate formulation. Recently, a consistent, straightforward Eulerian rate formulation of finite elastoplasticity based upon two consistency criteria has been proposed and developed. Its basics will be summarized in this section.

4.1 The separation $\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}}$ revisited

A simple, almost evident fact concerning the total stress power (59) is that the latter is composed of a recoverable part and an irrecoverable part during every process of elastoplastic deformation. The former will be stored as recoverable elastic-like potential energy, whereas most of the latter is dissipated. Let $\dot{w}^{\mathrm{e}}$ and $\dot{w}^{\mathrm{p}}$, respectively, be these parts of the total stress power $\dot{w}$. Then, we have

$$\dot{w} = \mathrm{tr}(\boldsymbol{\tau}\mathbf{D}) = \dot{w}^{\mathrm{e}} + \dot{w}^{\mathrm{p}} . \tag{74}$$

From this and the bilinear form of the scalar product for the stress power, we deduce that an elastic deformation rate $\mathbf{D}^{\mathrm{e}}$ and a plastic deformation rate $\mathbf{D}^{\mathrm{p}}$ may be introduced such that

$$\dot{w}^{\mathrm{e}} = \mathrm{tr}(\boldsymbol{\tau}\mathbf{D}^{\mathrm{e}}), \quad \dot{w}^{\mathrm{p}} = \mathrm{tr}(\boldsymbol{\tau}\mathbf{D}^{\mathrm{p}}) . \tag{75}$$

Then, Eqs. (74) and (75) yield the separation (2). To render this separation consistent with the physical motivation in the foregoing, the constitutive formulations for $\mathbf{D}^e$ and $\mathbf{D}^p$ should be established such that $\mathbf{D}^e$ is indeed elastic-like (recoverable), while $\mathbf{D}^p$ is plastic-like (dissipative).

4.2 Yielding stationarity and corotational rates

A necessary consistency criterion comes from an observation made earlier by Prager [27]. To explain this, let us examine the rate equations for plastic behaviour as given by the flow rule (67) and the evolution equations (68) and (69). It may be clear that $\overset{\circ}{\boldsymbol{\tau}} = \mathbf{0}$ (and likewise $\overset{\diamond}{\boldsymbol{\alpha}} = \mathbf{0}$ and $\dot{\kappa} = 0$) implies that the plastic flow should keep stationary. Consequently, to avoid contradiction, it is required that a simultaneous vanishing of the rates $\overset{\circ}{\boldsymbol{\tau}}$, $\overset{\diamond}{\boldsymbol{\alpha}}$ and $\dot{\kappa}$ should render the yield function $f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ stationary.

As mentioned before, the earlier “elementary discussion of definitions of stress rate” in Prager [29] led to the wide acceptance of the Jaumann rate. However, the unsuitability of this rate has been rendered clear by the basic counterexamples discovered by Lehmann [9], Dienes [8], Nagtegaal and de Jong [28] and many other researchers.

The above situation might cause some confusion and doubt about the reasonableness of Prager’s criterion. Towards an understanding, it is noted that there should be no doubt about the universal applicability of Prager’s criterion, but the conclusion derived from the early discussion would be limited because only several classical rates were involved.

With a fully general definition of objective rates covering both corotational and non-corotational rates, it has been demonstrated most recently ([30, 22]) that Prager’s criterion in a general sense implies that the definitions of the two rates $\overset{\circ}{\boldsymbol{\tau}}$ and $\overset{\diamond}{\boldsymbol{\alpha}}$ should be the same and corotational. It turns out that, being restricted to the four classical rates of which only the Jaumann rate is corotational, Prager’s conclusion was true but limited in scope.

4.3 Elastic rate equation and logarithmic rate

Now we come to the significant issue how to establish a self-consistent elastic rate formulation for $\mathbf{D}^e$ which can really characterize recoverable (dissipationless) elastic behaviour. As mentioned before, the difficulty may be that the standard elastic formulation of total stress-deformation relation could not be used as a guide, since only the elastic deformation rate $\mathbf{D}^e$ and the Kirchhoff stress $\boldsymbol{\tau}$ are at our disposal.

Fortunately, there exists an alternative Eulerian rate formulation for the notion of elasticity which establishes a linear relationship between the deformation rate $\mathbf{D}$ and an objective stress rate $\overset{\circ}{\boldsymbol{\tau}}$, namely the hypoelastic theory introduced by Truesdell [1, 31]. It is known that the notion of hypoelasticity is broader than the conventional notion of elasticity. Specifically, certain types of hypoelastic rate relations would be exactly integrable to deliver conventional path-independent elastic stress-deformation relations, whereas others would be non-integrable and represent rate-independent but path-dependent deformation behaviour. There exist explicit integrability conditions by means of which the exact integrability or the non-integrability property of any given hypoelastic rate relation may be judged. These conditions are furnished by Bernstein's integrability theorem (see [32]).

For our purpose, we introduce the general elastic rate relation (64) for $\mathbf{D}^e$. The tensor of moduli $\mathbb{H}(\boldsymbol{\tau})$ herein may be interpreted as elastic tangent compliance tensor and, thus, may be given by the Hessian $\partial^2\bar{W}/\partial\boldsymbol{\tau}^2$ with the scalar function $\bar{W} = \bar{W}(\boldsymbol{\tau})$ being a complementary elastic potential. This results in the elastic rate equation

$$\mathbf{D}^e = \frac{\partial^2\bar{W}}{\partial\boldsymbol{\tau}^2} : \overset{\circ}{\boldsymbol{\tau}} \, . \tag{76}$$

In particular, the simplest form of the potential $\bar{W} = \bar{W}(\boldsymbol{\tau})$ is given by an isotropic quadratic function. Then the second gradient $\partial^2\bar{W}/\partial\boldsymbol{\tau}^2$ is just the classical isotropic elastic compliance tensor with two constants. This leads to the widely used elastic rate equation (63).

With the Eulerian rate equation (76) of hypoelastic type arises the integrability issue, which essentially relies on the definition of the stress rate $\overset{\circ}{\boldsymbol{\tau}}$. Inspired by the study of the non-integrability issue for rate equation (63) with several classical rates by Simo and Pister [4], we introduce the integrability criterion for rate equation (76). Namely, for every process of elastic deformation with $\mathbf{D}^e = \mathbf{D}$, the rate equation should be exactly integrable to deliver a dissipationless elastic relation and hence characterize recoverable elastic behaviour (refer to Bruhns et al. [33]) and Xiao and al. [34, 30]). Evidently, the integrability conditions for the rate equations (76), in particular, the widely used relation (63), rely on the definition of the objective rate and prove to be extremely complicated even for the simplest Jaumann rate. In fact, Bernstein's integrability theorem supplies a nonlinear coupled system of a number of partial differential equations involving the stress-dependent tensor $\mathbb{H}(\boldsymbol{\tau})$ together with the Jaumann rate.

Recently, it has been demonstrated in the foregoing references that there is one and only one choice for the stress rate such that rate equation (63) satisfies the integrability criterion. This rate is the logarithmic rate $\overset{\circ}{\boldsymbol{\tau}} = \overset{\circ}{\boldsymbol{\tau}}{}^{\log}$, discovered independently by several researchers ([16], [17], [5, 23]). Via exact integrability it is revealed ([5, 10, 33, 34]) that there exists a unique, intrinsic relationship between Truesdell's hypoelastic rate equation (63) and Hencky's elastic relation (77), independently introduced 50 and 75 years ago

$$\mathbf{h} = \frac{\boldsymbol{\tau}}{2\mu} - \nu \frac{\operatorname{tr}\boldsymbol{\tau}}{E}\,\mathbf{I}\,. \tag{77}$$

For the more general case, i.e., Eq. (76), it is also demonstrated in the foregoing references that the logarithmic rate results in a natural, explicit solution. For initial elastic deformation, the rate equation (76) is exactly integrable to yield the hyperelastic equation

$$\mathbf{h} = \frac{\partial \bar{W}}{\partial \boldsymbol{\tau}}\,. \tag{78}$$

In the foregoing references, the uniqueness of the logarithmic rate in the above solution is verified among a general class of objective corotational rates. Here we further explain that this uniqueness property may be extended to the fully general cases covering both corotational and non-corotational rates. Towards this goal, we show that use of a non-corotational rate $\overset{\triangledown}{\boldsymbol{\tau}}$ in Eq. (64), in particular (76), may result in inconsistency. In fact, for this non-corotational rate there generally exists a stress $\boldsymbol{\tau}$ with changing principal values, such that this rate may vanish, i.e., $\overset{\triangledown}{\boldsymbol{\tau}} = \mathbf{0}$.[5] However, from Eq. (64), in particular (63), then follows $\mathbf{D}^{\mathrm{e}} = \mathbf{0}$. The latter implies that no elastic deformation increment should be induced for a stress $\boldsymbol{\tau}$ with changing principal values, which may be inconsistent with realistic material behaviour.

[5] The same may not be true for a corotational rate $\overset{\circ}{\boldsymbol{\tau}}$, since in this case the condition $\overset{\circ}{\boldsymbol{\tau}} = \mathbf{0}$ always leaves the principal values of the stress $\boldsymbol{\tau}$ stationary.

With rate equation (76) and the logarithmic rate, the recoverable part of the stress power is given by

$$\dot{w}^e = \boldsymbol{\tau} : \mathbf{D}^e = \dot{\bar{\Sigma}}, \quad \bar{\Sigma} = \bar{\Sigma}(\boldsymbol{\tau}) \equiv \frac{\partial \bar{W}}{\partial \boldsymbol{\tau}} : \boldsymbol{\tau} - \bar{W}(\boldsymbol{\tau}) \,. \tag{79}$$

This shows that the recoverable energy increment $\dot{w}^e\,\mathrm{d}t$ is indeed derivable from the complementary elastic potential $\bar{W} = \bar{W}(\boldsymbol{\tau})$ as an exact differential increment.

Combining the above results, we arrive at a unique choice for the two rates $\overset{\circ}{\boldsymbol{\tau}}$ and $\overset{\diamond}{\boldsymbol{\alpha}}$ in the rate equations (76), (67) and (69), among all possible corotational and non-corotational rates, namely,

$$\overset{\circ}{\boldsymbol{\tau}} = \overset{\circ}{\boldsymbol{\tau}}{}^{\log}, \qquad \overset{\diamond}{\boldsymbol{\alpha}} = \overset{\circ}{\boldsymbol{\alpha}}{}^{\log} \,. \tag{80}$$

Therefore, the elastic rate equation (63) should be given by

$$\mathbf{D}^e = \frac{\overset{\circ}{\boldsymbol{\tau}}{}^{\log}}{2\mu} - \frac{\nu}{E} \operatorname{tr} \overset{\circ}{\boldsymbol{\tau}}{}^{\log}\, \mathbf{I} \,, \tag{81}$$

and the general flow rule (67) and evolution equations (68) and (69) by

$$\mathbf{D}^p = \xi\, \mathbb{A}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} \,, \tag{82}$$

$$\dot{\kappa} = \mathbf{K}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) : \mathbf{D}^p \,, \tag{83}$$

$$\overset{\circ}{\boldsymbol{\alpha}}{}^{\log} = \mathbb{K}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) : \mathbf{D}^p \,. \tag{84}$$

With the logarithmic rate as unique choice, a new general framework for Eulerian rate theory of finite elastoplasticity is accordingly established by Eq. (76) with the logarithmic rate and Eqs. (82)-(84). Now, the exactly integrable Eulerian rate equation indeed characterizes the dissipationless elastic behaviour with any given complementary elastic potential $\bar{W} = \bar{W}(\boldsymbol{\tau})$, in an explicit, self-consistent manner, and Prager's criterion is rigorously satisfied in a general sense. Thus, the resultant constitutive formulation is endowed with a self-consistent composite structure.

4.4 Essential structure implied by the work postulate

With a general, consistent composite structure of Eulerian rate formulation for finite elastoplasticity as established above, it is possible to derive the essential structure implied by the work postulate. From a weakened form of Ilyushin's postulate, it has been recently shown [35] that the normality rule

$$\mathbf{D}^{\mathrm{p}} = \xi \, \frac{1}{h} \left(\frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} \right) \frac{\partial f}{\partial \boldsymbol{\tau}} \tag{85}$$

should hold true and, furthermore, that each elastic domain enclosed by the yield surface Eq. (65) should be convex. The above normality flow rule and the exactly integrable elastic rate equation produce a total relation

$$\mathbf{D} = \mathbb{C}^{ep} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} = \left(\frac{\partial^2 \bar{W}}{\partial \boldsymbol{\tau}^2} + \xi \, \frac{1}{h} \frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}} \right) : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} , \tag{86}$$

where the instantaneous elastoplastic tangent compliance tensor $\mathbb{C}^{ep}$ is given by

$$\mathbb{C}^{ep} = \frac{\partial^2 \bar{W}}{\partial \boldsymbol{\tau}^2} + \xi \, \frac{1}{h} \frac{\partial f}{\partial \boldsymbol{\tau}} \otimes \frac{\partial f}{\partial \boldsymbol{\tau}} . \tag{87}$$

Note that this compliance tensor $\mathbb{C}^{ep}$ is simple and endowed with both minor and major symmetry properties which may be important for efficient numerical implementations.

With the normality rule (85), now the plastic deformation rate $\mathbf{D}^{\mathrm{p}}$ is related to the essential, representative feature of plastic behaviour, namely, the yield surface, in a straightforward manner. Thus, the quantity $\mathrm{tr}(\boldsymbol{\tau}\mathbf{D}^{\mathrm{p}})$ as the dissipative part of the stress power is indeed endowed exactly with the physical feature as expected in introducing the separation (2). That is also the case for the recoverable part $\mathrm{tr}(\boldsymbol{\tau}\mathbf{D}^{\mathrm{e}})$, since the latter is just given by subtracting the dissipative part $\mathrm{tr}(\boldsymbol{\tau}\mathbf{D}^{\mathrm{p}})$ from the total stress power $\mathrm{tr}(\boldsymbol{\tau}\mathbf{D})$.

Finally, to render the elastoplastic constitutive formulation complete, the hardening modulus h should be derived from the plastic consistency condition $\dot{f} = 0$, and the loading-unloading criteria should be presented to specify the value of the plastic indicator ξ. The result for the hardening modulus h is as follows

$$h = -\frac{\partial f}{\partial \boldsymbol{\alpha}} : \mathbb{K} : \frac{\partial f}{\partial \boldsymbol{\tau}} - \frac{\partial f}{\partial \kappa} \frac{\partial f}{\partial \boldsymbol{\tau}} : \mathbf{K} , \tag{88}$$

where $\mathbb{K} = \mathbb{K}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ and $\mathbf{K} = \mathbf{K}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$. On the other hand, the traditional loading-unloading criterion has been known to be limited to the hardening case. Unified criteria have been studied by Hill [36, 37]. An explicit form of such unified criteria is given by [38]:

$$\xi = \begin{cases} 1 & \text{if} \quad f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) = 0 \quad \text{and} \quad \Phi > 0\,, \\ 0 & \text{if} \quad f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) < 0 \quad \text{or} \quad f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa) = 0 \quad \text{and} \quad \Phi \le 0\,. \end{cases} \tag{89}$$

where

$$\Phi = \frac{\partial f}{\partial \boldsymbol{\tau}} : \left(\frac{\partial^2 \bar{W}}{\partial \boldsymbol{\tau}^2} \right)^{-1} : \mathbf{D}\,.$$

The above results can be simplified for the widely treated case when the instantaneous elastic compliance tensor $\partial^2 \bar{W} / \partial \boldsymbol{\tau}^2$ is given by the constant isotropic elastic compliance tensor.

5 Eulerian formulation of initial anisotropy

The last issue is related with the question how to achieve a consistent treatment for all types of initial material symmetry, i.e., how to remove the isotropy limitation as indicated at the end of section 3. It seems worthwhile to point out that, in accordance with the material symmetry principle, any prescribed type of initial material symmetry of a solid material, described by a subgroup of the full orthogonal group, should place corresponding restrictions on the form of each constitutive function, e.g., those given by Eq. (77). Within the framework of an Eulerian formulation, the relevant matter might be how to arrive at a consistent mathematical treatment of the requirements both from the objectivity principle and from the material symmetry principle. A general Eulerian rate theory obeying both requirements has been put forward with [38]. The general idea is summarized in this section.[6]

Consider an elastoplastic solid with its initial material symmetry described by an orthogonal subgroup $\mathscr{G} \subset \mathscr{O}$, called the initial material symmetry group

[6]For more detailed information including examples, we refer to [38]. Moreover, we would emphasize that here an Eulerian description is discussed with the advantage that no possibly incompatible intermedite configuration has to be introduced. As has been mentioned in the introduction, different approaches of finite elastoplasticity are in use. Their respective assets and drawbacks are extensively discussed in [3].

of this solid. Here, $\mathscr{O}$ is the full orthogonal group. The initial configuration is assumed to be in a natural stress-free state. Let **S** be an objective Lagrangean stress measure, and let κ and $\mathbf{\Pi}$ be a scalar internal variable and an objective Lagrangean tensorial internal variable, respectively, which may characterize, e.g., the isotropic and kinematic hardening behaviour. The yield function is then defined by

$$F = F(\mathbf{S}, \mathbf{\Pi}, \kappa)\,. \tag{90}$$

As a typical constitutive function, the yield function F should fulfill the restrictions imposed by the material objectivity and the material symmetry. Since an objective Lagrangean quantity remains unchanged under any change of the observing frame, a constitutive function in terms of objective Lagrangean quantities automatically satisfies the restriction of material objectivity. Moreover, the material symmetry restriction requires

$$F(\mathbf{QSQ}^{\mathrm{T}}, \mathbf{Q\Pi Q}^{\mathrm{T}}, \kappa) = F(\mathbf{S}, \mathbf{\Pi}, \kappa), \quad \forall \mathbf{Q} \in \mathscr{G}\,. \tag{91}$$

Thus, with a Lagrangean formulation, the yield function F is an invariant scalar function under the initial material symmetry group $\mathscr{G}$. For an initially anisotropic solid, its material symmetry group is characterized by certain symmetry axes. A transversely isotropic solid, e.g., is described by an ∞-fold symmetry axis, an orthotropic solid by three mutually perpendicular two-fold symmetry axes and a crystalline solid by its crystallographic symmetry axes. Owing to these facts, a consequence of restriction (91) is that these symmetry axes will enter into the reduced form of yield function F.

Towards our goal, we first introduce a class of proper orthogonal tensors **P** and require that, under a change of the observing frame, **P** has the same transformation property as the rotation tensor **R** in the polar decompositions (7),

$$\mathbf{P}^* = \mathbf{QP}\,. \tag{92}$$

Moreover, we define the **P**-conjugate group of the initial material group $\mathscr{G}$ by

$$\mathbf{P}\mathscr{G}\mathbf{P}^{\mathrm{T}} = \{\mathbf{PQP}^{\mathrm{T}} \mid \mathbf{Q} \in \mathscr{G}\}\,. \tag{93}$$

Each orthogonal tensor in the conjugate group $\mathbf{P}\mathscr{G}\mathbf{P}^{\mathrm{T}}$ is an objective Eulerian tensor. Hence, the **P**-conjugate group $\mathbf{P}\mathscr{G}\mathbf{P}^{\mathrm{T}}$ is Eulerian type.

With these definitions the following result was verified: Let **P** be a given proper orthogonal tensor with the transformation property (92) and let the Eulerian pair $(\mathbf{s}, \boldsymbol{\pi})$ and the Lagrangean pair $(\mathbf{S}, \boldsymbol{\Pi})$ be **P**-conjugate. Then, the Lagrangean type yield function $F(\mathbf{S}, \boldsymbol{\Pi}, \kappa)$ satisfying the invariance restriction (91) under the group $\mathscr{G}$ and the Eulerian type yield function $f(\mathbf{s}, \boldsymbol{\pi}, \kappa)$ satisfying the invariance restriction under the **P**-conjugate group $\mathbf{P}\mathscr{G}\mathbf{P}^{\mathrm{T}}$, i.e.,

$$f(\mathbf{Q}\mathbf{s}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\boldsymbol{\pi}\mathbf{Q}^{\mathrm{T}}, \kappa) = f(\mathbf{s}, \boldsymbol{\pi}, \kappa), \quad \forall \mathbf{Q} \in \mathbf{P}\mathscr{G}\mathbf{P}^{\mathrm{T}}, \tag{94}$$

are equivalent to each other.

According to this result, with any given proper orthogonal tensor **P** meeting Eq. (92), we may reformulate a Lagrangean type yield function $F(\mathbf{S}, \boldsymbol{\Pi}, \kappa)$ with the invariance property (91) under the initial material group $\mathscr{G}$ as an equivalent objective Eulerian type yield function $f(\mathbf{s}, \boldsymbol{\pi}, \kappa)$ with the invariance property (94) under the **P**-conjugate group $\mathbf{P}\mathscr{G}\mathbf{P}^{\mathrm{T}}$, and vice versa. It should be pointed out that the Eulerian yield function $f(\mathbf{s}, \boldsymbol{\pi}, \kappa)$ is defined and introduced in its own right, independent of any Lagrangean formulation. In fact, the Eulerian stress measure **s** and the Eulerian tensorial internal variable $\boldsymbol{\pi}$, e.g., the Kirchhoff stress $\boldsymbol{\tau}$ and the back stress $\boldsymbol{\alpha}$, may be just the well-defined basic quantities introduced in section 3. Thus, introducing a suitable Eulerian stress measure **s** and internal variables $\boldsymbol{\pi}$ and κ, as well as a suitable proper orthogonal tensor **P**, e.g., the rotation tensor **R**, one can formulate the yield condition by means of an Eulerian yield function $f(\mathbf{s}, \boldsymbol{\pi}, k)$ obeying the invariance restriction (94), which fulfills both the objectivity requirement and the material symmetry restriction for an elastoplastic material with any given type of initial material symmetry. The aforementioned difficulty thus disappears.

The foregoing full **P**-conjugate correspondence relation implies that the general reduced forms or representations for the two kinds of invariant yield functions are also **P**-conjugate. Since the **P**-conjugate group $\mathbf{P}\mathscr{G}\mathbf{P}^{\mathrm{T}}$ is completely the same as the initial material symmetry group $\mathscr{G}$ in the sense of algebraic construction, a representation for $f(\mathbf{s}, \boldsymbol{\pi}, \kappa)$ is obtainable from a representation for $F(\mathbf{S}, \boldsymbol{\Pi}, \kappa)$ by simply replacing the Lagrangean tensor variables **S** and $\boldsymbol{\Pi}$ as well as relevant initial material symmetry axes with their Eulerian **P**-conjugate counterparts.

6 Cyclic element deformation

As mentioned in the introduction, the work is completed with some calculations of the stesses within the purely hypoelastic model (63), i.e. with $\mathbf{D}^{\mathrm{e}} = \mathbf{D}$, subjected to constantly repeated strain cycles. Different objective stress rates have been used. Their different influences on the stresses are discussed.

Consider an unstressed, quadratic element of side length H and subject it to a combined elongation and shearing process such that the upper element corners are moved along an ellipse-like curve of radii a and b (Fig. 1). The param-

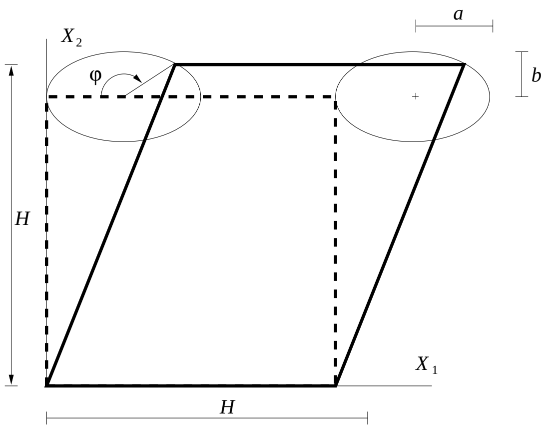

Figure 1: Deformed (full line) and undeformed element (dashed line)

eter φ then describes the deformation process. We introduce dimensionless parameters $\eta = b/H$ $(0 < \eta < 1)$ as measure for the elongation/compression of the element and $\xi = a/b$ as measure for related rotation, respectively, and find

$$x_1 = X_1 + \eta \xi \frac{1 - \cos\varphi}{1 + \eta \sin\varphi} X_2 , \quad x_2 = (1 + \eta \sin\varphi) X_2 , \quad x_3 = X_3 . \tag{95}$$

In the next section, three different cases of related rotation are discussed with "small rotations" $\xi = 0.1$), "moderate rotations" ($\xi = 1$) and "large rotations" ($\xi = 5$), superimposed on two cases of elongation/compression, namely $\eta = 0.02$ (i.e. 2% elongation/compression) and 0.1 (10%). From Eq. (5) we compute the deformation gradient $\mathbf{F}$ and its partial derivative

$$\begin{aligned}\mathbf{F} &= \begin{pmatrix} 1 & \eta\xi\,\dfrac{1-\cos\varphi}{1+\eta\sin\varphi} & 0 \\ 0 & 1+\eta\sin\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathbf{F}' = \frac{\partial \mathbf{F}}{\partial\varphi} &= \eta \begin{pmatrix} 0 & \xi\,\dfrac{\sin\varphi+\eta(1-\cos\varphi)}{(1+\eta\sin\varphi)^2} & 0 \\ 0 & \cos\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned} \tag{96}$$

We now introduce time independent quantities $\bar{\mathbf{L}}$, $\bar{\mathbf{D}}$, $\bar{\mathbf{W}}$ and $\bar{\boldsymbol{\Omega}}^*$, with

$$\bar{\mathbf{L}} = \mathbf{F}'\mathbf{F}^{-1}, \quad \bar{\mathbf{D}} = \frac{1}{2}(\bar{\mathbf{L}}+\bar{\mathbf{L}}^{\mathrm{T}}), \quad \bar{\mathbf{W}} = \bar{\mathbf{L}}-\bar{\mathbf{D}}, \quad \boldsymbol{\Omega}^* = \bar{\boldsymbol{\Omega}}^*\dot{\varphi} \tag{97}$$

and arrive at:

$$\mathbf{L} = \bar{\mathbf{L}}\dot{\varphi}, \quad \mathbf{D} = \bar{\mathbf{D}}\dot{\varphi}, \quad \mathbf{W} = \bar{\mathbf{W}}\dot{\varphi}, \quad \dot{\boldsymbol{\tau}} = \boldsymbol{\tau}'\dot{\varphi}, \quad \dot{\mathbf{F}} = \mathbf{F}'\dot{\varphi}.$$

With these relations, we may eliminate $\dot{\varphi}$ from the foregoing constitutive relations and stress rates and get a set of ordinary differential equations for τ_{ik} with independent variable φ.

Moreover, with reference to the purely hypoelastic case (refer to the first paragraph of this section) and the different definitions of velocity gradient $\mathbf{L}$, strain rate (stretching) $\mathbf{D}$, vorticity $\mathbf{W}$ and corotational logarithmic spin $\boldsymbol{\Omega}^{\log}$, given in section 2, these equations are indeed straightforwardly derived. Finally, the thus derived set of ordinary differential equations is integrated with standard numerical integration procedure.

7 Numeric results

7.1 Small rotations ($\xi = 0.1$)

In all following examples, we take Poisson's ratio $\nu = 0.3$ to calculate the relation between Young's modulus E and shear modulus μ. The stresses in the different graphs are thus depicted as dimensionless quantities τ_{ik}/μ.

With the dimensionless geometrical ratio $\xi = 0.1$ the rotation is almost negligible; the resulting shear deformation is relatively small compared to the elongation/compression in 2-direction. As would be expected the development of normal stresses τ_{11}, τ_{22} and shear stress τ_{12} versus the deformation angle

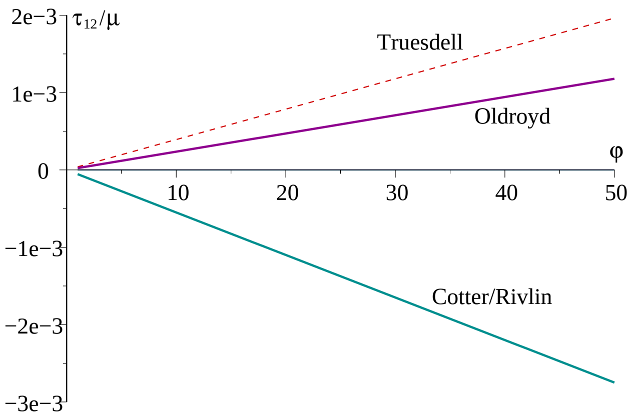

Figure 2: Residual shear stress at cycle end, $\xi = 0.1, \eta = 0.02$

φ for a small elongation of $\eta = 0.02$ are almost congruent. A closer look to the shear stress τ_{12} at the end of the deformation cycle would, however, evidence, that, in spite of the small rotation, τ_{12} will not return to 0 when non-corotational rates are used.

In course of repeated cycles this error may increase to unacceptable values (Fig. 2). E.g., for the non-corotational rates the residual stress τ_{12} accumulates in course of 50 cycles to the order of the first cycle's maximum value.

For increased elongations, e.g. for $\eta = 0.1$, i.e. 10 % compression in 2-direction, the normal stresses in 2-direction τ_{22} show larger deviations (Fig. 3). The distribution of shear stress τ_{12} is doubtful in case of non-corotational rates, cf. Fig. 4.

7.2 Moderate rotations ($\xi = 1$)

The question arises whether results would stay reasonable for larger rotations in case of corotational rates. In order to examine this, we investigate the case of moderate rotations ($\xi = 1$), i.e., we consider strain cycles, where the upper points of the element are moved along a circle.

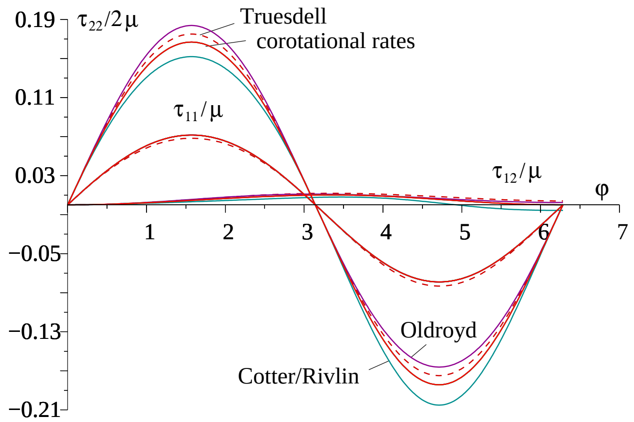

Figure 3: τ_{22} development over a deformation cycle, $\xi = 0.1, \eta = 0.1$

Fig. 5 shows the development of Kirchhoff stresses over a deformation cycle. In the case of the non-corotational rates, the residual stresses τ_{12} at the end of the cycle are considerable. When increasing the elastic deformation to $\eta = 0.1$, we notice also in the case of corotational rates that there are remaining stresses. Only the logarithmic rate shows reliable results, i.e. all stresses are returning to zero at the end of the cycle.

7.3 Large rotations ($\xi = 5$)

The case of larger elastic strains may be important for several applications, and for softer materials. Figs. 6 and 7 show stresses τ_{11} and τ_{12} for $\xi = 5$ and $\eta = 0.1$, i.e. a case where shear deformation is predominant. Apparently, the normal stress τ_{11} differs considerably from one rate to another. It may be noticed that for all non-corotational rates the extreme values are too high or too low. The non-zero end values are unacceptable in the case of non-corotational and corotational rates with exception of the logarithmic rate. Generally, at the end of a cycle stress error accumulates in course of cycles. This is true for all rates, except for the logarithmic rate, where no error occurs and the Jaumann rate, where we observe a sort of error oscillation.

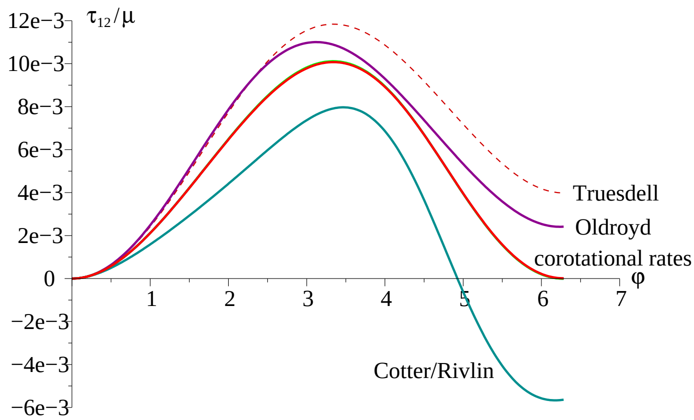

Figure 4: τ_{12} development over a deformation cycle, $\xi = 0.1, \eta = 0.1$

8 Conclusion

In the present paper, the general form of constitutive equations for Eulerian elastoplasticity at finite deformations has been discussed. Several as yet still unsafe fundamental issues have been clarified. We propose a simple, closed strain path for investigating the behavior of different objective rates in the hypoelastic deformation and Eulerian description. This strain cycle can equally be used for large normal deformations and/or large rotation. It turns out that there is only a single corotational rate that may be esteemed to be reliable in the proposed cycle, namely the logarithmic rate. All other rates fail whenever rotations tend to non-negligible values. This confirms the theoretical results obtained by Xiao et al. [33].

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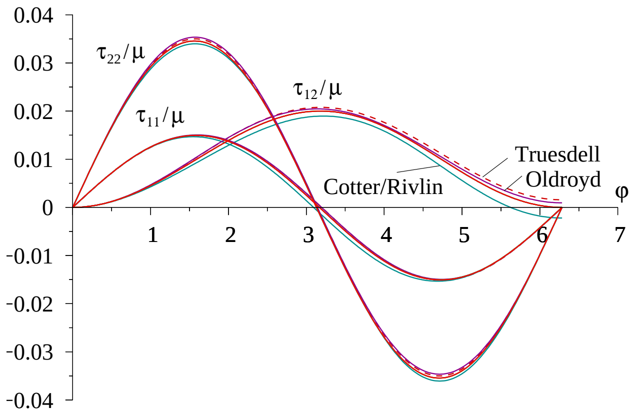

Figure 5: Stress development over a deformation cycle, $\xi = 1, \eta = 0.02$

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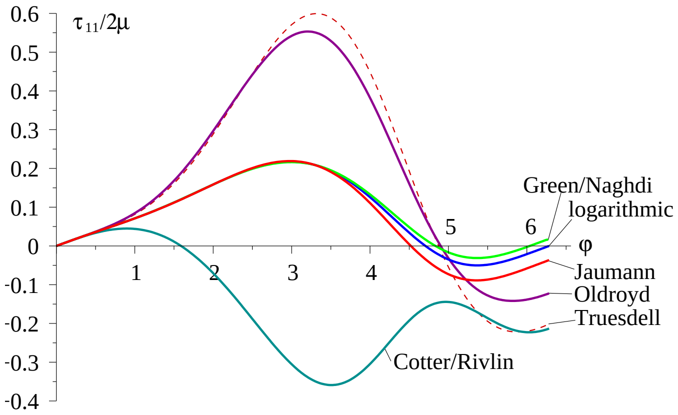

Figure 6: Normal stress τ_{11} development, $\xi = 5, \eta = 0.1$

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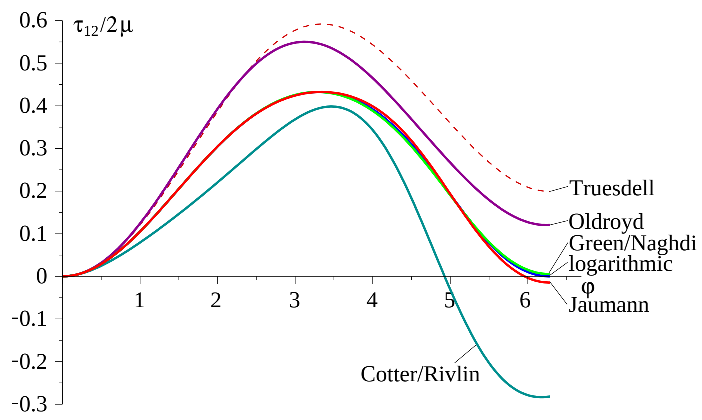

Figure 7: Shear stress τ_{12} development, $\xi = 5, \eta = 0.1$

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Submitted on May 2009, revised on September 2009.

Ojlerovska elastoplastičnost: osnovne postavke i nedavni rezultati

Tradicionalne formulacije elastoplastičnosti u prisustvu konazv cne deformacije i velikog obrtanja su Ojlerovskog tipa i naširoko se koriste; zasnovane su, pored ostalog, na aditivnoj dekompoziciji brzine deformacije (ili njene Ojlerovskog predstavnika) u elastične i plastične delove. Pri takvim formulacijama, funkcije tečenja i konstitutivne jednačine, u kojima se pojavljuju objektive brzine, su izražene preko objektivnih Ojlerovskih tenzorskih veličina uključujući brzinu deformacije, Kirhofov napon, unutrašnje promenljive stanja itd. Svaka od ovih veličina se tranformišen na korotacioni način pri promeni posmatračkog sistema. Saglasno principu objektivnosti, svaka konstitutivna funkcija treba da bude invarijantna, kadgod se posmatrački sistem menja bilo kojim obrtanjem promenljivim u vremenu. U ovom radu se diskutuje opšti oblik konstitutivnih jednačina. Nekoliko često korišćenih objektivnih brzina se analizira u odnosu na njihovu uslužnost razvoju samousaglašavajuće formulacije, tj. integrabilnosti koja daje elastičnu i, posebno, hiperelastičnu, relaciju pri isčezavajućoj plastičnoj deformaciji. Ovo je od velike važnosti, naprimer, za takozvana odskočna izračunavanja u slučaju obrade deformacijom.

doi:10.2298/TAM0903167B Math.Subj.Class.: 74C15, 74E10