

# Multi-temperature mixture of fluids

Tommaso Ruggeri \*

## Abstract

We present a survey on some recent results concerning the different models of a mixture of compressible fluids. In particular we discuss the most realistic case of a mixture when each constituent has its own temperature ( $MT$ ) and we first compare the solutions of this model with the one with a unique common temperature ( $ST$ ). In the case of Eulerian fluids it will be shown that the corresponding ( $ST$ ) differential system is a *principal subsystem* of the ( $MT$ ) one. Global behavior of smooth solutions for large time for both systems will also be discussed through the application of the Shizuta-Kawashima condition.

Then we introduce the concept of the average temperature of mixture based upon the consideration that the internal energy of the mixture is the same as in the case of a single-temperature mixture. As a consequence, it is shown that the entropy of the mixture reaches a local maximum in equilibrium. Through the procedure of Maxwellian iteration a new constitutive equation for non-equilibrium temperatures of constituents is obtained in a classical limit, together with the Fick's law for the diffusion flux.

Finally, to justify the Maxwellian iteration, we present for dissipative fluids a possible approach of a classical theory of mixture with multi-temperature and we prove that the differences of temperatures between the constituents imply the existence of a new *dynamical pressure* even if the fluids have a zero bulk viscosity.

**Keywords:** Multi-temperature mixture of fluids; Maxwellian iteration; dynamical pressure; non-equilibrium thermodynamics.

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\*Department of Mathematics and Research Center of Applied Mathematics (C.I.R.A.M.) University of Bologna, Via Saragozza 8, 40123 Bologna, Italy, e-mail: ruggeri@ciram.unibo.it

# 1 Mixtures in rational thermodynamics

In the context of rational thermodynamics, the description of a homogeneous mixture of  $n$  constituents is based on the postulate that each constituent obeys to the same balance laws that a single fluid does [1], [2]. The laws express the equations of balance of masses, momenta and energies

$$\left\{ \begin{array}{l} \frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha) = \tau_\alpha, \\ \frac{\partial(\rho_\alpha \mathbf{v}_\alpha)}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - \mathbf{t}_\alpha) = \mathbf{m}_\alpha, \quad (\alpha = 1, 2, \dots, n) \\ \frac{\partial(\frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha)}{\partial t} \\ \quad + \operatorname{div}\left\{(\frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha) \mathbf{v}_\alpha - \mathbf{t}_\alpha \mathbf{v}_\alpha + \mathbf{q}_\alpha\right\} = e_\alpha. \end{array} \right. \quad (1)$$

On the left hand side,  $\rho_\alpha$  is the density,  $\mathbf{v}_\alpha$  is the velocity,  $\varepsilon_\alpha$  is the internal energy,  $\mathbf{q}_\alpha$  is the heat flux and  $\mathbf{t}_\alpha$  is the stress tensor of the constituent  $\alpha$ . The stress tensor  $\mathbf{t}_\alpha$  can be decomposed into a pressure part  $-p_\alpha \mathbf{I}$  and a viscous part  $\boldsymbol{\sigma}_\alpha$  as

$$\mathbf{t}_\alpha = -p_\alpha \mathbf{I} + \boldsymbol{\sigma}_\alpha.$$

On the right hand sides  $\tau_\alpha$ ,  $\mathbf{m}_\alpha$  and  $e_\alpha$  represent the production terms related to the interactions between constituents. Due to the total conservation of mass, momentum and energy of the mixture, the sum of production terms over all constituents must vanish

$$\sum_{\alpha=1}^n \tau_\alpha = 0, \quad \sum_{\alpha=1}^n \mathbf{m}_\alpha = \mathbf{0}, \quad \sum_{\alpha=1}^n e_\alpha = 0.$$

Mixture quantities  $\rho$ ,  $\mathbf{v}$ ,  $\varepsilon$ ,  $\mathbf{t}$  and  $\mathbf{q}$  are defined as

$$\begin{aligned}
 \rho &= \sum_{\alpha=1}^n \rho_{\alpha} && \text{total mass density,} \\
 \mathbf{v} &= \frac{1}{\rho} \sum_{\alpha=1}^n \rho_{\alpha} \mathbf{v}_{\alpha} && \text{mixture velocity,} \\
 \varepsilon &= \varepsilon_I + \frac{1}{2\rho} \sum_{\alpha=1}^n \rho_{\alpha} u_{\alpha}^2 && \text{internal energy,} \\
 &\left( \varepsilon_I = \frac{1}{\rho} \sum_{\alpha=1}^n \rho_{\alpha} \varepsilon_{\alpha} \right) && \\
 \mathbf{t} &= -p\mathbf{I} + \sigma_I - \sum_{\alpha=1}^n (\rho_{\alpha} \mathbf{u}_{\alpha} \otimes \mathbf{u}_{\alpha}) && \text{stress tensor,} \\
 \mathbf{q} &= \mathbf{q}_I + \sum_{\alpha=1}^n \rho_{\alpha} \left( \varepsilon_{\alpha} + \frac{p_{\alpha}}{\rho_{\alpha}} + \frac{1}{2} u_{\alpha}^2 \right) \mathbf{u}_{\alpha} && \text{flux of internal energy,}
 \end{aligned} \tag{2}$$

where  $\mathbf{u}_{\alpha} = \mathbf{v}_{\alpha} - \mathbf{v}$  is the diffusion velocity of the component  $\alpha$ ,  $p = \sum_{\alpha=1}^n p_{\alpha}$  is the total pressure,  $\varepsilon_I$  is the total intrinsic internal energy,  $\mathbf{q}_I = \sum_{\alpha=1}^n \mathbf{q}_{\alpha}$  is the total intrinsic heat flux and  $\sigma_I = \sum_{\alpha=1}^n \sigma_{\alpha}$  is the total intrinsic shear stress. We obtain by summation of Eqs. (1),

$$\left\{ \begin{array}{l}
 \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \\
 \frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}) = \mathbf{0}, \\
 \frac{\partial \left( \frac{1}{2} \rho v^2 + \rho \varepsilon \right)}{\partial t} + \operatorname{div} \left\{ \left( \frac{1}{2} \rho v^2 + \rho \varepsilon \right) \mathbf{v} - \mathbf{t} \mathbf{v} + \mathbf{q} \right\} = 0,
 \end{array} \right.$$

that are the conservation laws of mass, momentum and energy of the mixture. They are in the same form as for a single fluid.

In order to compare the balance equations of mixture and single fluid, we write Eqs. (1) in the equivalent form,

$$\left\{ \begin{array}{l} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{t}) = \mathbf{0}, \\ \frac{\partial(\frac{1}{2}\rho v^2 + \rho \varepsilon)}{\partial t} \\ \quad + \operatorname{div}\left\{(\frac{1}{2}\rho v^2 + \rho \varepsilon)\mathbf{v} - \mathbf{t}\mathbf{v} + \mathbf{q}\right\} = 0, \\ \frac{\partial \rho_b}{\partial t} + \operatorname{div}(\rho_b \mathbf{v}_b) = \tau_b, \quad (b = 1, \dots, n-1) \\ \frac{\partial(\rho_b \mathbf{v}_b)}{\partial t} + \operatorname{div}(\rho_b \mathbf{v}_b \otimes \mathbf{v}_b - \mathbf{t}_b) = \mathbf{m}_b, \\ \frac{\partial(\frac{1}{2}\rho_b v_b^2 + \rho_b \varepsilon_b)}{\partial t} \\ \quad + \operatorname{div}\left\{(\frac{1}{2}\rho_b v_b^2 + \rho_b \varepsilon_b)\mathbf{v}_b - \mathbf{t}_b \mathbf{v}_b + \mathbf{q}_b\right\} = e_b, \end{array} \right. \quad (3)$$

where the index  $b$  runs from 1 to  $n-1$ .

In this multi-temperature model (*MT*), used in particular in plasma physics [3], we have  $5n$  independent field variables  $\rho_\alpha$ ,  $\mathbf{v}_\alpha$  and  $T_\alpha$  ( $\alpha = 1, 2, \dots, n$ ), where  $T_\alpha$  is the temperature of constituent  $\alpha$ . To close the system (3) of the field equations of the mixture process, we must write the constitutive equations for the quantities  $p_\alpha$ ,  $\varepsilon_\alpha$ ,  $\mathbf{q}_\alpha$ ,  $\sigma_\alpha$  ( $\alpha = 1, 2, \dots, n$ ) and  $\tau_b$ ,  $\mathbf{m}_b$ ,  $e_b$  ( $b = 1, \dots, n-1$ ) in terms of the field variables  $\rho_\alpha$ ,  $\mathbf{v}_\alpha$  and  $T_\alpha$  ( $\alpha = 1, 2, \dots, n$ ).

## 1.1 Galilean invariance of field equations

The system (3) is a particular case of the balance law one:

$$\partial_t \mathbf{F}^0 + \partial_i \mathbf{F}^i = \mathbf{F}, \quad (4)$$

where  $\mathbf{F}^0$  (densities),  $\mathbf{F}^i$  (fluxes) and  $\mathbf{F}$  (productions) are  $N$ -vectors functions of the field variables and  $\partial_t = \partial/\partial t$ ,  $\partial_i = \partial/\partial x_i$ . It is convenient to divide field variables into a pair  $\mathbf{u} = (\mathbf{w}, \mathbf{v})$ ,  $\mathbf{v}$  being the velocity field and  $\mathbf{w}$  objective quantities. Moreover, the non-convective fluxes can be defined as  $\mathbf{G}^i = \mathbf{F}^i - \mathbf{F}^0 v^i$ , thus putting equation (4) in the form:

$$\partial_t \mathbf{F}^0 + \partial_i (\mathbf{F}^0 v^i + \mathbf{G}^i) = \mathbf{F}. \quad (5)$$

Principle of relativity requires that field equations are invariant with respect to Galilean transformations:

$$\mathbf{x} \mapsto \mathbf{x}^* - \mathbf{c}t^*; \quad t \mapsto t^*; \quad \mathbf{v} \mapsto \mathbf{v}^* - \mathbf{c}, \quad \text{for any constant velocity } \mathbf{c},$$

i.e. the system (5) has the same form in the new frame:

$$\partial_{t^*} \mathbf{F}^0(\mathbf{w}, \mathbf{v}^*) + \partial_{i^*} (\mathbf{F}^0(\mathbf{w}, \mathbf{v}^*) v^{i^*} + \mathbf{G}^i(\mathbf{w}, \mathbf{v}^*)) = \mathbf{F}(\mathbf{w}, \mathbf{v}^*).$$

As a consequence, (see Ruggeri [4]), there exists a linear operator  $\mathbf{X}(\mathbf{v})$  such that :

$$\begin{aligned} \mathbf{F}^0(\mathbf{w}, \mathbf{v}) &= \mathbf{X}(\mathbf{v}) \hat{\mathbf{F}}^0(\mathbf{w}); \\ \mathbf{G}^i(\mathbf{w}, \mathbf{v}) &= \mathbf{X}(\mathbf{v}) \hat{\mathbf{G}}^i(\mathbf{w}); \\ \mathbf{F}(\mathbf{w}, \mathbf{v}) &= \mathbf{X}(\mathbf{v}) \hat{\mathbf{F}}(\mathbf{w}), \end{aligned}$$

where:

$$\hat{\mathbf{F}}^0(\mathbf{w}) = \mathbf{F}^0(\mathbf{w}, \mathbf{0}); \quad \hat{\mathbf{G}}^i(\mathbf{w}) = \mathbf{G}^i(\mathbf{w}, \mathbf{0}); \quad \hat{\mathbf{F}}(\mathbf{w}) = \mathbf{F}(\mathbf{w}, \mathbf{0}),$$

are objective "internal" densities, fluxes and productions, while  $\mathbf{X}(\mathbf{v})$  has the following properties:

$$\mathbf{X}(\mathbf{a} + \mathbf{b}) = \mathbf{X}(\mathbf{a})\mathbf{X}(\mathbf{b}) = \mathbf{X}(\mathbf{b})\mathbf{X}(\mathbf{a}); \quad \mathbf{X}(\mathbf{0}) = \mathbf{I},$$

i.e.

$$\mathbf{X}(\mathbf{v}) = e^{\mathbf{A}^r v_r}$$

with  $\mathbf{A}^r$  constant ( $3 \times 3$ ) matrix such that  $\mathbf{A}^r \mathbf{A}^s = \mathbf{A}^s \mathbf{A}^r$ , ( $r, s = 1, 2, 3$ ).

In the case of a mixture, the Galilean invariance dictates the velocity dependence in the source terms [5]:

$$\begin{aligned}\tau_b &= \hat{\tau}_b; \\ \mathbf{m}_b &= \hat{\tau}_b \mathbf{v} + \hat{\mathbf{m}}_b; \quad (b = 1, \dots, n-1) \\ e_b &= \hat{\tau}_b \frac{v^2}{2} + \hat{\mathbf{m}}_b \cdot \mathbf{v} + \hat{e}_b,\end{aligned}\tag{6}$$

where  $\hat{\tau}_b$ ,  $\hat{\mathbf{m}}_b$  and  $\hat{e}_b$  are independent of  $\mathbf{v}$ .

## 1.2 The entropy principle and its restrictions

Another important restriction comes from the entropy inequality, i.e. there exists a supplementary balance law of entropy with an entropy production non negative::

$$\partial_t \rho S + \partial_i (\rho S v^i + \varphi^i) = \Sigma \geq 0,$$

where  $\rho S = \sum_{\alpha=1}^n \rho_\alpha S_\alpha$ ,  $\varphi^i$  and  $\Sigma$  are the entropy density, the non-convective entropy flux and the entropy production, respectively. For example in the case of a mixture of Eulerian fluids, the entropy production becomes [5]:

$$\begin{aligned}\Sigma &= \sum_{b=1}^{n-1} \left( \frac{\mu_n - \frac{1}{2} u_n^2}{T_n} - \frac{\mu_b - \frac{1}{2} u_b^2}{T_b} \right) \hat{\tau}_b \\ &+ \left( \frac{\mathbf{u}_n}{T_n} - \frac{\mathbf{u}_b}{T_b} \right) \cdot \hat{\mathbf{m}}_b + \left( \frac{1}{T_b} - \frac{1}{T_n} \right) \hat{e}_b \geq 0.\end{aligned}\tag{7}$$

This inequality allows to obtain the following structure of production terms.

The internal parts of production terms (6) are chosen in such a way that the residual inequality (7) is actually a quadratic form. In particular

in agreement with kinetic theory, we obtain:

$$\begin{aligned}\hat{\tau}_b &= \sum_{c=1}^{n-1} \varphi_{bc} \left( \frac{\mu_n - \frac{1}{2}u_n^2}{T_n} - \frac{\mu_c - \frac{1}{2}u_c^2}{T_c} \right) + \sum_{c=1}^{n-1} g_{bc} \left( \frac{1}{T_c} - \frac{1}{T_n} \right); \\ \hat{\mathbf{m}}_b &= \sum_{c=1}^{n-1} \psi_{bc} \left( \frac{\mathbf{u}_n}{T_n} - \frac{\mathbf{u}_c}{T_c} \right); \\ \hat{e}_b &= \sum_{c=1}^{n-1} \theta_{bc} \left( \frac{1}{T_c} - \frac{1}{T_n} \right) + \sum_{c=1}^{n-1} g_{bc} \left( \frac{\mu_n - \frac{1}{2}u_n^2}{T_n} - \frac{\mu_c - \frac{1}{2}u_c^2}{T_c} \right),\end{aligned}\quad (8)$$

where:

$$\mu_\alpha = \varepsilon_\alpha - T_\alpha S_\alpha + \frac{p_\alpha}{\rho_\alpha}$$

( $\alpha = 1, \dots, n$ ), are the chemical potentials of the constituents and

$$\begin{bmatrix} \varphi_{bc} & g_{bc} \\ g_{bc} & \theta_{bc} \end{bmatrix}, \quad \psi_{bc}$$

are phenomenological symmetric positive definite matrices ( $b, c = 1, \dots, n-1$ ). In the sequel, our analysis will be restricted to a model of non-reacting mixtures, for which  $\tau_b = 0$ .

## 2 Coarser theories: single temperature model and classic mixture

Due to the difficulties to measure the temperature of each component, a common practice among engineers and physicists is to consider only one temperature for the mixture. When we use a single temperature ( $ST$ ), Eq. (3)<sub>6</sub> disappears and we get a unique global conservation of the total energy in the form (3)<sub>3</sub> (see for example [2]).

A further step of coarsening theory is the classical approach of mixtures ( $CT$ ), in which the independent field variables are the density, the mixture velocity, the individual temperature of the mixture and the concentrations of constituents. In that case system (3) reduces to the equa-

tions

$$\left\{ \begin{array}{l} \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0, \\ \rho \frac{d\mathbf{v}}{dt} - \operatorname{div} \mathbf{t} = 0, \\ \rho \frac{d\varepsilon}{dt} - \mathbf{t} \operatorname{grad} \mathbf{v} + \operatorname{div} \mathbf{q} = 0, \\ \rho \frac{dc_b}{dt} + \operatorname{div} \mathbf{J}_b = 0, \quad (b = 1, \dots, n-1), \end{array} \right. \quad (9)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}}$$

represents the material derivative of the mixture motion

$$c_\alpha = \frac{\rho_\alpha}{\rho}, \quad \left( \sum_{\alpha=1}^n c_\alpha = 1 \right)$$

are the components concentrations, and

$$\mathbf{J}_\alpha = \rho_\alpha \mathbf{u}_\alpha = \rho_\alpha (\mathbf{v}_\alpha - \mathbf{v}), \quad \left( \sum_{\alpha=1}^n \mathbf{J}_\alpha = 0 \right)$$

are the diffusion fluxes of the components.

In the classical approach the stress tensor - as in a single fluid - splits up into the pressure isotropic part and the viscosity stress tensor  $\sigma$  (for Stokesian fluids this is a deviatoric tensor)

$$\mathbf{t} = -p\mathbf{I} + \sigma.$$

The system (9) determines the field variables  $\rho, T, \mathbf{v}$  and  $c_b$  ( $b = 1, \dots, n-1$ ). Consequently, we need constitutive relations for  $\varepsilon, \sigma, \mathbf{q}$  and  $\mathbf{J}_b$  ( $b = 1, \dots, n-1$ ).

We consider the pressure  $p(\rho, T, c_b)$  and the internal energy  $\varepsilon(\rho, T, c_b)$  as given by the equilibrium equations of state as they appear in the Gibbs



equations for mixture, viz.

$$TdS = d\varepsilon - \frac{p}{\rho^2} d\rho - \sum_{b=1}^{n-1} (\mu_b - \mu_n) dc_b \quad (10)$$

where  $\mu_\alpha = \mu_\alpha(\rho, T, c_b)$ , with  $\alpha = 1, \dots, n$ , denote the chemical potentials of components at equilibrium and  $S$  is the entropy density of the mixture. The entropy balance law is a consequence of equation (10) and system (9). For dissipative fluids, by using arguments of the thermodynamics of irreversible processes (*TIP*) presented in [6] and [2] chapter 5, we obtain the classical constitutive equations of mixtures

$$\begin{aligned} \sigma &= \lambda (\operatorname{div} \mathbf{v}) \mathbf{I} + 2\nu \mathbf{D}^D, \\ \mathbf{q} &= L \operatorname{grad} \left( \frac{1}{T} \right) + \sum_{b=1}^{n-1} L_b \operatorname{grad} \left( \frac{\mu_b - \mu_n}{T} \right), \\ \mathbf{J}_a &= \tilde{L}_a \operatorname{grad} \left( \frac{1}{T} \right) - \sum_{b=1}^{n-1} L_{ab} \operatorname{grad} \left( \frac{\mu_b - \mu_n}{T} \right) \end{aligned} \quad (11)$$

where  $\mathbf{D}^D$  denotes the deviatoric part of the strain velocity tensor  $\mathbf{D} = \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$ . The *phenomenological coefficients*  $L, L_b, \tilde{L}_a$  and  $L_{ab}$  ( $a, b = 1, \dots, n-1$ ) are the transport coefficients of heat conduction and diffusion.

Let us note that relation (11)<sub>1</sub> are the classical Navier-Stokes equations of a Newtonian (Stokesian and isotropic) fluid, while (11)<sub>2,3</sub> are generalizations of the original phenomenological laws of Fourier and Fick according to which the heat flux and the diffusion flux depend on the gradients of temperature and concentrations respectively (*but not on both*). *TIP* permits the temperature gradient to influence the diffusion fluxes and concentration gradients to influence the heat flux; both effects are indeed observed and they are called respectively thermo-diffusion effect and diffusion-thermo effect or Soret effect. Additively, the Onsager conditions of symmetry yield the following symmetries of coefficients [7]

$$L_{ab} = L_{ba}, \quad \tilde{L}_b = L_b \quad (a, b = 1, \dots, n-1)$$

and the following inequalities must be satisfied

$$\begin{bmatrix} L & L_b \\ \tilde{L}_a & L_{ab} \end{bmatrix} \quad \text{is a positive definite form} \quad (12)$$

and  $\nu \geq 0$ ,

such that the entropy inequality can be satisfied.

### 3 Euler fluids and comparison between $MT$ and $ST$ models

First we consider the case in which all the constituents of the mixture are Eulerian fluids, i.e. neither viscous nor heat-conducting, i.e.:

$$\mathbf{t}_\alpha = -p_\alpha \mathbf{I}, \quad \mathbf{q}_\alpha = \mathbf{0} \quad ; \quad (\alpha = 1, \dots, n).$$

#### 3.1 Symmetric hyperbolic system and principal subsystems

For a generic hyperbolic system (4), when the entropy density  $h^0 = -\rho S$  is a convex function of  $\mathbf{u} \equiv \mathbf{F}^0$ , it is possible to prove that there exists a privileged set of field variables, the *main field*

$$\mathbf{u}' = \frac{\partial h^0}{\partial \mathbf{u}}$$

such that original balance laws could be transformed in a symmetric form. In fact introducing the four potentials [8], [9]

$$h'^0 = \mathbf{u}' \cdot \mathbf{F}^0 - h^0; \quad h'^i = \mathbf{u}' \cdot \mathbf{F}^i - h^i \quad (i = 1, 2, 3)$$

the original system can be put in the special symmetric form:

$$\begin{aligned} \partial_i \left( \frac{\partial h'^0}{\partial \mathbf{u}'} \right) + \partial_i \left( \frac{\partial h'^i}{\partial \mathbf{u}'} \right) &= \mathbf{F} \\ &\iff \\ \frac{\partial^2 h'^0}{\partial \mathbf{u}' \partial \mathbf{u}'} \partial_t \mathbf{u}' + \frac{\partial^2 h'^i}{\partial \mathbf{u}' \partial \mathbf{u}'} \partial_i \mathbf{u}' &= \mathbf{F} \quad ; \quad (i = 1, 2, 3). \end{aligned} \quad (13)$$

The Boillat proof [8, 10], has the advantage with respect to the symmetrization of Friedrichs and Lax [11] that the symmetric system is the original one. Moreover this includes, as a particular case, the example discovered first by Godunov for the fluid dynamics case and the Euler-Lagrange systems [12]. Symmetric structure of the system of balance laws is highly desirable due to hyperbolicity and local well-posedness of initial-value problems (see e.g. [13]). The main field components for the mixture of Euler fluids described by the system (3) have the form [5]:

$$\begin{aligned}
\Lambda^\rho &= \frac{1}{T_n} \left( \mu_n - \frac{1}{2} (\mathbf{u}_n + \mathbf{v})^2 \right) \\
\Lambda^{\mathbf{v}} &= \frac{1}{T_n} (\mathbf{u}_n + \mathbf{v}) \\
\Lambda^\varepsilon &= -\frac{1}{T_n} \\
\Lambda^{\rho_b} &= \frac{1}{T_b} \left( \mu_b - \frac{1}{2} (\mathbf{u}_b + \mathbf{v})^2 \right) - \frac{1}{T_n} \left( \mu_n - \frac{1}{2} (\mathbf{u}_n + \mathbf{v})^2 \right) \\
\Lambda^{\mathbf{v}_b} &= \frac{\mathbf{u}_b}{T_b} - \frac{\mathbf{u}_n}{T_n} - \left( \frac{1}{T_b} - \frac{1}{T_n} \right) \mathbf{v} \\
\Lambda^{\varepsilon_b} &= \frac{1}{T_n} - \frac{1}{T_b}.
\end{aligned} \tag{14}$$

The use of the main field has still another advantage: the possibility of recognition of principal and equilibrium subsystems. Let us give a brief review of the results which can be found in [14]. Let us split the main field  $\mathbf{u}' \in R^N$  into two parts  $\mathbf{u}' \equiv (\mathbf{v}', \mathbf{w}')$ ,  $\mathbf{v}' \in R^M$ ,  $\mathbf{w}' \in R^{N-M}$ , ( $0 < M < N$ ) and the system (13) with  $\mathbf{F} \equiv (\mathbf{f}, \mathbf{g})$ , reads:

$$\partial_t \left( \frac{\partial h^0(\mathbf{v}', \mathbf{w}')}{\partial \mathbf{v}'} \right) + \partial_i \left( \frac{\partial h^i(\mathbf{v}', \mathbf{w}')}{\partial \mathbf{v}'} \right) = \mathbf{f}(\mathbf{v}', \mathbf{w}'), \tag{15}$$

$$\partial_t \left( \frac{\partial h^0(\mathbf{v}', \mathbf{w}')}{\partial \mathbf{w}'} \right) + \partial_i \left( \frac{\partial h^i(\mathbf{v}', \mathbf{w}')}{\partial \mathbf{w}'} \right) = \mathbf{g}(\mathbf{v}', \mathbf{w}'). \tag{16}$$

Given an assigned constant value  $\mathbf{w}'_*$  to  $\mathbf{w}'$ , we call principal subsystem

of (13) the system:

$$\partial_t \left( \frac{\partial h^0(\mathbf{v}', \mathbf{w}'_*)}{\partial \mathbf{v}'} \right) + \partial_i \left( \frac{\partial h^{i'}(\mathbf{v}', \mathbf{w}'_*)}{\partial \mathbf{v}'} \right) = \mathbf{f}(\mathbf{v}', \mathbf{w}'_*) \quad (17)$$

In other words, a principal subsystem coincide with the first block of the system (15) putting  $\mathbf{w}' = \mathbf{w}'_*$ . In this case we have

**Sub-Entropy Law:** *The solutions of a principal subsystem (17) satisfy a supplementary subentropy law: The subentropy  $\bar{h}^0$  is convex and therefore every principal subsystem is symmetric hyperbolic [14].*

Let  $\lambda^{(k)}(\mathbf{v}', \mathbf{w}', \mathbf{n})$  and  $\bar{\lambda}^{(\bar{k})}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n})$ , be the characteristic velocities of the total system (15), (16) and of the subsystem (17), respectively,  $\mathbf{n} \in R^3$  being a unit vector. In general the solutions of the subsystem are not particular solutions of the system (for  $\mathbf{w}' = \mathbf{w}'_*$ ) and the spectrum of  $\bar{\lambda}$ 's is not a part of the spectrum of  $\lambda$ 's. However, if

$$\lambda_{\max} = \max_{k=1,2,\dots,N} \lambda^{(k)}, \quad \bar{\lambda}_{\max} = \max_{\bar{k}=1,2,\dots,M} \bar{\lambda}^{(\bar{k})},$$

and similarly for the minima, one obtains the following result

**Sub-characteristic Conditions:** *Under the assumption that  $h^0$  is a convex function, the following sub-characteristic conditions hold for every principal subsystem [14]:*

$$\begin{aligned} \lambda_{\max}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n}) &\geq \bar{\lambda}_{\max}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n}), \\ \lambda_{\min}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n}) &\leq \bar{\lambda}_{\min}(\mathbf{v}', \mathbf{w}'_*, \mathbf{n}), \end{aligned} \quad (18)$$

$\forall \mathbf{v}' \in R^M$  and  $\forall \mathbf{n} \in R^3, \|\mathbf{n}\| = 1$ .

Taking into account (14) and (3), we can recognize the following interesting principal subsystems:

**Case 1** - *The single-temperature model is a principal subsystem of the multi-temperature.* Let us suppose that  $\Lambda^{\varepsilon_b} = 0$  for  $b = 1, \dots, n-1$ , then

$$T_1 = \dots = T_n = T.$$

This principal subsystem contains only the energy conservation equation for the mixture, while energy balance equations for the constituents are

dropped. Thus, one may conclude that single-temperature model naturally appears as a principal subsystem of the multi-temperature system.

**Case 2** - *The equilibrium subsystem.* If we set

$$\Lambda^{\varepsilon b} = \Lambda^{\mathbf{v}b} = \Lambda^{\rho b} = 0 \quad \forall b = 1, \dots, n-1$$

i.e.:

$$T_b = T ; \quad \mathbf{u}_b = 0 ; \quad \mu_b = \mu \quad \forall b = 1, \dots, n-1,$$

we have the equilibrium Euler subsystem (a single fluid system) with concentrations  $c_b$  being solutions of  $\mu_1 = \mu_2 = \dots = \mu_n$ .

### 3.2 Characteristic velocities and their upper bound in the ST model

The characteristic velocities for the  $MT$  are simple to evaluate. Since, for each constituent, they are the same as the ones of a single fluid, i.e.:

$$\lambda_{\alpha}^{(1)} = v_{\alpha n} - c_{s\alpha}; \quad \lambda_{\alpha}^{(2,3,4)} = v_{\alpha n}; \quad \lambda_{\alpha}^{(5)} = v_{\alpha n} + c_{s\alpha};$$

where  $v_{\alpha n} = \mathbf{v}_{\alpha} \cdot \mathbf{n}$  are the normal component of the velocities on the wave front and  $c_{s\alpha} = \sqrt{\left(\frac{\partial p_{\alpha}}{\partial \rho_{\alpha}}\right)_{S_{\alpha}}}$  are the sound velocities. For an ideal gas,

$$p_{\alpha} = \frac{k}{m_{\alpha}} \rho_{\alpha} T_{\alpha}, \quad \varepsilon_{\alpha} = c_V^{(\alpha)} T_{\alpha}, \quad c_V^{(\alpha)} = \frac{k}{m_{\alpha}(\gamma_{\alpha} - 1)} \quad (19)$$

( $k, m_{\alpha}, c_V^{(\alpha)}, \gamma_{\alpha}$ , are respectively the Boltzmann constant, the atomic mass, the specific heat at constant volume and the ratio of the specific heats), we have

$$c_{s\alpha} = \sqrt{\frac{k\gamma_{\alpha}}{m_{\alpha}} T_{\alpha}}.$$

Instead, in the case of a ST system the evaluation of the velocities is very difficult also in an equilibrium state due to the fact that the characteristic polynomial is, in general, irreducible (see e.g. [2], [15]) but thanks to the subcharacteristic property (18) of principal subsystems we are able now to establish the following lower and upper bound for the characteristic velocities of the ST model:

$$\min_{\alpha} (v_{\alpha n} - c_{s\alpha}^*) \leq \lambda_{\min}^{ST}; \quad \max_{\alpha} (v_{\alpha n} + c_{s\alpha}^*) \geq \lambda_{\max}^{ST},$$

where now

$$c_{s\alpha}^* = \sqrt{\frac{k\gamma_\alpha}{m_\alpha} T}.$$

### 3.3 Qualitative analysis

The system (3), is a particular case of a system of balance laws (4) and it is dissipative due to the presence of the productions that satisfy the entropy principle. Moreover we have verified that  $h^0$  is a convex function of the densities  $\mathbf{u} \equiv \mathbf{F}^0$ . On the other hand the system (3) is of mixed type, some equations are conservation laws and the other ones are real balance laws, i.e., we are in the case in which

$$\mathbf{F}(\mathbf{u}) \equiv \begin{pmatrix} 0 \\ \mathbf{g}(\mathbf{u}) \end{pmatrix}; \quad \mathbf{g} \in \mathbb{R}^{N-M}.$$

In this case the coupling condition discovered for the first time by Shizuta-Kawashima (K-condition) [16] plays a very important role in the analysis of global existence of smooth solutions. If it is satisfied the dissipation present in the second block of equations (balance laws) have effect also on the first block of equations (conservation laws). Hanouzet and Natalini [17] in one-space dimension and Wen-An Yong [18] in the multidimensional case, have proved the following theorem:

*Assume that the system (4) is strictly dissipative and the K-condition is satisfied. Then there exists  $\delta > 0$ , such that, if  $\|\mathbf{u}(x, 0)\|_2 \leq \delta$ , there is a unique global smooth solution, which verifies*

$$\mathbf{u} \in \mathcal{C}^0([0, \infty); H^2(\mathbb{R})) \cap \mathcal{C}^1([0, \infty); H^1(\mathbb{R})).$$

Recently, Ruggeri and Serre [19] have proved in the one-dimensional case the stability of constant states:

*Under natural hypotheses of strongly convex entropy, strict dissipativeness, genuine coupling and “zero mass” initial for the perturbation of the equilibrium variables, the constant solution stabilizes*

$$\|\mathbf{u}(t)\|_2 = O(t^{-1/2}).$$

Lou and Ruggeri [20] have observed that exists a weaker form of K-condition, valid only for the genuine non-linear eigenvalues, that is a necessary (but, in general, not sufficient) condition for the global existence of smooth solutions.

### 3.4 The K-condition in the mixture theories

It is important to observe that in general, under the same initial data, the solutions of the subsystem are not particular solutions of the full system! Therefore also if the  $ST$  appears a particular physical case of the  $MT$ , the solutions starting from the same initial data (with  $T_\alpha(\mathbf{x}, 0) = T(\mathbf{x}, 0) = T_0(\mathbf{x})$ ) are different and with different regularity. In fact, for  $ST$  theory without chemical reactions it was proven [21], [22] that the K-condition is violated also for some genuinely non linear eigenvalues. Therefore, taking into account the results [20], in general, global smooth solutions do not exist, even if the initial data are small enough. Instead, for  $MT$  system it is possible to verify that the K-condition is satisfied for all eigenvalues. This means, roughly speaking, that the dissipation in  $ST$  is too weak with respect to the hyperbolicity and we do not have global smooth solutions for all the time; instead if we add the multi-temperature effect together with mechanical diffusion, the dissipation becomes enough to win the effect of hyperbolicity. Therefore we can conclude [5]:

*If the initial data of the  $MT$  model are perturbations of equilibrium state, smooth solutions exist for all time and tends to the equilibrium constant state.*

Also from this point of view the  $MT$  model provides a description more realistic than the  $ST$  model.

## 4 Average temperature

The ( $MT$ ) theory is of course the most realistic one and also in agreement with the kinetic theory [23] and it is a necessary theory in several physical situations, in particular in plasma physics [3]. Nevertheless, from the theoretical point of view, the main problem remains how it is possible to measure the temperatures of each constituent. Therefore, a question

of definition of a macroscopic average temperature has to be posed. In this paper we reconsider the definition of average temperature recently proposed by Ruggeri and co-workers [24]-[28]. The main idea is to exploit the definition of internal energy to introduce (average) temperature  $T$  as a state variable for the mixture, and to do it in such a way that the intrinsic internal energy  $\varepsilon_I$  (see (2)<sub>3</sub>) of the multi-temperature mixture resembles the structure of intrinsic internal energy of a single-temperature mixture. Therefore, *the following implicit definition of an average temperature is adopted*:

$$\rho\varepsilon_I(\rho_\beta, T) = \sum_{\alpha=1}^n \rho_\alpha \varepsilon_\alpha(\rho_\alpha, T) = \sum_{\alpha=1}^n \rho_\alpha \varepsilon_\alpha(\rho_\alpha, T_\alpha), \quad (20)$$

By expanding this relation in the neighborhood of the average temperature we have:

$$T = \frac{\sum_{\alpha=1}^n \rho_\alpha c_V^{(\alpha)} T_\alpha}{\sum_{\alpha=1}^n \rho_\alpha c_V^{(\alpha)}}, \quad (21)$$

where

$$c_V^{(\alpha)} = \left. \frac{\partial \varepsilon_\alpha(\rho_\alpha, T_\alpha)}{\partial T_\alpha} \right|_{T_\alpha=T}$$

is the specific heat at constant volume of constituent  $\alpha$ . We observe that Eq. (21) gives the exact value of the average temperature in the case of the mixture of ideal gases for which the  $c_V^{(\alpha)}$  are constant.

This definition of average temperature has several advantages with respect to usual ones used in the literature, as we can see in the following. First, as a consequence of the definition, the conservation law for the energy of mixture (3)<sub>3</sub> becomes an evolution equation for the average temperature  $T$  as in the case of  $(ST)$  and  $(CT)$ . So, in the case of spatial homogeneous solution of the differential system (3) (whose solutions only depend on time),  $T$  is constant and all the non-equilibrium temperatures of each constituent  $T_\alpha$  converge to  $T$  for large time, as we can see in the sequel.

The second advantage is related to the entropy of the whole mixture that, thanks to the introduction of this average temperature, reaches its maximum value when  $T_\alpha = T$ . In fact Ruggeri and Simić have proved in [26], using the Gibbs equations for each constituent, that the entropy



density near the equilibrium becomes a negative definite quadratic form of the non-equilibrium variables  $\Theta_\alpha = T_\alpha - T$  (*diffusion temperature flux*):

$$\begin{aligned} \rho S &= \sum_{\alpha=1}^n \rho_\alpha S_\alpha(\rho_\alpha, T_\alpha) = \\ &= \sum_{\alpha=1}^n \rho_\alpha S_\alpha(\rho_\alpha, T) - \frac{1}{2T^2} \sum_{\alpha=1}^n \rho_\alpha c_V^{(\alpha)} \Theta_\alpha^2 + O(\Theta_\alpha^3). \end{aligned} \quad (22)$$

#### 4.1 Alternative form of the differential system

It is convenient for the following to rewrite the system (3) using the material derivatives

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \quad \frac{d_b}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_b \cdot \nabla$$

for  $b = 1, \dots, n-1$ , and taking into account the definition of the average temperature given by (20):

$$\left\{ \begin{array}{l} \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0, \\ \rho \frac{d\mathbf{v}}{dt} - \operatorname{div} \mathbf{t} = 0, \\ \rho \frac{\partial \varepsilon_I}{\partial T} \frac{dT}{dt} = \rho^2 \frac{\partial \varepsilon_I}{\partial \rho} \operatorname{div} \mathbf{v} + \sum_{b=1}^{n-1} \frac{\partial \varepsilon_I}{\partial c_b} \operatorname{div} \mathbf{J}_b + \mathbf{t} \cdot \operatorname{grad} \mathbf{v} - \operatorname{div} \mathbf{q} \\ \frac{d_b \rho_b}{dt} + \rho_b \operatorname{div} \mathbf{v}_b = 0 \\ \rho_b \frac{d_b \mathbf{v}_b}{dt} - \operatorname{div} \mathbf{t}_b = \hat{\mathbf{m}}_b, \\ \rho_b \frac{d_b \varepsilon_b}{dt} - \mathbf{t}_b \cdot \nabla \mathbf{v}_b + \operatorname{div} \mathbf{q}_b = \hat{e}_b. \end{array} \right. \quad (23)$$

These differential equations give the evolution equations respectively of  $\rho$ ,  $\mathbf{v}$ ,  $T$ ,  $\rho_b$ ,  $\mathbf{J}_b$  and  $\Theta_b$ , provided we assign the constitutive equations

$p_\alpha, \varepsilon_\alpha$  and for dissipative fluids also the heat fluxes  $\mathbf{q}_\alpha$  and the viscous stress tensors  $\sigma_\alpha$ .

## 5 Examples of spatially homogenous mixture and static heat conduction

In this section two simple examples will be provided in order to support previous theoretical considerations, and to stress the main features of multi temperature approach and the role of the average temperature.

### 5.1 Solution of a spatially homogenous mixture

First we consider a non-reacting mixture of gases in the special case of spatially homogeneous fields, i.e. the case in which field variables depend solely on time [27], [26]. The governing equations (23) can be written in the following form:

$$\frac{d\rho}{dt} = 0; \quad \frac{d\mathbf{v}}{dt} = \mathbf{0}; \quad \frac{dT}{dt} = 0; \quad (24)$$

$$\frac{d\rho_b}{dt} = 0; \quad \rho_b \frac{d\mathbf{v}_b}{dt} = \hat{\mathbf{m}}_b; \quad \rho_b \frac{d\varepsilon_b}{dt} = \hat{e}_b. \quad (25)$$

where now  $d/dt = \partial/\partial t$ . From (24), (25), it is easy to conclude:

$$\begin{aligned} \rho &= \text{const.}; & \mathbf{v} &= \text{const.}; & T &= \text{const.}; \\ \rho_b &= \text{const.}; & b &= 1, \dots, n, \end{aligned}$$

and due to Galilean invariance we may choose  $\mathbf{v} = \mathbf{v}_0 = \mathbf{0}$  without loss of generality. It is also remarkable that the average temperature of the mixture remains constant during the process:  $T(t) = T_0$ .

In the sequel we shall regard only small perturbations of equilibrium state,  $\mathbf{v}_\alpha = \mathbf{v}_0 = \mathbf{0}$ ,  $T_\alpha = T_0$ ,  $\alpha = 1, \dots, n$ , and analyze their behavior. Therefore, the r.h.s. of (25)<sub>2,3</sub> could be linearized in the neighborhood of equilibrium and taking into account (8), we obtain:

$$\rho_b \frac{d\mathbf{v}_b}{dt} = - \sum_{c=1}^{n-1} \frac{\psi_{bc}^0}{T_0} (\mathbf{v}_c - \mathbf{v}_n); \quad (26)$$

$$\rho_b c_V^{(b)} \frac{dT_b}{dt} = - \sum_{c=1}^{n-1} \frac{\theta_{bc}^0}{T_0^2} (T_c - T_n), \quad (27)$$

where  $\psi_{bc}^0$  and  $\theta_{bc}^0$  are entries of positive definite matrices evaluated in equilibrium. Note that  $\mathbf{v}_b - \mathbf{v}_n = \mathbf{u}_b - \mathbf{u}_n$  and  $T_b - T_n = \Theta_b - \Theta_n$ . For what concerns the  $n$ -species  $\mathbf{v}_n$  and  $T_n$  are obtained by the algebraic equations (2)<sub>2</sub> and (21).

In the particular case of a binary mixture the explicit solution of equations (26), (27), can be obtained and it reads:

$$\mathbf{v}_1(t) = \mathbf{v}_1(0)e^{-\frac{t}{\tau_v}}; \quad T_1(t) = T_0 + (T_1(0) - T_0)e^{-\frac{t}{\tau_T}},$$

where  $\tau_v$  and  $\tau_T$  represent relaxation times that for ideal gas assume the expression

$$\tau_v = \frac{\rho_1 \rho_2 T_0}{\psi_{11}^0 \rho};$$

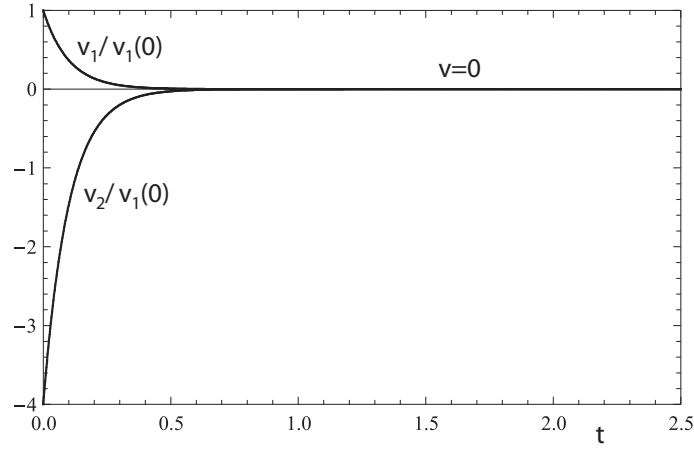
$$\tau_T = \frac{k \rho_1 \rho_2 T_0^2}{\theta_{11}^0 (\rho_1 m_2 (\gamma_2 - 1) + \rho_2 m_1 (\gamma_1 - 1))}.$$

Starting from these solutions, other field variables can be obtained by means of defining equations:

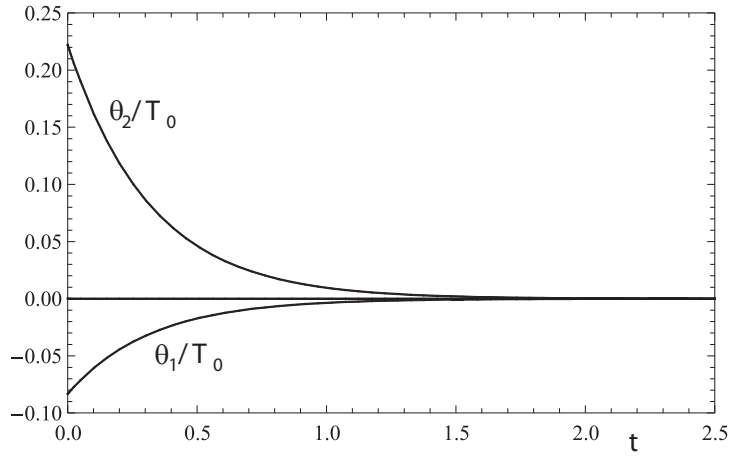
$$\begin{aligned} \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2 &= \rho \mathbf{v} = \mathbf{0}. \\ \rho_1 c_V^{(1)} T_1 + \rho_2 c_V^{(2)} T_2 &= (\rho_1 c_V^{(1)} + \rho_2 c_V^{(2)}) T \\ &= (\rho_1 c_V^{(1)} + \rho_2 c_V^{(2)}) T_0. \end{aligned}$$

It is obvious that, due to dissipative character of the system, *all the non-equilibrium variables exponentially decay and converge to their equilibrium values*. In order to compare the values of  $\tau_v$  and  $\tau_T$  for ideal gases, and also to compute the actual values of variables in numerical example, the relations from kinetic theory has to be recalled [3]:

$$\theta_{11}^0 = \frac{3m_1 m_2}{(m_1 + m_2)^2} k T_0^2 \Gamma'_{12}; \quad \psi_{11}^0 = \frac{2m_1 m_2}{m_1 + m_2} T_0 \Gamma'_{12},$$



(a) Velocities



(b) Fluxes

Figure 1: Dimensionless velocities and diffusion temperature fluxes of the constituents versus time.

where  $\Gamma'_{12}$  represents volumetric collision frequency, and the following estimate can be obtained:

$$\frac{\tau_T}{\tau_v} = \frac{2}{3} \frac{\rho(m_1 + m_2)}{\rho_1 m_2 (\gamma_2 - 1) + \rho_2 m_1 (\gamma_1 - 1)} > \frac{2}{3(\gamma_{\max} - 1)} \geq 1, \quad (28)$$

$$(\gamma_{\max} = \max\{\gamma_1, \gamma_2\} \leq 5/3).$$

In Figures 1, we present the graphs of normalized velocities, diffusion temperature fluxes [26]. It can be observed that, due to inequality (28), *the mechanical diffusion vanishes more rapidly than the thermal one*. This is in sharp contrast with widely adopted approach which ignores the influence of multiple temperature of each constituent of the mixture.

## 5.2 Static heat conduction solution

Another simple example is the one-dimensional mixture of gas at rest ( $\mathbf{v}_\alpha = 0$ ), without chemical reactions ( $\tau_\alpha = 0$ ) between two walls  $0 \leq x \leq L$ , maintained at two different temperatures  $T(0) = T_0, T(L) = T_L$  [28].

In both (CT) and (ST), the static field equation reduces to the global energy equation (23)<sub>3</sub> that reads  $\text{div } \mathbf{q} = 0$ . In the one-dimensional case, this equation combined with the Fourier law with constant heat conductivity, yields the classical result of a linear behavior temperature profile as for a single fluid:

$$T'' = 0 \iff T = (T_L - T_0)\xi + T_0$$

with  $\xi = x/L$  and ' denotes  $d/d\xi$ . For what concerns the densities they are obtained by the conditions that the pressure of each constituent must be constant due to the momentum equations.

In the (MT) model, the situation is quite different. In fact if we consider the simple case of a binary mixture ( $n = 2$ ), by taking into account of Eqs. (23), and system (8), in the linear case, is reduced to

$$\begin{cases} \frac{dp_1}{dx} = 0, & \frac{dp_2}{dx} = 0, \\ \frac{dq_1}{dx} = \beta (T_2 - T_1), \\ \frac{dq_2}{dx} = \beta (T_1 - T_2), \end{cases} \quad (29)$$

where  $\beta = \theta_{11}/T_0^2$ . By using the Fourier law, Eqs. (29)<sub>2,3</sub> can be rewritten as

$$\begin{cases} T_1'' = \nu_1(T_1 - T_2), \\ T_2'' = \nu_2(T_2 - T_1), \end{cases} \quad (30)$$

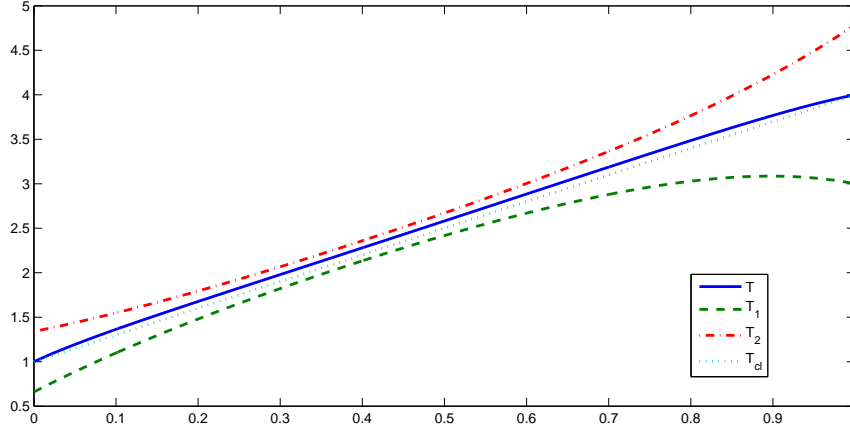


Figure 2: Graphs of the average temperature  $T$  and constituent temperatures  $T_1, T_2$  in terms of the dimensionless distance  $x/L$ .  $T_d$  represents the classical straight line solution.  $T_0$  is the temperature unit.

where we assume that the dimensionless quantities

$$\nu_1 = \frac{\beta L^2}{\chi_1}, \quad \nu_2 = \frac{\beta L^2}{\chi_2}, \quad (31)$$

are constant. The system (30) is equivalent to

$$\widehat{T}'' = 0, \quad \Theta'' - \omega^2 \Theta = 0$$

with  $\widehat{T} = \nu T_1 + (1 - \nu)T_2$ ,  $\Theta = T_2 - T_1$  and

$$\nu = \frac{\nu_2}{\nu_1 + \nu_2} = \frac{\chi_1}{\chi_1 + \chi_2}, \quad \omega = \sqrt{\nu_1 + \nu_2}.$$

Consequently, we get the solution in the form

$$T_1 = \widehat{T} - (1 - \nu)\Theta, \quad T_2 = \widehat{T} + \nu\Theta \quad (32)$$

with

$$\widehat{T} = A\xi + B, \quad \Theta = \frac{1}{\sinh(\omega)} \{ \Theta_L \sinh(\omega\xi) + \Theta_0 \sinh(\omega(1 - \xi)) \}, \quad (33)$$

and  $A, B, \Theta_0, \Theta_L$  are constants of integration. In the case of ideal gas, equations (29)<sub>1</sub> and (19) yield the constance of the internal energy densities of each constituent:

$$\rho_\alpha T_\alpha c_V^{(\alpha)} = P_\alpha = \text{Const.}, \quad (\alpha = 1, 2). \quad (34)$$

and Eq. (21) yield the average temperature

$$\frac{1}{T} = \frac{\pi}{T_1} + \frac{1 - \pi}{T_2}, \quad \text{with } \pi = \frac{P_1}{P_1 + P_2}. \quad (35)$$

The constant  $\pi$  belongs to  $[0, 1]$ . It is interesting to observe that the *coldness*  $1/T$  (inverse of the average temperature) belongs to the convex envelope of the component coldness  $1/T_1$  and  $1/T_2$ . Equations (32), (33) and (35) give the explicit solution of  $T_1, T_2$  and  $T$  as function of  $\xi$  and five constants of integration:  $(A, B, \Theta_0, \Theta_L, \pi)$ . We observe the behavior of  $T$  is not a straight line as in the classic case of  $(CT)$  or  $(ST)$  theories; the multi-temperature effect is that the temperature is not a linear function of  $x$  (see Figure 2). Due to Eqs. (29)-(31), when  $\epsilon = 1/\beta$  tends towards zero, the solution of Eqs. (32), (33) and (35) converges towards the classical solution  $T_1 = T_2 = T = \bar{T}$  for any  $\xi \in ]0, 1[$ . This result is true also at the boundary when  $\Theta_0$  and  $\Theta_1$  are of same order as  $\epsilon$ .

Let  $c = \rho_1/(\rho_1 + \rho_2)$  the concentration and  $c(0) = c_0$ . Equations (34), (32), (33), imply

$$c = \frac{c_0}{c_0 + \Omega(1 - c_0)}, \quad \text{with } \Omega = \frac{T_1 T_{20}}{T_2 T_{10}}$$

and

$$T_{10} = B - (1 - \nu)\Theta_0, \quad T_{20} = B + \nu\Theta_0.$$

The concentration is function of the position  $x$  whereas in classical case  $\Omega = 1$  and  $c = c_0$ . Ruggeri and Lou [28] have studied how is possible to determine in a unique way the constant of integration and they proved that for a mixture of  $n$  constituents the measure of the average temperature at  $2(n - 1)$  points allows to know the temperature behavior of each constituent in all points.

## 6 Maxwellian iteration

To find a connection between the extended models of  $MT$  or  $ST$  with the classical theory  $CT$  it is necessary to use the procedure that is known as *Maxwellian iteration* (see [2]). In the present case the Maxwellian iterative procedure requires to put in the left side of the system (23)<sub>5,6</sub> the zero-th iterate, i.e. the equilibrium state and in the right side the first iterate. Taking into account that in zeroth iteration  $\mathbf{v}_\alpha^{(0)} = \mathbf{v}$  and consequently  $d_b^{(0)}/dt = d/dt$ ,  $\mathbf{J}_b^{(0)} = \mathbf{u}_b^{(0)} = \mathbf{0}$  and moreover  $T_\alpha^{(0)} = T$ ,  $\mathbf{q}^{(0)} = \mathbf{q}_b^{(0)} = \mathbf{0}$ ,  $\mathbf{t}^{(0)} = -p^{(0)}\mathbf{I} = -p_0\mathbf{I}$ ,  $\mathbf{t}_b^{(0)} = -p_b^{(0)}\mathbf{I}$ , we obtain after some straightforward calculations, the Fick law for what concerns the momentum equations (11)<sub>3</sub>, while for what concerns the energy equations we obtain a new constitutive equations:

$$\Theta_a^{(1)} = -k_a \operatorname{div} \mathbf{v}. \quad (36)$$

Equation (36), obtained by means of Maxwellian iteration gives the temperatures of each species as a constitutive equation in the same manner as the Fick law gives the velocities of each species.

For a mixture of ideal gases  $k_a$  is a linear combination of

$$\Omega_b = \rho_b T c_V^{(b)} \frac{\sum_{\alpha=1}^n \rho_\alpha c_V^{(\alpha)} (\gamma_b - \gamma_\alpha)}{\sum_{\alpha=1}^n \rho_\alpha c_V^{(\alpha)}}.$$

When all the constituents have the same ratio of specific heats we have  $\Omega_b = 0$ , and consequently  $\Theta_b^{(1)} = 0$ ,  $a = 1, \dots, n-1$ . In this case the diffusion temperature flux cannot be observed in the first approximation.

## 7 A classical approach of multi-temperature mixtures

Therefore the  $ST$  model using the Maxwellian iteration converges to the classic theory but when we start from the  $MT$  case we obtain a new constitutive equation that is not present in the classical approach of mixture theory.

To justify the results of the Maxwellian iteration, Gouin and Ruggeri [25] have constructed a classical theory of mixture with multi-temperature.



The idea is to use the usual equation of the classical approach (9), but now we suppose that each constituent has its own temperature. In this approach the role of the average temperature previously defined by (20) is fundamental. In fact the multi-temperature effect appears through the pressure, that now near equilibrium has the form:

$$p = p_0 + \pi_\theta,$$

where

$$p_0 = \sum_{\alpha=1}^n p_\alpha(\rho_\alpha, T), \quad \pi_\theta = \sum_{b=1}^{n-1} r_b \Theta_b$$

and

$$r_b = \frac{1}{\rho_n c_V^{(n)}} \left\{ \rho_n c_V^{(n)} \frac{\partial p_b}{\partial T_b}(\rho_b, T) - \rho_b c_V^{(b)} \frac{\partial p_n}{\partial T_n}(\rho_n, T) \right\}. \quad (37)$$

Therefore, the total pressure  $p$  of the mixture is the sum of the equilibrium part  $p_0$  depending on  $T$  and a new dynamical pressure part (as a non-equilibrium term)  $\pi_\theta$  due to the difference of temperatures between the constituents.

We assume the internal energy  $\varepsilon(\rho, T, c_b)$  and the equilibrium pressure  $p_0(\rho, T, c_b)$  satisfy the Gibbs equation

$$T dS = d\varepsilon - \frac{p_0}{\rho^2} d\rho - \sum_{b=1}^{n-1} (\mu_b - \mu_n) dc_b. \quad (38)$$

The differences between Eq. (10) and Eq. (38) consist in the fact that in Eq. (38)  $T$  means the average temperature when each component  $\alpha$  has its own temperature  $T_\alpha$  and  $p_0$  takes the place of  $p$ . Let us consider first the case of a Stokesian fluid  $\text{tr } \sigma = 0$ . Taking into account that from (22)  $\rho S$  until first order expansion depends only on the average temperature and using Eqs. (9) to eliminate the time derivative, we obtain from (38) the following entropy balance:

$$\rho \frac{dS}{dt} + \text{div} \left\{ \frac{1}{T} \left( \mathbf{q} - \sum_{b=1}^{n-1} (\mu_b - \mu_n) \mathbf{J}_b \right) \right\} =$$

$$\mathbf{q} \cdot \text{grad} \left( \frac{1}{T} \right) - \sum_{b=1}^{n-1} \mathbf{J}_b \cdot \text{grad} \left( \frac{\mu_b - \mu_n}{T} \right) + \frac{1}{T} \text{tr} (\mathbf{J}_{mech} \mathbf{D}), \quad (39)$$

where the mechanical flux is

$$\mathbf{J}_{mech} = \sigma - \pi_\theta \mathbf{I}.$$

Eq. (39) can be interpreted as a balance of entropy, if we accept

$$\Phi = \frac{1}{T} \left( \mathbf{q} - \sum_{b=1}^{n-1} (\mu_b - \mu_n) \mathbf{J}_b \right)$$

and

$$\Sigma = \mathbf{q} \cdot \text{grad} \left( \frac{1}{T} \right) - \sum_{b=1}^{n-1} \mathbf{J}_b \cdot \text{grad} \left( \frac{\mu_b - \mu_n}{T} \right) + \text{tr} \left( \mathbf{J}_{mech} \frac{\mathbf{D}}{T} \right) \quad (40)$$

as the entropy flux and the entropy production respectively.

We observe that the entropy production is the sum of products of the following quantities:

| <b>thermodynamic fluxes</b>         | <b>thermodynamic forces</b>  |
|-------------------------------------|--|
| heat flux $\mathbf{q}$              | temperature gradient $\text{grad} \left( \frac{1}{T} \right)$ ,                      |
| diffusion fluxes $\mathbf{J}_b$     | chemical potential gradients $-\text{grad} \left( \frac{\mu_b - \mu_n}{T} \right)$ , |
| mechanical flux $\mathbf{J}_{mech}$ | velocity gradient $\frac{\mathbf{D}}{T}$ .   |

In accordance with the case of a single temperature model [29] and [2] chapter 5, in *TIP* near equilibrium, the fluxes depend linearly on the associated forces (see also for the general methodology of *TIP* [6, 7, 30, 31]):

- For the heat flux and the diffusion fluxes, we obtain the constitutive equations in the form of Eqs (11)<sub>2,3</sub>.

- For Stokesian fluids, the last term of Eq. (40) corresponding to the mechanical production of entropy can be written in a separated form

$$\frac{1}{T} \text{tr} (\mathbf{J}_{mech} \mathbf{D}) = \frac{1}{T} \text{tr} (\sigma \mathbf{D}^D) - \frac{1}{T} \pi_\theta \text{div} \mathbf{v}.$$

We obtain the constitutive equation of the viscous stress tensor in the form of Eq. (11)<sub>1</sub> (with  $\lambda = 0$ ), and the dynamical pressure part due to

the difference of temperatures yields

$$\pi_\theta = \sum_{b=1}^{n-1} r_b \Theta_b = -L_\pi \operatorname{div} \mathbf{v}, \quad (41)$$

where  $L_\pi$  is a scalar coefficient of proportionality.

The production of entropy must be non-negative (40) and therefore the *phenomenological coefficients* must satisfy the inequalities (12) and

$$L_\pi \geq 0.$$

Taking into account that terms  $r_b$  given by Eq. (37) depend on  $(\rho_b, T)$ , from Eq. (41) we deduce that constitutive quantities  $\Theta_a$  (depending *a priori* on  $\nabla \mathbf{v}$ ) must be proportional to  $\operatorname{div} \mathbf{v}$ :

$$\Theta_a = -k_a \operatorname{div} \mathbf{v} \quad (a = 1, \dots, n-1).$$

This in perfect agreement with the Maxwellian iteration procedure presented in the previous section.

Let  $\|M_{ab}\|$  be the matrix such that  $k_a = \sum_{b=1}^{n-1} M_{ab} r_b$ , we have

$$\Theta_a = - \sum_{b=1}^{n-1} M_{ab} r_b \operatorname{div} \mathbf{v} \quad (a = 1, \dots, n-1). \quad (42)$$

Introducing expression (42) into Eq. (41), we obtain

$$L_\pi = \sum_{a,b=1}^{n-1} M_{ab} r_a r_b \geq 0,$$

and assuming the Onsager symmetry property,  $M_{ab} = M_{ba}$  ( $a, b = 1, \dots, n-1$ ), we deduce that coefficients  $M_{ab}$  are *associated with a positive definite quadratic form*.

Finally, the results are the same as in the classical theory, but moreover we get new constitutive equations (42) for the difference of temperatures.

We have considered the simple case of Stokes fluids. If the fluid is non Stokesian, the Navier-Stokes stress tensor of viscosity is given by (11)<sub>1</sub> where  $\lambda$  is the bulk viscosity. The stress tensor  $\mathbf{t}$  becomes

$$\mathbf{t} = -(p_0 + \pi_\theta) \mathbf{I} + \sigma = -p \mathbf{I} + 2\nu \mathbf{D}^D,$$

with

$$p = p_0 + \pi_\theta + \pi_\sigma.$$

The non-equilibrium pressure  $p - p_0$  is separated in two different parts. The first one  $\pi_\sigma = -\lambda \operatorname{div} \mathbf{v}$  is related to the bulk viscosity and the second one  $\pi_\theta = -L_\pi \operatorname{div} \mathbf{v}$  is related to the multi-temperature effects between components. Due to a non-zero dynamical pressure even for Stokes fluids, we conclude that multi-temperature mixtures of fluids have a great importance. Perhaps such a model may be used to analyze the evolution of the early universe in which the dynamical pressure seems essential [32, 33].

## 8 Conclusions

In this survey we have reconsidered the definition of an average temperature in the context of multi-temperature approach to the theory of mixtures of fluids. It was based upon the assumption that internal energy of the mixture should retain the same form as in a single-temperature approach. The supremacy of this definition is supported by a simple derivation of entropy maximization result in equilibrium as its consequence and the result that the average temperature remains constant for spatially homogenous mixture. Furthermore, by means of Maxwellian first iterative procedure we have derived constitutive equations for non-equilibrium variables, mechanical diffusion flux  $\mathbf{J}_a$  and diffusion temperature flux  $\Theta_a$ , in the neighborhood of equilibrium. It was shown that the first iterate of diffusion flux  $\mathbf{J}_a^{(1)}$  coincides with the classical generalized Fick's laws which can be obtained in TIP. However, diffusion temperature flux  $\Theta_a^{(1)}$  is found to be proportional to  $\operatorname{div} \mathbf{v}$  – a new result which is in accordance with recent observations within classical TIP framework of multi-temperature mixture [25].

Moreover we have studied in the case of the one-dimensional steady heat conduction between two walls. We have verified that the main effect of multi-temperature is that the average temperature is not a linear function of the distance as in the case of the *ST* theory. These last result could be used during some experiments to show the order of magnitude of the difference between *ST* and *MT*.

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## Višetemperaturna mešavina fluida

Daje se pregled nekih nedavnih rezultata koji se odnose na različite modele smese stivšljivih fluida. Posebno se diskutuje najrealniji slučaj smese kada svaki sastojak ima svoju temperaturu ( $MT$ ), te se prvo upoređuju rešenja ovog modela sa modelom koji ima jedinstvenu zajedničku temperaturu ( $ST$ ). U slučaju Ojlerovskih fluida se zatim pokazuje da je odgovarajući ( $ST$ )-diferencijalni sistem neki *glavni podsistem* of the ( $MT$ )-sistema. Globalno ponašanje glatkih rešenja za velika vremena oba sistema se tada diskutuje primenom Shizuta-Kawashima uslova.

Tada se uvodi koncept srednje temperature smese zasnovan na razmatranju da je unutrašnja energija smese ista kao u slučaju ( $ST$ )-smese. Kao posledica, pokazano je da entropija smese dostiže lokalni maksimum u ravnoteži. Postupkom Meksvelovske iteracije dobija se nova konstitutivna jednačina za neravnotežne temperature sastojaka u klasičnom graničnom slučaju zajedno sa Fikovim zakonom za fluks difuzije.

Konačno, da bi Meksvelovska iteracija bila zadovoljena, daje se za disipativne fluide jedan moguć pristup klasične teorije smesa sa više temperatura, te se pokazuje da razlike temperatura sastojaka uzrokuju postojanje novog *dinamičkog pritiska* čak i ako fluidi imaju nultu zapreminsku viskoznost.