Theoret. Appl. Mech., Vol.36, No.4, pp. 261-274, Belgrade 2009

Unsteady natural convection flow past an accelerated vertical plate in a thermally stratified fluid

Rudra Kt. Deka * **Bhaben Ch. Neog*** †

Abstract

An exact solution to one-dimensional unsteady natural convection flow past an infinite vertical accelerated plate, immersed in a viscous thermally stratified fluid is investigated. Pressure work term and the vertical temperature advection are considered in the thermodynamic energy equation. The dimensionless governing equations are solved by Laplace Transform techniques for the Prandtl number unity. The velocity and temperature profiles as well as the skin-friction and the rate of heat transfer are presented graphically and discussed the effects of the Grashof number Gr, stratification parameter S at various times t.

Keywords: Unsteady flow, vertical accelerated plate, thermal stratification.

*Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India. e-mail: rkdgu@yahoo.com

†2 Department of Mathematics, Jagiroad College, Jagiroad, Morigaon, Assam, India. e-mail: neogbc@gmail.com

Nomenclature:

A	Constant acceleration
C_p	Specific heat at constant pressure
g	Acceleration due to gravity
Gr	Grashof number
K	Thermal conductivity of the fluid
Pr	Prandtl number
S	Non-dimensional Stratification parameter
T'_w	Temperature at the plate
T'_∞	Temperature of the mainstream fluid.
t'	Time
t	Dimensionless Time
x'	Co-ordinate normal to the plate
w'	Velocity of fluid in the z$'$ direction
W	Non-dimensional vertical velocity
α	Thermal diffusivity
β	Co-efficient of thermal expansion
μ	Co-efficient of viscosity.
ν	Kinematic viscosity
ϱ	Fluid density
γ	Thermal stratification parameter.
θ	Dimensionless temperature
τ	Dimensionless skin-friction
ξ	Dimensionless co-ordinate normal to the plate

1 Introduction

Thermal stratification of fluid is a natural process that takes place at higher temperature in which a warmer and less dense layer overlies a colder but denser layer. It occurs mainly because of temperature variations, concentration differences or for the presence of different fluids of different density. This natural process creates a transition zone of temperature gradient between cold and hot fluid zones and hence in case of vertical natural convection it plays an important role in vertical temperature distribution. Recently, the dynamics of thermally stratified fluid has attracted attention of researchers and immersed as an important topic for scientific enquiry because of its wide spread applications in a number of industrial, engineering and environmental applications [1], [5], [10] and [7]. Inclusion of thermal stratification term in the energy equation in vertical equations of motion is quite a recent concept, as a result of which the flow situation comes out as steady at large times, whereas without thermal stratification it grows continuously against time.

Unsteady natural convection flows have been studied by several authors imposing different restrictions in flow fields. Simplest of them is the unsteady laminar one-dimensional natural convection flow along a vertical plate in which the momentum equation with Boussinesq approximation and the energy equation reduce to a set of linear partial differential equations, which can be used as benchmark for developing the results of computational fluid dynamics. Some of the pioneers in this field are Illingworth [18], Gebhart [4], Schetz and Eichhorn [2], Soundalgekar [19] etc. They obtained analytical solutions for unsteady one-dimensional natural convection flows along an infinite vertical plate for different surface conditions. Goldstein and Briggs [13] have obtained the solutions of unsteady natural convection flow from an impulsively heated circular cylinder. Soundalgekar [19] studied the Stoke's problem for flow past an impulsively started infinite vertical plate and also for vertical oscillating plate [20], while Das et. al. [17], and Soundalgekar et. al. [14] have obtained the one-dimensional solutions for natural convection flow with temporally periodic surface temperature and surface heat flux respectively.

In all these studies pressure work has been neglected and ambient thermal stratification was not considered. It was Hyun [16] and Lin and Armfield [21] who showed experimentally that an initially unstratified convective motion leads to self stratifications. Gill and Davey [9] and Bergholz [11] suggest that stratification may exert a stabilizing influence on convection flow past vertical plates. Shapiro and Fedorovich [3] have recently refined the classical theory of one-dimensional flow by introducing the stratification parameter in the energy equation. Introduction of thermal stratification leads to the coupling effect on both the momentum equation with Boussinesq approximation and the energy equation in boundary layer theory. They obtained analytical solutions for the cases of impulsive change in plate perturbation temperature, sudden application of plate heat flux, and for arbitrary temporal variations in plate perturbation or plate heat flux. They found that as a result of thermal stratification a negative feedback mechanism appears whereby warm fluid rises and cooled relative to the environment, while subsiding cool fluid warmed relative to the environment. They [8] again extended their work for the Prandtl number dependence convection of a stably stratified fluid along a vertical plate by regular perturbation method. Magyari et. al [10] studied the flow in a stably stratified fluid in porous medium.

In this paper it is proposed to study the thermal stratification effects on unsteady natural convection flow past an infinite vertical accelerated plate. Our main interest is to observe how stratification affects the flow past an infinite vertical accelerated plate. The solutions are obtained by Laplace transform technique for the velocity and temperature fields for Prandtl number unity and these are presented graphically for different values of parameters viz; Grashof number Gr, thermal stratification parameter S and time t. Skin friction and the rate of heat transfer are also presented graphically. We have also compared the solutions thus obtained with the solutions of the flow field without thermal stratification effects.

2 Mathematical Analysis

We consider a Cartesian co-ordinate system where z' -axis is in vertically upward direction, $y'z'$ -plane coincides with the vertical plate,x' -axis is horizontal, normal to the plate and fluid fills the region $x' \geq 0$. At time $t' > 0$, the plate is given an impulsive constant acceleration A and the plate temperature raised from the environment temperature T'_∞ to a constant wall temperature T'_w. The motion is one-dimensional with the only non-zero vertical velocity component w', varying with x' and t' only. Due to one-dimensional nature, the equation of continuity is trivially satisfied.

Thus the Boussinesq form of vertical equation of motion becomes

$$\frac{\partial w'}{\partial t'} = g\beta(T' - T'_\infty) + \nu \frac{\partial^2 w'}{\partial x'^2} \tag{1}$$

Corresponding energy equation with the inclusion of pressure work and vertical temperature advection [3], becomes-

$$\frac{\partial T'}{\partial t'} = -\gamma w' + \alpha \frac{\partial^2 T'}{\partial x'^2} \tag{2}$$

where $\gamma = \frac{dT'_\infty}{dz'} + \frac{g}{C_p}$. Here $\frac{dT'_\infty}{dz'}$ is thermal stratification and $\frac{g}{C_p}$ is the pressure work term. The environment is statically stable, neutral or unstable if $\gamma >, = or < 0$. We consider the cases of stable and neutral conditions only.

And the initial and boundary conditions are considered as follows:

$$\begin{aligned} w' = 0, \quad T' = T'_\infty \quad &at \qquad t' \leq 0 \\ w' = At', \quad T' = T'_w \quad &at \qquad x' = 0 \\ w' \to 0, \quad T' = T'_\infty \quad &as \qquad x' \to \infty \end{aligned} \tag{3}$$

where $A(> 0)$ is the constant acceleration.

Now with the introduction of following non-dimensional quantities,

$$\begin{gathered} \theta = \frac{T' - T'_\infty}{T'_w - T'_\infty},\ \xi = \frac{x' A^{1/3}}{\nu^{2/3}},\ t = \frac{t' A^{2/3}}{\nu^{1/3}},\ W = \frac{w'}{(A\nu)^{1/3}}, \\ Pr = \frac{\mu C_p}{K},\ S = \frac{\gamma \nu^{2/3}}{A^{1/3}(T'_w - T'_\infty)},\ Gr = \frac{b\beta(T'_w - T'_\infty)}{A} \end{gathered} \tag{4}$$

equations (1) and(2) reduce to:

$$\frac{\partial W}{\partial t} = Gr\theta + \frac{\partial^2 W}{\partial \xi^2} \tag{5}$$

$$\frac{\partial \theta}{\partial t} = -SW + \frac{1}{Pr} \frac{\partial^2 \theta}{\partial \xi^2} \tag{6}$$

And the initial and boundary conditions (3) then reduce to:

$$
\begin{aligned}
W = \theta = 0 \quad forall \quad \xi, \quad t \leq 0 \\
W = t, \quad \theta = 1 \qquad at \quad \xi = 0 \\
W \to 0, \quad \theta \to 0 \quad as \quad \xi \to \infty
\end{aligned} \tag{7}
$$

2.1 Method of Solutions

We use the Laplace Transform techniques to solve the above coupled equations. Thus eliminating the transformed velocity $\bar{W}$ from the resulting transformed equations, we obtain a fourth order linear differential equation in $\bar{\theta}$ as follows:

$$
[D^4 - (Pr+1)sD^2 + Pr(s^2 + SGr)]\bar{\theta} = 0 \quad whereD = \frac{d}{d\xi} \tag{8}
$$

Now to determine the exact solutions of the equations we assume $Pr = 1$. Thus, taking account of the Laplace Transform of (7), we obtained the expressions for $\bar{\theta}$ and $\bar{W}$ as follows:

$$
\bar{\theta} = \frac{1}{2}\left[\frac{e^{-\xi\sqrt{(s+iQ)}}}{s} + \frac{e^{-\xi\sqrt{(s-iQ)}}}{s}\right] + \frac{iS}{2Q}\left[\frac{e^{-\xi\sqrt{(s-iQ)}}}{s^2} - \frac{e^{-\xi\sqrt{(s+iQ)}}}{s^2}\right] \tag{9}
$$

$$
\bar{W} = \frac{1}{2}\left[\frac{e^{-\xi\sqrt{(s+iQ)}}}{s^2} + \frac{e^{-\xi\sqrt{(s-iQ)}}}{s^2}\right] + \frac{iQ}{2S}\left[\frac{e^{-\xi\sqrt{(s+iQ)}}}{s} - \frac{e^{-\xi\sqrt{(s-iQ)}}}{s}\right] \tag{10}
$$

Using Hetnarski's [6] algorithm and [12] for inverse Laplace Transform of $\bar{\theta}$ and $\bar{W}$ we obtain the solutions for θ and W as follows:

$$
\begin{aligned}
\theta = & \frac{1}{4}\left[e^{\xi\sqrt{iQ}} erfc\left(\frac{\xi}{2\sqrt{t}} + \sqrt{iQt}\right)\left\{1 - \frac{iSt}{Q} - \frac{\sqrt{i}S\xi}{2Q^{3/2}}\right\}\right] \\
& + \frac{1}{4}\left[e^{\xi\sqrt{iQ}} erfc\left(\frac{\xi}{2\sqrt{t}} + \sqrt{iQt}\right)\left\{1 - \frac{iSt}{Q} - \frac{\sqrt{i}S\xi}{2Q^{3/2}}\right\}\right] \quad + cc
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
W = & \frac{1}{4}\left[e^{-\xi\sqrt{iQ}} erfc\left(\frac{\xi}{2\sqrt{t}} - \sqrt{iQt}\right)\left\{t + \frac{iQ}{S} - \frac{\xi}{2\sqrt{iQ}}\right\}\right] \\
& + \frac{1}{4}\left[e^{\xi\sqrt{iQ}} erfc\left(\frac{\xi}{2\sqrt{t}} + \sqrt{iQt}\right)\left\{t + \frac{iQ}{S} + \frac{\xi}{2\sqrt{iQ}}\right\}\right] \quad + cc
\end{aligned} \tag{12}
$$

where 'cc' is the complex conjugate and $Q = \sqrt{GrS}$. The non-dimensional skin-friction and the rate of heat transfer i.e. the Nusselt number at the plate are obtained as follows:

$$\tau = \frac{1}{\sqrt{\pi t}}\left\{t\, cos(Qt) + \frac{Q}{S} sin(Qt)\right\}$$
$$+\frac{1}{2}\left\{\sqrt{iQ}\left(t + \frac{iQ}{S}\right) + \frac{1}{2\sqrt{iQ}}\right\} erf\left(\sqrt{iQt}\right) + cc \qquad (13)$$

$$Nu = \frac{1}{\sqrt{\pi t}}\left\{cos(Qt) - \frac{St}{Q} sin(Qt)\right\}$$
$$+\frac{1}{2}\left\{\sqrt{iQ}\left(1 - \frac{iSt}{Q}\right) - \frac{\sqrt{i}S}{2Q^{3/2}}\right\} erf\left(\sqrt{iQt}\right) + cc \qquad (14)$$

3 Solutions without thermal stratification

To facilitate the comparison of solutions of equations with and without thermal stratification we have also obtained the solutions of equations without thermal stratification by putting $\gamma = 0$ in equation (2). They are non-dimensionalised with the same set of non-dimensional quantities (4).Thus the solutions of velocity, temperature, skin-friction and Nusselt number are obtained as follows-

$$W(\xi, t) = \left\{t - (Gr - 1)\frac{\xi^2}{2}\right\} erfc\left(\frac{\xi}{2\sqrt{t}}\right) + (Gr - 1)\xi\sqrt{\frac{t}{\pi}}\, e^{-\xi^2/4t} \qquad (15)$$

$$\theta(\xi, t) = erfc\left(\frac{\xi}{2\sqrt{t}}\right) \qquad (16)$$

$$\tau = (2 - Gr)\sqrt{\frac{t}{\pi}} \qquad (17)$$

$$Nu = \frac{1}{\sqrt{\pi t}} \qquad (18)$$

4 Result and Discussion

In order to discuss the effect of physical parameters on the vertical velocity, temperature, skin-friction and the rate of heat transfer, computations of the solutions are carried out and they are represented graphically in figures $1 - 5$. In all the figures dotted lines correspond to solutions of equations without thermal stratification. Since the arguments in the complementary error function are complex we separate them into real and imaginary parts following Abramowitz and Stegum [15]

Figure 1 represents the temperature profile θ against ξ for different values of Gr and S. It is clear form the figure that temperature decreases with the increase of

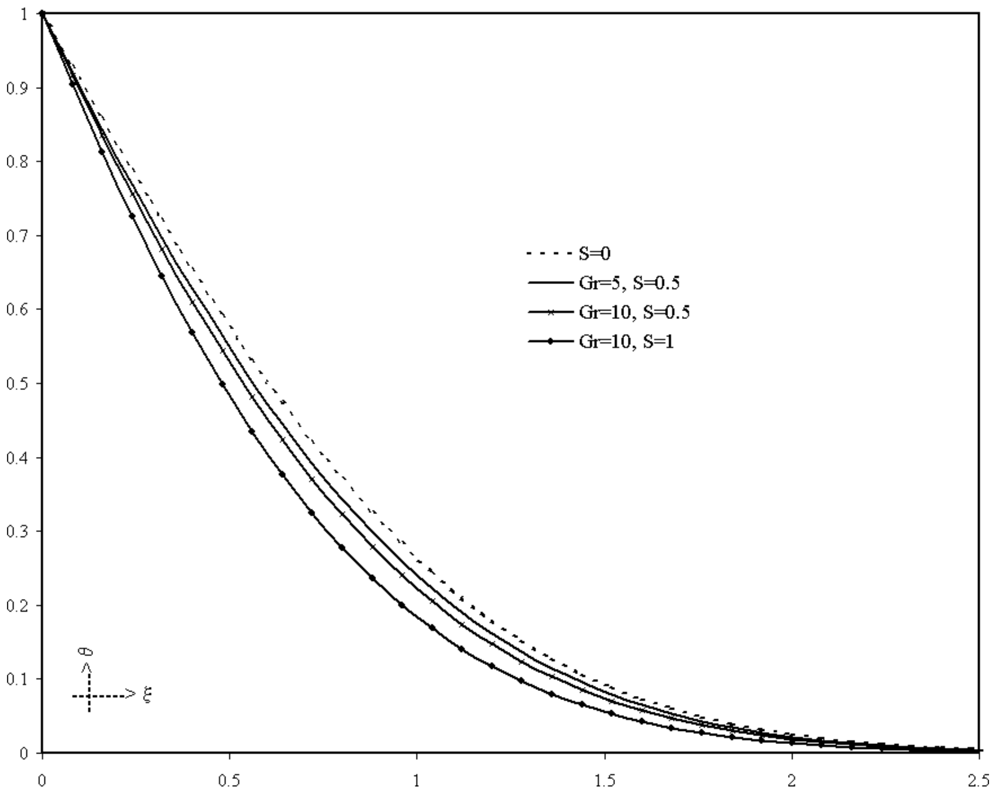

Figure 1: Temperature profile for different values of Gr and S.

stratification parameter S and Grashof number Gr. Also the fall of temperature is more rapid when S is made larger along with Gr.

Figures 2 and 3 represent the cross-sections of the vertical velocity profile W against ξ for different values of S, Gr and t. From figures it is observed that thermal stratification has significant impact on the vertical velocity field. Increase of Grashof number Gr and time t leads to an increase of velocity; which is obvious because more Gr means more heating and less density. But increase of stratification parameter S leads to decrease of velocity, which is due to the layering effect of thermal stratification as it acts like a resistive force. Also since the plate is accelerated and the flow is unsteady hence velocity increases with time t.

In figure 4 skin-frictions is presented for different values of Gr and S. From the figure clear distinction can be observed between the skin-frictions with and without thermal stratification. In the absence of thermal stratification, skin-friction decreases continuously against time but in stratified fluid though initially decreases but as time

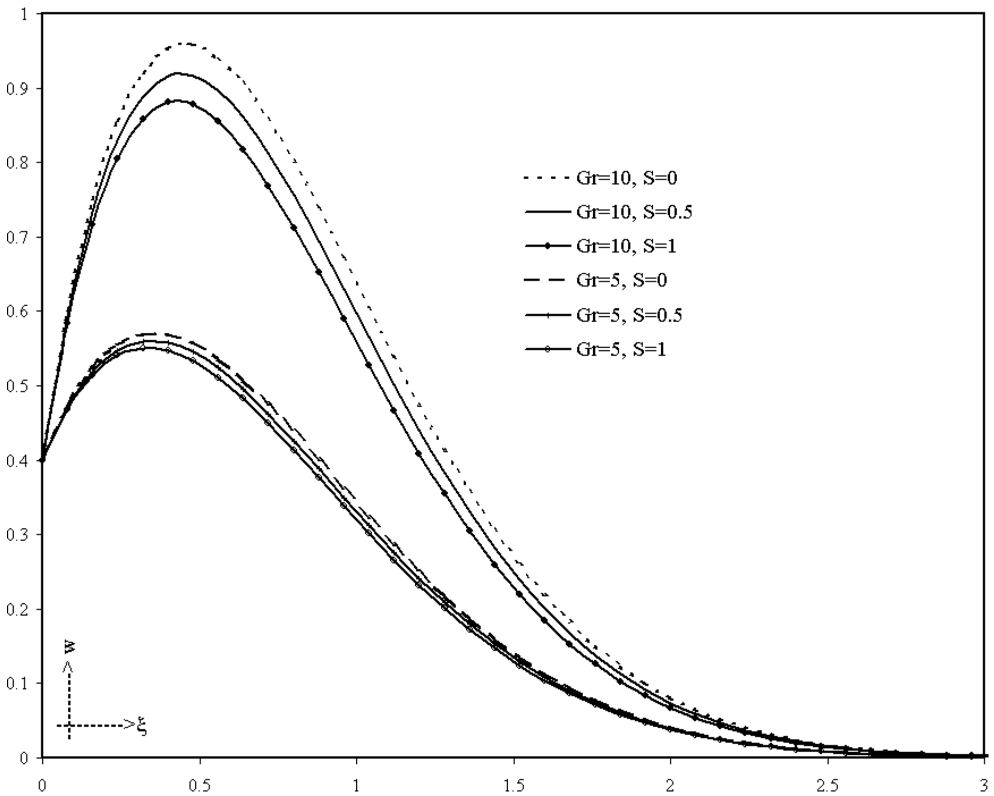

Figure 2: Velocity profile for different values of Gr and S.

progresses it gradually increases. Also it decreases when Gr increases and increases when S increases.

Rate of heat transfer i.e. the Nusselt number Nu is presented in figure 5. Near the plate the Nusselt number is infinite and in the absence of thermal stratification, it decreases continuously as time increases and approaches to zero as $t \to \infty$. But in stratified fluid it decreases for smaller time and thereafter increases as time progresses. Also Nusselt number increases when S and Gr increase.

5 Conclusions

In this paper, we have investigated the problem of unsteady natural convection flow past an accelerated vertical plate in a thermally stratified fluid medium. Exact solutions are obtained by Laplace transform technique and represented them graphically for various values of parameters. It is clear from the above discussions that the ther-

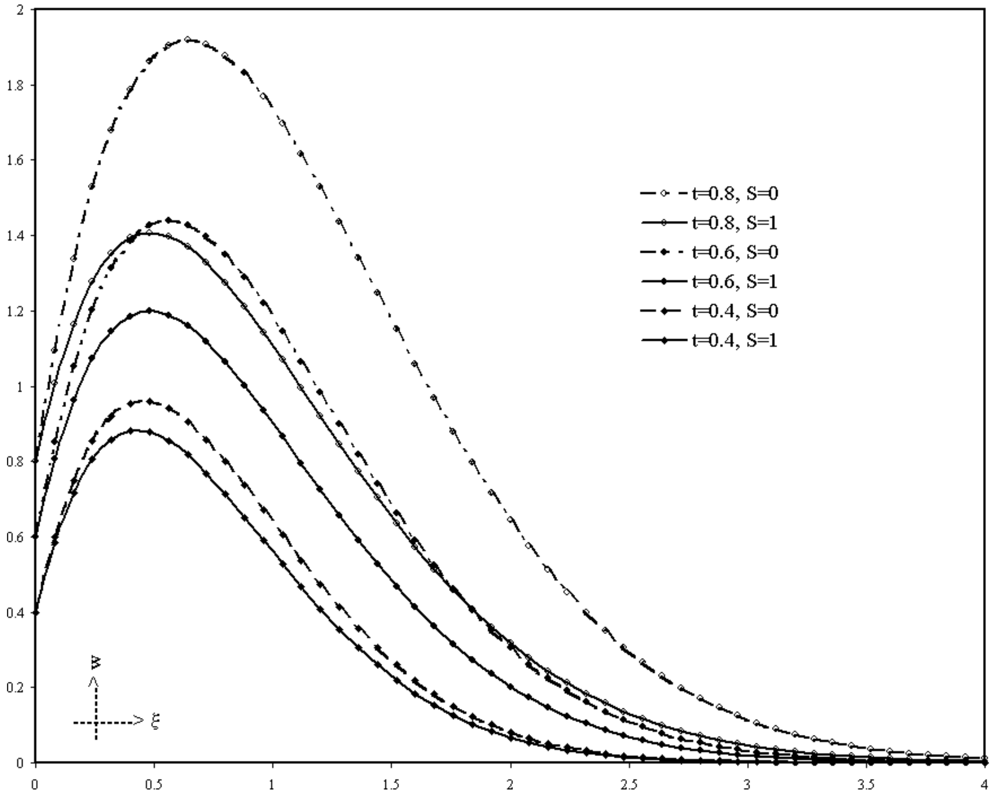

Figure 3: Velocity profile for different values of t and S.

mal stratification parameter S plays an important role on unsteady vertical natural convection flow. We hope that this investigation can be utilized for further more complex vertical natural convection flows in scientific and engineering applications. We summarize the following observations:

- Velocity in thermally stratified fluid is less than unstratified fluid. It decreases significantly with the increase of stratification parameter S and increases when Gr and t increase.
- Temperature decreases with the increase of S and Gr.
- Skin-friction initially decreases but finally it increases as time progresses. Also it decreases when Gr increases and increases when S increases.
- Nusselt number initially decreases for a small time but increases finally. It also increases with the increase of S and Gr.

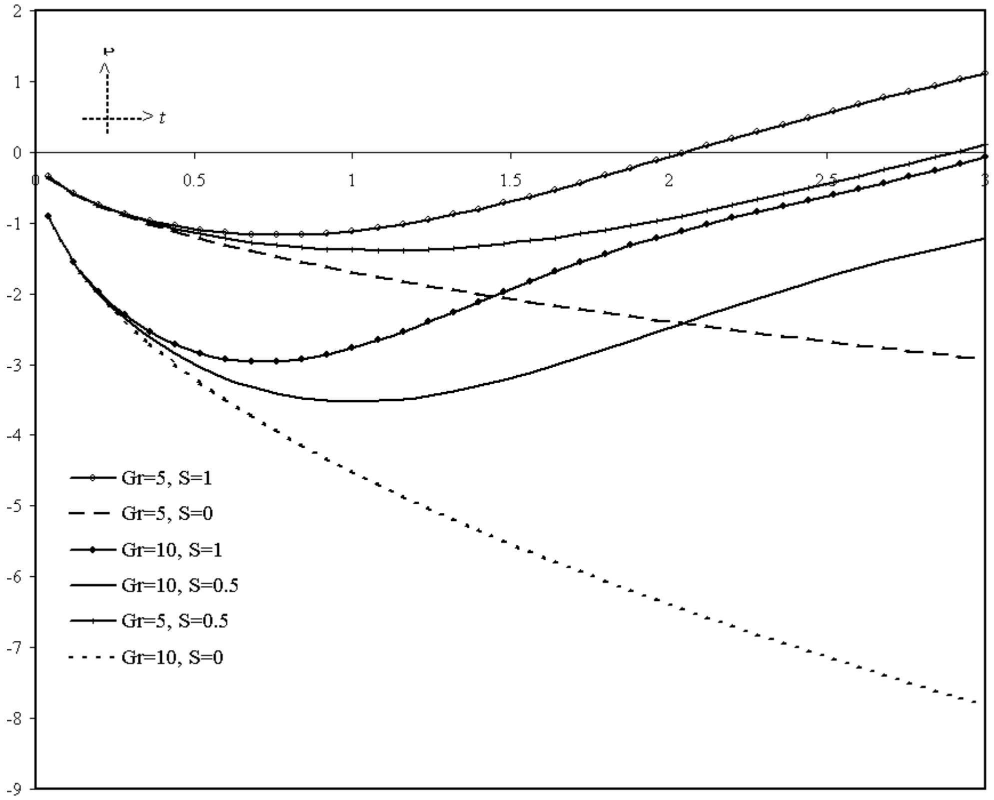

Figure 4: Skin-friction for different values of Gr and S.

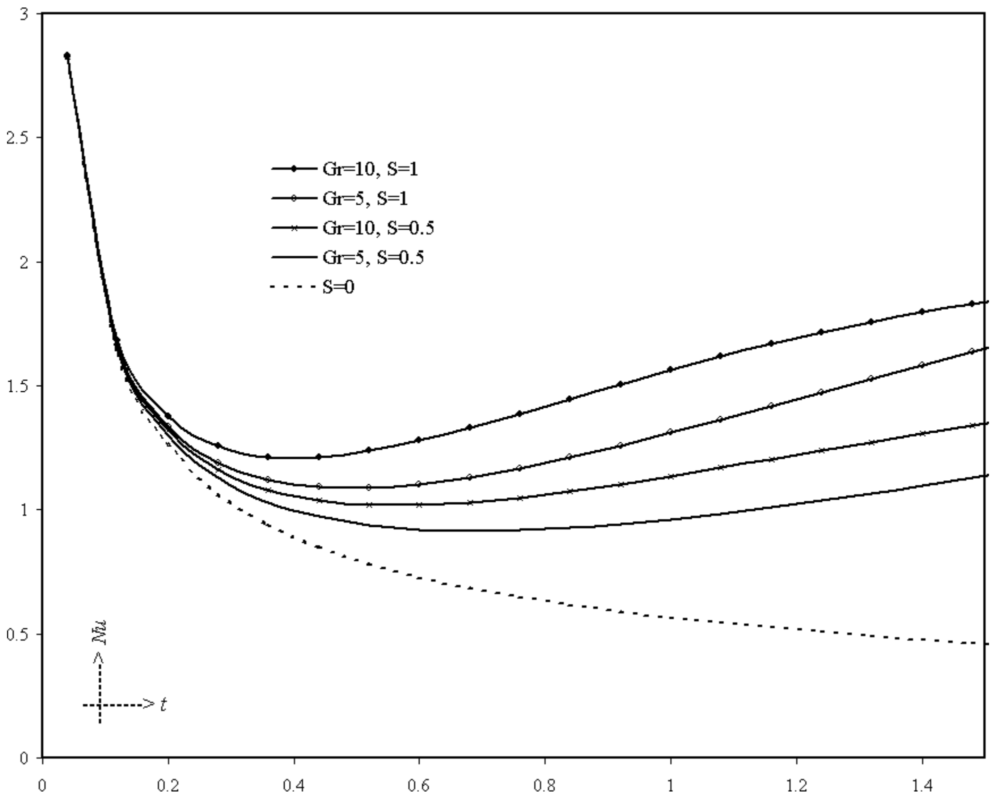

Figure 5: Nusselt Number for different values of Gr and S.

Acknowledgements

This work has been partially funded to the author by the UGC(NER),India

References

[1] Bejan A., *Convection heat transfer.*, John Wiley and Sons., New York, 1995.

[2] Schetz J. A. and Eichhorn R., *Unsteady natural convection in the vicinity of a doubly infinite vertical plate.*, Journal of Heat Transfer **84** (1962), 334–338.

[3] Shapiro A. and Fedorovich E., *Unsteady convectively driven flow along a vertical plate immersed in a stably stratified fluid.*, Journal of Fluid Mechanics. **498** (2004), 333–352.

[4] Gebhart B., *Transient natural convection from vertical elements.*, Journal of Heat Transfer (ASME). **83C** (1961), 61–70.

[5] Gebhart B., Jaluria Y., Mahajan R. L., and Sammakia B., *Buoyancy-induced flows and transport.*, Hemisphere., New York, 1988.

[6] Hetnarski R. B., *An algorithm for generating some inverse laplace transforms of exponential form.*, Applied Mathematics and Physics (ZAMP) **26** (1975), 249–253.

[7] Torgersen C.E., Faux. R.N., McIntosh B.A., Poage N.J., and Norton D.J., *Airborne thermal remote sensing for water sensing for water temperature assessment in rivers and streams.*, Remote Sensing of Environment (Elsevier). **76** (2001), 386–398.

[8] Fedorovich E. and Shapiro A., *Prandtl number dependence of unsteady natural convection along a vertical plate in a stably stratified fluid.*, International Journal of Heat and Mass Transfer (Elsevier). **47** (2004), 4911–4927.

[9] Gill A. E. and Davey A., *Instabilities of a buoyancy driven system.*, Journal of Fluid Mechanics. **35** (1969), 775–798.

[10] Magyari E., Pop I., and Keller B., *Unsteady free convection along an infinite vertical flat plate embedded in a stably stratified fluid-saturated porous medium.*, Transport in Porous Media **62** (2006), 233–249.

[11] Bergholz R. F., *Instability of steady natural convection in a vertical fluid layer.*, Journal of Fluid Mechanics **84** (1978), 743–768.

[12] Bateman Harry., *Tables of integral transferms.*, McGraw-Hill., New York, 1954.

[13] Goldstein R. J. and Briggs D. G., *Transient free convection about vertical plates and circular cylinders.*, Journal of Heat Transfer (ASME) **86** (1964), 490–500.

[14] Soundalgekar V. M.and Deka R. K. and Das U. N., *Transient free convection flow of a viscous incompressible fluid past an infinite vertical plate with periodic heat flux.*, Indian Journal of Engineering and Material Sciences **10** (2003), 390–396.

[15] Abramowitz M. and Stegun I. A., *Handbook of mathematical functions.*, Dover Publications, New York, 1965.

[16] Hyun J. M., *Transient process of thermally stratifying an initially homogeneous fluid in an enclosure.*, International Journal of Heat and Mass Transfer. **27** (1984), 1936–1938.

[17] Das U. N., Deka R. K., and Soundalgekar V. M., *Transient free convection flow past an infinite vertical plate with periodic temperature variation.*, Journal of Heat Transfer.(ASME) **121** (1999), 1091–1094.

[18] Illingworth C. R., *Unsteady laminar flow of a gas near an infinite flat plate.*, Proc. of Cambridge Philosophical Society. **46** (1950), 603–613.

[19] Soundalgekar V.M., *Free convection effects on stokes problem for an infinite vertical plate.*, Journal of Heat Transfer (ASME) **99C** (1977), 499–501.

[20] .Soundalgekar V.M., *Free convection effects on the flow past an infinite vertical oscillating plate.*, Astrophysics and Space Science **64** (1979), 165–171.

[21] Lin W. X. and Armfield S. W., *Natural convection cooling of rectangular and cylindrical containers.*, International Journal of Heat Fluid Flow **22** (2001), 72–81.

Submitted on March 2009

Nestacionarno prirodno konvekciono tečenje po ubrzavajućoj vertikalnoj ploči u termički slojevitom fluidu

Proučava se jedno egzaktno rešenje jednodimenzionog nestacionarnog prirodnog konvekcionog tečenja po ubrzavajućoj vertikalnoj ploči u termički slojevitom fluidu. U termodinamičkoj energijskoj jednačini se posmatraju pritisak kao član rada kao i vertikalna temperaturna advekcija. Bezdimenzione jednačine problema su rešene tehnikom Laplasove transformacije za jedinični Prantlov broj. Brzinski i temperaturni profili, trenje na zidu kao i brzina prenosa toplote su prikazani grafički te se diskutuju efekti Grashofovog broja Gr, parametra slojevitosti S za različita vremena t.

doi:10.2298/TAM0904261D Math.Subj.Class.: 76R10, 76D50