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Comments on the paper:

A contribution to the theory of the extended Lagrangian formalism for rheonomic systems

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Reply by the author - Djordje Mušicki [†]

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Comments by Veljko A. Vujičić

1. Rheonomic

In the paper [1] the number of differential equations of motion of rheonomic systems as well as the notion of variation for generalized coordinates are formally extended. We here point out at the shortages of [1]. In the classical analytical mechanics *rheonomic systems* are systems of N material points, whose motion is bounded by $k \leqslant 3N$ constraints, presented by equations:

$$f_\mu(y_1, \ldots, y_{3N}, t) = 0, \Longleftrightarrow f_\mu(x_1, \ldots, x_{3N}, t) = 0, \tag{1.1}$$

where y_i, $(i = 1, ..., 3N)$, rectangular coordinates of material points positions, and x_i are curvilinear coordinates, and t is time. The f_μ may be differentiable functions, so that there exists a system of differential equations as follows:

$$df_\mu = \sum_{i=1}^{3N} \frac{\partial f_\mu}{\partial y_i} dy_i + \frac{\partial f_\mu}{\partial t} dt = 0, \quad \mu = 1, \ldots, k = 3N. \tag{1.3}$$

In several papers (see Reference) it has been shown that it is more correct to write constraints equations (1.1) as functional equations

$$f_\mu[y_1(t), \ldots,\ y_{3N}(t); \tau(t)] = 0, \tag{1.3}$$

where $y_i(t)$ are unknown functions of time and $\tau(t)$ is some known (prescribed) function of time. For the known conditions

$$\left|\partial f_\mu / \partial y_i\right|_k^k \neq 0,$$

Now, k coordinates y_j can be obtained from the system of equations (1.3) as the functions of $3N - k$ *independent generalized coordinates* $q^1(t), \ldots, q^n(t)$ and known additional function $q^0 = \tau(t)$.

By means of independent generalized coordinates q^α $(\alpha = 0, 1, \ldots, n = 3N - k)$ Lagrangian differential equations of motion read:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} - \frac{\partial \mathcal{L}}{\partial q^\alpha} = 0, \quad (\alpha = 1, 2, \ldots, n). \tag{1.4}$$

Here all the coordinates $q^0, q^1, \ldots q^0$ are functions of parameters γ_a, γ_0, but time t is the unique independent variable:

$$q^\alpha = f^\alpha(\gamma_a, t), \quad q^0 = \tau(\gamma_0, t), \tag{1.5}$$

2. A scientific critics

By a formal approach, instead of a single equation for q^0, the system of differential equations of motion of a rheonomic system:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = R_a^0, \quad a = n+1, \dots, n+A. \tag{2.1}$$

is introduced in [1]. In order to prove it, Mušicki defined the new variation of functions, which is not consonant with general theory of variation calculus and with variation of the functions (1.5).

Basic ideas of Mušicki's formalism is described by the author in the second chapter of his paper ([1], p. 50): "*Let us consider the motion of a mechanical system of N material points under the influence of arbitrary active forces, bounded by k non stationary holonomic constraints, in which time appears though one or several functions $\varphi_a(t)$,*

$$f_\mu[\mathbf{r}_\nu, \varphi_a(t)] = 0, \quad \mu = 1, 2, \dots, k;\ \nu = 1, 2, \dots, N),$$

which is affirmed in all real examples," or in the form (1.1),

$$f_\mu(y_1, \dots, y_{3N}, t) = 0, \quad \mu = 1, \dots, k \leqslant 3N. \tag{2.2}$$

"*The fundamental idea of this formulation of mechanics (Dj. Mušicki) is based on the introduction of new quantities suggested by the form of the constraints, which change in the course of time according to the law $\tau_a = \varphi_a(t)$, and on extension of the chosen set of generalized coordinates by these quantities.*"

The *"change in the course of time according to the law $\tau_a = \varphi_a(t)$"* is very formalistic. Really:

$$df_\mu = \sum_{i=1}^{3N} \frac{\partial f_\mu}{\partial y_i} dy_i + \frac{\partial f_\mu}{\partial \varphi_a} d\varphi_a = 0. \tag{2.3}$$

$$\frac{\partial f_\mu}{\partial \varphi_a} d\varphi = \frac{\partial f_\mu}{\partial \tau_a} d\tau_a = \frac{\partial f_\mu}{\partial \tau_a}\frac{\partial \tau_a}{\partial t} dt = \frac{\partial f_\mu}{\partial t} dt,$$

and by substituting them into the equations (2.3) the classical system equations (1.2) are obtained.

Introduction of the additional function $y^0(t)$ is not consonant as in [1]. Instead, an additional coordinate $y^0 = \tau(t)$, from which it is possible to determine time t as the function of $y^0, \rightarrow t = t(y^0)$. Substituting it into functions φ_a, $\varphi_a(t) = \varphi(y^0)$ is obtained. Likewise, differential equations (2.3) are reduced to the simpler form

$$df_\mu = \sum_{j=0}^{3N} \frac{\partial f_\mu}{\partial y_j} dy_j = 0, \quad \mu = 1, \dots, k \leqslant 3N, j = 0, 1, \dots, 3N. \tag{2.4}$$

Then the complete set of generalized coordinates will be: $(q^0, q^1, \dots, q^{3N-k}) \in M^{3N-k+1}$ so that $y^i = y^i(q^0, q^1, \dots, q^n); \quad n = 3N - k$.

This differs considerably from the Mušicki's equations

$$q^\alpha = \big(q^i(i = 1, 2, \dots, n);\, q^a(a = n + 1, \dots, n + A)\big); \quad (2.2, M).$$

Let us notice that the author of [1]1 has not defined what is number A in his relation (2.2,M), as, immediately after that equation author defines as A the center of the sphere in the mentioned example. The simple example in [1], where a material point on a sphere "*whose center moves*" moves uniformly along a horizontal line is not correct.

It is very important to find out the numbers that "a" is made of, as well as inertial tensor a_{ab} as with indices which are not defined,so that kinetic energy using the formula (2.7,M), cannot be determined:

$$E_k = \frac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}\beta = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + a_{ia} \dot{q}^i \dot{q}^a + \frac{1}{2} a_{ab} \dot{q}^a \dot{q}^b,$$

3. On the extension of the notion of variation

Let us notice that Mušicki has not defined neither the notion of variation of a functional nor of a parametric expressed function in which are clearly delimited parameter and the independent coordinate. Without clear definition of these terms it is hard to tell about on the clear term of variation. The generalized coordinates q^α are independent functions of time t, as independent variable, and of other geometrical, kinematical and dynamical parameters γ. However, Dj. Mušicki writes: *"So extended notion of variation can be formulated for the extended generalized coordinates"* $q^\alpha(\alpha = 1, 2, \dots, n + A)$ *as well, where the family of varied paths can be described by* $\quad q^\alpha = q^\alpha(t, \gamma, \tau_a), \quad (3.6, M)$, *with the quantities* τ_a *having double role: they are the additional parameters and also the additional generalized coordinates* $q^a = \tau_a$*. The simultaneous variation of the generalized coordinate* q^α *will be defined by*

$$\delta q^\alpha := q^\alpha(t, \gamma + \delta\gamma, \tau_a + \delta\tau_a) - q^\alpha(t, \gamma, \tau); \quad (3.7, M)''$$

If we expand the first function in (3.7,M) in Taylor's series, in a analogous way as in the previous case, we obtain

$$\delta q^\alpha = \big(\frac{\partial q^\alpha}{\partial \gamma}\big)_0 \delta\gamma + \big(\frac{\partial q^\alpha}{\partial \tau_a}\big)_0 \delta\tau_a; \quad (3.8, M).''$$

If we notice that $q^0(\gamma, t) = \tau(\gamma, t)$, as well as that the quantities τ_a do not have a double role, instead of relations (3.8,M), we obtain, [2] - [8] $\delta q^\alpha = (\partial q^\alpha / \partial \gamma)\delta\gamma, \quad (\alpha = 0, 1, \dots, n)$.

Let's explain our conclusions by the following example: The rheonomic constraint is $f(x, y, t) = y - tx = 0$, where x, y are Cartesian coordinates, and

t is time. This constraint cannot exist, as it is not dimensionally homogenous. Dimensions of x and y are L, and $\dim t = T$; thus $L \neq TL$. In order that the given equation can be a constraint it is necessary to have some parameter γ with t, as $y - \gamma\, tx = 0, \quad \gamma = 1T^{-1}; \quad y - \tau(\gamma, t)x = 0$, where, as seen $\tau(\gamma, t) = q^0$ rheonomic coordinate function of parameter γ and time t.

Formally one can write:([1], p.78), $q^0 := t, \quad q^0 \stackrel{def}{=} t, \quad q^0 \equiv t$. In this case an illogical formality is obtained $\delta q^0 := \delta\, t, \quad \delta q^0 \stackrel{def}{=} \delta t, \quad \delta q^0 \equiv \delta\, t$.

In the case of a parametric representation of the function $q^0(t) = \gamma t$, $\gamma \approx 1$ the variation is

$$\delta q^0(t) = \frac{\partial q^0}{\partial \gamma} \delta\gamma = t \cdot \delta\gamma = \epsilon t.$$

In *chapter 5* of [1] the same symbol δ is used for both *variation* and *virtual displacement*. It is not correct. The concept of possible displacement implies any, no matter how small, diversion from the material point's real position that could be archieved by that point. This concept is not consonant with the differential $d\mathbf{r}$ or the variation $\delta\mathbf{r}$ of the position vector. ([2], p 82-84). Thus, there not exist two groups of equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_1*, \quad (i = 1, \dots, n),$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = Q_a*, \quad (a = n+1, \dots, N+A), \quad (5.7, M).,$$

There exist only $n+1$ differential equations in the form (1.4).

In the chapter **6. Energy relation in this formulation of mechanics** with equations (6.2,M) the author makes an important mistake. From these equations follow the relations:

$$\delta q^\alpha = \frac{dq^\alpha}{dt} = dq^\alpha,$$

which is an absurd. Variation of the independent variable, does not exist. As here time t is independent variable; it not depend on any parameter γ. Thus, in his case (see ([1], p 78, eq.(8.5,M) - (8.7,M)), $\tau = q^0 = t$ the variation is $\delta\tau = \delta q^0 = \delta t = 0$.

The conservative law was obtained in our papers and monograph using various methods. It is also shown that Painleve's integral(6.12,M) is not a general conservation law of energy of the system, but just one from possible co-cyclic integrals; [6].

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Reply by the author - Djordje Mušicki

1. Reply to V. Vujičić's basic remark

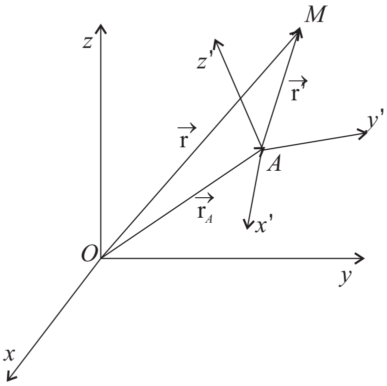
Prof. Vujičić's basic remark refers to the fact that in this paper I expressed the opinion that there could be more than one additional generalized coordinate, the so-called rheonomic coordinate. This remark probably comes from Vujičić's failure to realize that this was about a different interpretation of additional generalized coordinates, based on my proof of the geometrical sense of these quantities. There had been earlier indications and corroborations of this theory in all Vujičić's and my examples, which enabled a more logical structure and a confirmation of the theory's validity. Although this thesis substantiates Vujičić's results, it was not in accordance with his interpretation of the problem, and hence was probably unacceptable for him.

In my paper [R3] I proved the attitude that if we present non–stationary constraints in the form $f_\mu[\overrightarrow{r}_\nu, \varphi_a(t)] = 0$ and take the quantities $\tau_a = \varphi_a(t)$ for additional generalized coordinates, they define the position of the associated frame of reference to which the chosen generalized coordinates refer, in respect to an immobile frame of reference. In this way the introduced quantities got a geometrical sense, and it is precisely due to this characteristic that this theory, which was started by Vujičić's modification of the mechanics of rheonomic systems and continued through my work, acquires the true sense and confirmation. Namely, in the common Lagrangian formulation for rheonomic systems, the chosen generalized coordinates always refer to a mobile frame of reference, whereas the dynamic quantities and the energy laws refer to an immobile system, which brings about unnatural conservation laws (e.g. Painlevé's energy integral). On the other hand, in this formulation of mechanics by extended generalized coordinates $q^\alpha(\alpha = 1, 2, ..., n + A)$ we have the completely defined position of the observed mechanical system in respect to **the same**, immobile frame of reference all the dynamic quantities and the energy laws refer to. Owing to this, the energy laws in this formulation of mechanics are totally in accordance with the corresponding laws in vector formulation, if they are expressed through the quantities introduced in this formulation of mechanics. Thus the correctness of this formulation of mechanics and its energy laws is absolutely confirmed, and due to the said characteristic these energy laws are more consistent, more general and more natural than the corresponding laws in the common Lagrangian formulation, including the influence of the non–stationary constraints.

In the critic V. Vujičić truly believes what he states in the above mentioned remarks, he should establish my proof wrong, although it passed by all means strict review of the said magazine without a single observation. However, he failed to even mention this paper in his remarks, let alone tried to prove the inaccuracy of my allegations, which arose from the said proof. Besides, should Vujičiś's remark and his view on rheonomic coordinate be adopted, the said geometrical sense of additional generalized coordinates would have to be discarded, along with all the mentioned results and the advantages of this formulation of mechanics, and consequently this whole theory, as well as Vujičić's modification itself, would remain without a firm foundation. In this matter, the fact that there is most often only one additional generalized coordinate is of no importance at all, but it also has a completely different meaning from Vujičić's rheonomic coordinate, as it always defines the position of associated frame of reference $Ax'y'z'$ in respect to the immobile system $Oxyz$, and the corresponding theory must consider the general case. Such a case can be found in all Vujičić's and my examples, where due to the simplicity of the problem, the position of this associated system in respect to the immobile one can be defined by just one scalar quantity at any given moment. For the sake of argument, let us outline here just one example, where we would have two additional generalized coordinates: the motion of a particle across a plane, which performs a uniform translatory motion at the velocity V and a simultaneous rotatory motion around a vertical axis at a constant angular velocity ω. In this case the time in the non–stationary constraints would appear through the function $\varphi_1(t) = Vt$ due to the translatory motion, and through the function $\varphi_2(t) = \omega t$ due to the rotary motion. In this way we would get two additional generalized coordinates: $q^{n+1} = \tau_1 = Vt$ and $q^{n+2} = \tau_2 = \omega t$, but such cases are very rare and we most often have only one additional generalized coordinate.

Due to the great importance of this remark, let us give a rough outline of my proof for the said statement, published in the mentioned magazine. This proof is based on the transformation of constraints from the associated frame of reference $Ax'y'z'$ [R3], in respect to which the chosen generalized coordinates refer, to the immobile system $Oxyz$ (fig. 1).

This immobile system $Oxyz$ can be brought into the associated system $Ax'y'z'$ through one translation for $\Delta \vec{r}_A = \overrightarrow{OA}$, where $\vec{r}_A$ is a position vector of the pole A, and then through one rotation around an axis which goes through the point A at an angle α. The position vector $\vec{r} = \overrightarrow{OM}$ of any point M translates into the position AM' and then rotates into the position AM^*, defined by vector $\vec{r^*} = \overrightarrow{OM^*}$ in respect to the pole O of the immobile system (fig. 2).



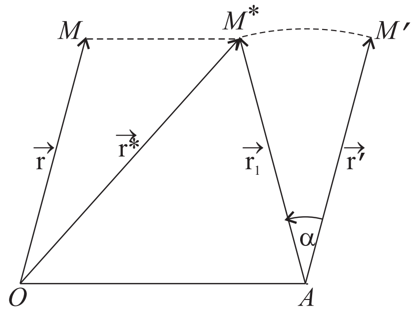


Figure 1 Figure 2

This translation will be defined if we know the vector $\overrightarrow{r}_A$ as the function of time and the rotation is defined by the so–called versor $\overrightarrow{w} = \mathrm{tg}\,\frac{\alpha}{2}\overrightarrow{\omega_0}$, where $\overrightarrow{\omega}_0$ is the unit vector of the rotation axis. So, in order that the position of the system $Ax'y'z'$ would be determined, the functions $\overrightarrow{r}_A(t)$ and $\overrightarrow{w}(t)$ must be given in advance. On the basis of kinematics of rigid bodies and especially the so-called Rodrigues's formula, which determines the change of the position vector at the rotation of a rigid body, it turns out that the position vector of any point after this translation and rotation (see Appendix of [R3] as well as [R4] pp. 81–92) is:

$$(1)\qquad \overrightarrow{r^*} = \overrightarrow{r}_A + \cos\alpha \cdot r + (1-\cos\alpha)(\overrightarrow{\omega}_0 \cdot \overrightarrow{r})\overrightarrow{\omega}_0 - \sin\alpha(\overrightarrow{\omega} \times \overrightarrow{r})$$

Since the transformation of the translated system $Axyz$ into the system $Ax'y'z'$ is equivalent to the transition of all the points M in the contrary sense for the angle $-\alpha$, the projections of the vector $\overrightarrow{r^*}$ onto the new coordinate axes are equal to the projections of the vector $\overrightarrow{r^*} = \overrightarrow{r^*}(-\alpha)$ onto the old ones

(2)

$$x_i' = \overrightarrow{r^*} \cdot \overrightarrow{e}_i' = \overrightarrow{r'^*} \cdot \overrightarrow{e}_i = x_{Ai} + \cos\alpha \cdot x_i + (1-\cos\alpha)(\omega_{0k}x_k)\omega_{0i} - \sin\alpha\varepsilon_{ijk}\omega_{0j}x_k,$$

where $\overrightarrow{e}_i$ and $\overrightarrow{e_i'}$ $(i = 1,2,3)$ are unit vectors of the coordinate axes of these two systems and ε_{ijk} is the Levi–Civita's symbol.

In frame of reference $Ax'y'z'$ attached to the observed system of particles, the constraints will not explicitly contain time

$$(3)\qquad f_\mu(\overrightarrow{r}_\nu) = 0 \quad \Leftrightarrow \quad f_\mu(x_{\nu i}) = 0 \quad (\nu = 1,2,...,k;\ i = 1,2,3),$$

as this system is not mobile in respect to that mechanical system. If we pass to the immobile system $Oxyz$, the coordinates of each particle should be replaced by the expression (2), and we must bear in mind that the functions $\overrightarrow{r}_A(t)$, $\alpha(t)$ and $\overrightarrow{\omega_0}(t)$ must be given in advance

(4)
$$f_\mu[x_{Ai}(t)+\cos\alpha(t)x_i+(1-\cos\alpha(t))(\omega_{0k}(t)x_k)\omega_{0i}(t)-\sin\alpha(t)\epsilon_{ijk}\omega_{0j}(t)x_k]=0$$
$$(\mu=1,2,...,k)$$

Hereby we can see that the constraints in the immobile frame of reference $Oxyz$ explicitly contain the functions $x_{Ai}(t)$, $\alpha(t)$ and $\omega_{0i}(t)$, so they have the following form

(5)
$$f_\mu[x_{\nu i};x_{Ai}(t),\alpha(t),\omega_{0i}(t)]=0 \quad (\mu=1,2,...,k)$$

By comparing this equation with the general shape of non–stationary holonomic constraints $f_\mu[x_{\nu i},\varphi_a(t)]=0$ we can conclude the following: 1) the function $\varphi_a(t)$ in non–stationary constraints are de facto the functions $x_{Ai}(t)$, $\alpha(t)$ or $\omega_{0i}(t)$. 2) the quantities defined by these functions $\tau_a=\{x_{Ai}(t),\alpha(t),\omega_{0i}(t)\}$ define the position of the associated frame of reference, referred to by the chosen generalized coordinates, in respect to an immobile (or more generally inertial) system and 3) the number of additional generalized coordinates is within the interval $1\le A\le 6$, depending on the problem (as six is the number of components of the vectors $\overrightarrow{r}_A$ and $\overrightarrow{\omega}$). Accordingly, the complete set of extended generalized coordinates in this extended Lagrangian formalism in the general case is

(6)
$$q^\alpha=\{\underbrace{q^i(i=1,2,...,n)}_{\substack{\text{define the position of}\\ \text{the mech. system in}\\ \text{respect to the assoc.}\\ \text{frame of reference}}}\;;\;\underbrace{q^a(t)(a=n+1,...,n+A)}_{\substack{\text{define the position of}\\ \text{this system in respect to}\\ \text{some immobile system}}}\},$$

so there are two groups of corresponding Lagrangian equations, and only the second one contains the characteristic quantities R^0_a.

2. Reply to the rest of Vujičić's remarks

2–1. Regarding the remark at the bottom of the page 2 (in front of (2.3c)[1]) that the manner of determining quantities $\tau_a=\varphi_a(t)$ is formalistic, I must say that I cannot see any basis for such a statement, and the relation that follows has nothing to do with the previous one.

[1] All formulae in this section with ending "c" refer to the comments of Vujicic like (2.1c).

2–2. As for the remark immediately after (2.3c), in front of (2.4c), that the introduction of the quantity $y^0 = \varphi(t)$ is not consistent, it is in accordance with Vujičić's view that there can be only one additional generalized coordinate, so this remark is out of question.

2–3. In the middle of the page 3 (page 332, fourth line up) there is a remark that the number A is not defined, which is true in terms of an explicit definition. Namely, I thought it was implicated by the formula (2.1) in [R2], i.e. that is the number of the functions $\varphi_a(t)$ i.e. the number of additional generalized coordinates, and in the paper [R3] which I referred to, it was shown that $1 \leq A \leq 6$, depending on the problem.

2–4. Regarding the remark following right behind the one I mentioned above, that the given example is incorrect, because I accidently marked the position of the centre of the sphere with the same letter A, so the number A can also imply the index in x_A, I have the following comment. In this example it is obvious that the number of additional generalized coordinates is $A = 1$ and does Vujičić truly believe that someone can consider the index in x_A a number?

2–5. In respect to the remark that the definition of the extended variation of the generalized coordinate is not complete (first sentence of the 3rd section of the comments), I believe all the vital elements have been included. The varied trajectories have been defined by the relation (3.6) in [R2], with the explanation why it is necessary to introduce quantities τ_a as additional parameters and with the usual characteristics of varied trajectories. The definition of the variation δq^{α_i} is given through the relation (3.7) in [R2] in the standard way as a difference between the value of the generalized coordinate on a varied trajectory, defined by the parameter $(\gamma + \delta\gamma, \tau_a + \delta\tau_a)$, and on the actual path with (γ, τ_a) at the same moment.

2–6. Vujičić's claim that the quantities $\tau_a = \varphi_a(t)$ do not have a double role (the text before the one I mentioned above) is not acceptable either, as it contradicts the facts. Namely, according to the very definition of the variation (3.7) in [R2] these quantities have the character of parameters, and according to the basic ideas of the extended Lagrangian formalism they are to be taken as additional generalized coordinates and then they have the characteristics of a function. In other words, for $\alpha = n+1, ..., n+A$ for each set of the values of parameters $\tau_a + \delta\tau_a$ we shall have the same value of the corresponding generalized coordinate as function: $q^a = \tau_a + \delta\tau_a = q^a(t, \gamma + \delta\gamma, \tau_a + \delta\tau_a)$ for any value of t and $\tau_a + \delta\tau_a$, so that $\delta q^\alpha = (\tau_a + \delta\tau_a) - \tau_a = \delta\tau_a$.

2–7. The remark about the formula (3.8) in [R2], where Vujičić (second formula in 3rd section of his comments) says that it is possible to put $q^0(\gamma, t) = \tau(\gamma, t)$ and $\delta q^\alpha = (\partial q^a/\partial\gamma)\delta\gamma$, is unrealistic, as we have several quantities $\tau_a(\gamma, t)$ here, and the parameters τ_a are by no means to be considered special cases of the parameter γ.

2–8. Regarding the remark on page 333 of the comments (immediately after the first formula up) that I used the same symbol δ for both variation and virtual displacement, Vujičić has probably failed to notice that I previously (paragraph 3 of [R2], page 54) showed both the variation $\delta \overrightarrow{r}_\nu$, defined by the formula (3.5) in [R2] and virtual displacement, defined as $d' \overrightarrow{r}_\nu - d\overrightarrow{r}$ comme down to the same expression, and subsequently the variation $\delta \overrightarrow{r}_\nu$ is equal to the virtual displacement.

2–9. The remark in front of the 2nd formula of page 333 of the comments, that there are neither two groups of Lagrangian equations (5.7) nor the set (2.2), both from my paper [R2], but only $n+1$ of these equations and a set of $n+1$ elements, comes from Vujičić's conviction that there can be only one additional generalized coordinate. However, he did not realize that this is a case of a different understanding these quantities, based on their geometrical sense, and in the general case there are $n+A$ additional generalized coordinates.

2–10. As for remark given on page 333 of the comments escorting last formula down, referring to the formula (6.2) from [R2], that it was a significant mistake, I would like to point out the following, I presented here the variation of generalized coordinate as a difference between two corresponding possible changes $\delta q^\alpha = d'q^\alpha - dq^\alpha$, then I defined $d'q^\alpha$ as the realistic increment of this generalized coordinate in the time interval $(t, t+dt)$, i.e. $dq^\alpha = (dq^\alpha)_{real} = \dot{q}^\alpha dt$ and I used for dq^α the zero change $dq^\alpha = 0$, accordingly I got $\delta q^\alpha = \dot{q}^\alpha dt$. Where then have I written the nonsense relation $\delta q^\alpha = (dq^\alpha/dt) = dq^\alpha$ and where is that significant mistake?

2–11. The remark about the equations (8.5) – (8.7) from my paper (page 5, sixth line down) refers to a part of my proof that the energy laws from this formulation of mechanics are in total accordance with the corresponding laws from vector formulation, if they are expressed through the quantities introduced in this formulation of mechanics. If I understand this remark correctly, Vujičić seems to think that instead of differential of time we should here have the variation of time, and since $\delta t = 0$, this proof is incorrect. Then the equivalence of these energy laws with the corresponding laws from vector formulation would be out of question, and consequently the correctness of these energy laws would be become utterly problematic, including Vujičić's results as well. However, the fact of the matter here is that we are looking at the real, and not possible, changes of all the quantities in the time interval $(t, t+dt)$. Accordingly, these realistic changes can be presented only through differentials and not variations, which can also be seen from the procedure of inferring this proof.

2–12. Regarding the remark (coming immediately after the one I mentioned above) that Painlevé's energy integral does not represent the general law on energy conservation, it would be true if we talked about the so–called Jacobi's energy integral $\mathcal{E} = \sum(\partial L/\partial \dot{q}^\alpha)\dot{q}^\alpha - L = const$, but for rheonomic systems it comes down to Painlevś's energy integral. By no means is this contradictory to the fact that it is one of the possible co–cycling integrals, and it was Vujičić who showed that.

References

[R1] Vujičić V., Comments on [R2] on previous pages 330-334.

[R2] Mušicki Dj., A contribution to the theory of the extended Lagrangian formalism for rheonomic systems, *Theoret.Appl.Mech.*(Beograd), Vol 36, No 1, (2009), pp. 47-83.

[R3] Mušicki Dj., Extended Lagrangian formalism and the corresponding energy relations, *European Journal of Mechanics A / Solids*, vol. 23 (2004), p. 975–991,

[R4] Bilimović A.: RATIONAL MECHANICS, vol. III (in Serbian), Naučna knjiga, Beograd, 1954.