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Mathematical model to determine the surface stress acting on the tooth of gear

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Abstract

Surface stress on the surface contact of gear tooth calculated by the Buckingham equation constitutes the basis for The American Gear Manufacturers Association (AGMA) pitting resistance formula, which is based on a normal stress that does not cause failure since the yielding in contact problems is caused by shear stresses. An alternative expression based on the maximum-shear-stress is proposed in this paper. The new expression is obtained by using the maximum-shear-stress distribution and the Tresca failure criteria in order to know the maximum-shear-stress value and its location beneath the contact surface. Remarkable differences between the results using the proposed equation and those when the AGMA equation is applied are found.

Keywords: Surface stress; displacement equation; pressure distribution; maximum-shear-stress; gear design equation.

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1 Introduction

Contact process between gear and pinion is comparable with the one produced by two cylinders with the same radius of curvature loaded in rolling contact. Based in such comparison, the contact between two cylinders produced in loading can be solved by using: I) Hertzian polynomial equation to describe each tooth circular convexities belonging to gear and pinion;

II) Elasticity theory to know the surface displacements and its relation to the pressure distribution produced by the load and III) The Flamant generalized stress equation [1] to determine the pressure distribution and to calculate the state of stresses beneath the contact surface. By using the state of stresses equations, we can calculate the maximum-shear-stress distribution in gear tooth contact surface and, by considering the Tresca failure criteria, the proposed equation is obtained. The maximum-shear-stress distribution and the locus of its higher value show the difference between this proposed equation and that of Buckingham's.

2 Hertzian contact

Figure 1 shows Pinion and gear tooth in contact under the action of a load P. Dashed lines show the original shape of the two bodies and the continuous lines their shapes under the load P. From Figure, the gear and pinion tooth profile radiuses are R_2 and R_1 respectively, and the strip of the contact area is $2a$. Then, from the scheme the relative elastic displacements for each tooth surface can be expressed as

$$u_{Z_1} + u_{Z_2} = \delta - z_1 - z_2 \tag{1}$$

where u_{Z_1} and u_{Z_2} are the displacements of any points over the contact surface of body 1 and body 2 respectively, $\delta\mathrm{g} = \mathrm{g}\delta_{\mathrm{xg}} + \mathrm{g}\delta_{\mathrm{y}}$ is the total body displacement and z_1 and z_2 the positions of the points over the contact surface.

The Hertzian expressions whose plots approaches to each tooth circular convexities on the contact surface are

$$z_1 = \frac{x_1^2}{2R_1} \qquad \text{and} \qquad z_2 = \frac{x_2^2}{2R_2} \tag{2}$$

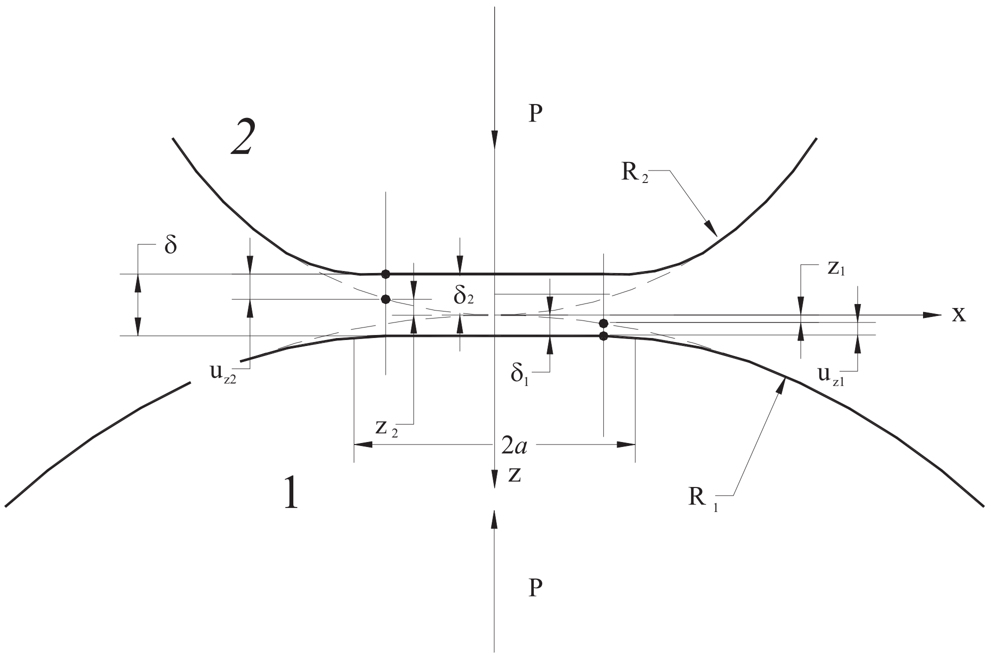

Figure 1: Scheme showing the contact phenomenon between the teeth of gear and pinion

At certain point in the contact surface $x_1 = x_2 = x$; then, substituting Equations (2) into Equation (1) and by making

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R} \tag{3}$$

it is obtained

$$u_{z_1} + u_{z_2} = u_z = \delta - \frac{x^2}{2R} \tag{4}$$

This equation is the *displacement equation* for whatever point in the contact surface. Then, the variation of the contact surface along the x-direction can be determined by partial differentiation of Equation (4) with respect to x, resulting in

$$\frac{\partial u_z}{\partial x} = -\frac{x}{R} \tag{5}$$

This displacement gradient must be equal to that produced by the pressure distribution on the contact area.

3 Displacements produced by the pessure distribution

Normal pressure distributed in arbitrary manner over elastic half-space is shown in Figure 2.

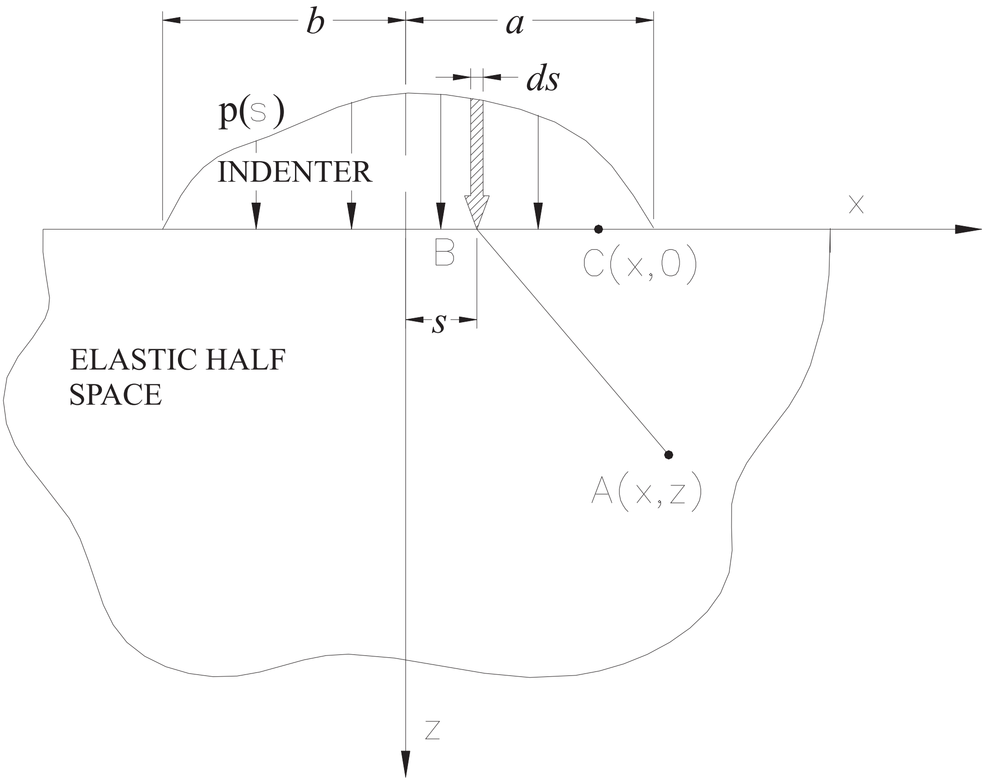

Figure 2: Plot that shows an arbitrary normal pressure distribution over an elastic half-space.

The load acting on the surface at B, distance s of O, on an elemental area of width ds can be assumed as a concentrated normal force P of magnitude $p(s)ds$ acting at B. The state of stresses produced by P at point A, are calculated using Flamant equation:

$$\sigma_r = -\frac{2P}{\pi}\frac{\cos\theta}{r}$$

this equation in rectangular coordinates become

$$\sigma_x = \sigma_r sen^2\theta = -\frac{2P}{\pi}\frac{x^2 z}{\left(x^2+z^2\right)^2}$$
$$\sigma_z = \sigma_r \cos^2\theta = -\frac{2P}{\pi}\frac{z^3}{\left(x^2+z^2\right)^2} \tag{6}$$
$$\tau_{zx} = \sigma_r sen\theta\cos\theta = -\frac{2P}{\pi}\frac{xz^2}{\left(x^2+z^2\right)^2}$$

Using Equations (6), replacing x by $x - s$ to relocate each point to the origin and integrating over the loaded region, $-b < s < a$, we get

$$\sigma_x = -\frac{2z}{\pi}\int_{-b}^{a}\frac{p\left(s\right)\left(x-s\right)^2 ds}{\left[\left(x-s\right)^2+z^2\right]^2}$$
$$\sigma_z = -\frac{2z^3}{\pi}\int_{-b}^{a}\frac{p(s)ds}{\left[\left(x-S\right)^2+z^2\right]^2} \tag{7}$$
$$\tau_{zx} = -\frac{2z^2}{\pi}\int_{-a}^{a}\frac{p(s)\left(x-s\right)ds}{\left[\left(x-s\right)^2+z^2\right]^2}$$

These equations are the basis to determine the maximum-shear-stress.

On the other hand, in order to know the displacements of points over the contact surface and the distortion under the load action, the Hook's law and the Flamant equation are used, which results in

$$\frac{\partial u_r}{\partial r} = \varepsilon_r = -\frac{\left(1-\nu^2\right)}{E}\frac{2P}{\pi}\frac{\cos\theta}{r}$$
$$\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \gamma_{r\theta} = \frac{\tau_{r\theta}}{G} = 0$$

after integration, the displacements can be obtained (as derived by Timoshenko & Goodier [2])

$$[u_r]_{\theta=\frac{\pi}{2}} = [u_r]_{\theta=-\frac{\pi}{2}} = -\frac{\left(1-2\nu\right)\left(1+\nu\right)P}{2E} \tag{8}$$
$$[u_\theta]_{\theta=\frac{\pi}{2}} = -[u_\theta]_{\theta=-\frac{\pi}{2}} = \frac{\left(1-\nu^2\right)}{\pi E}2P\ln\frac{r_0}{r} - \frac{\left(1+\nu\right)}{\pi E}P$$

by assuming that $u_\theta = u_x$ and $u_r = u_z$ for $\theta = \frac{\pi}{2}$ and that $P = \int_{-a}^{a} p(s)ds$ in our case, the displacements at A (as derived by Rekach [3]) can be write as

$$u_x = -\frac{(1-2\nu)(1+\nu)}{2E}\left[\int_{-a}^{x} p(s)ds - \int_{x}^{a} p(s)ds\right]$$

$$u_z = \frac{2(1-\nu^2)}{\pi E}\int_{-a}^{a} p(s)\ln|x-s|\,ds - \frac{(1+\nu)}{\pi E}P \tag{9}$$

Equations (9) take a much clear form if we choose to specify the displacement gradients at surface $\partial u_x/\partial x$ and $\partial u_z/\partial x$ rather than the absolute values of u_x y u_z [4]. The terms in curly brackets can be differentiated with respect to limit x and the other integrals can be differentiated under the integral operator to give

$$\begin{aligned} \frac{\partial u_x}{\partial x} &= -\frac{(1-2\nu)(1+\nu)}{E}p(x) \\ \frac{\partial u_z}{\partial x} &= -\frac{2(1-\nu^2)}{\pi E}\int_{-a}^{a} \frac{p(s)}{x-s}ds \end{aligned} \tag{10}$$

In equation (3), we can see that $\frac{\partial u_x}{\partial x} = \varepsilon_x$, whereas Equation (3) is the slope of the deformed contact surface of the cylinders due to the pressure acting on them, then

$$\frac{\partial u_{z_1}}{\partial x} = -\frac{2(1-\nu_1^2)}{\pi E_1}\int_{-a}^{a} \frac{p(s)}{x-s}ds$$

$$\frac{\partial u_{z_2}}{\partial x} = -\frac{2(1-\nu_2^2)}{\pi E_2}\int_{-a}^{a} \frac{p(s)}{x-s}ds$$

Adding each other the last equations, we can get

$$\frac{\partial u_z}{\partial x} = \frac{\partial u_{z_1}}{\partial x} + \frac{\partial u_{z_2}}{\partial x} = -\frac{2}{\pi}\left[\frac{(1-\nu_1^2)}{E_1} + \frac{(1-\nu_2^2)}{E_2}\right]\int_{-a}^{a} \frac{p(s)}{x-s}ds$$

and, by using the relation

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$

the variation of the contact surface along the x-direction can be wrote as

$$\frac{\partial u_z}{\partial x} = -\frac{2}{\pi E*} \int_{-a}^{a} \frac{p(s)}{x-s} ds \tag{11}$$

This slope, Equation (11), and the displacement gradient, Equation (5), must be equal to each other, and then we can write

$$\int_{-a}^{a} \frac{p(s)}{x-s} ds = -\frac{\pi E*}{2R} x$$

Solving this singular equation, we will be able to know the pressure distribution, which produces this displacement gradient. Therefore, in order to solve this equation, we divide by a(the half width of contact area) and we replace X by x/a and S by s/a, so the last equation became

$$\int_{-1}^{1} \frac{p(S)}{X-S} dS = -\frac{\pi E*}{2R} X \tag{12}$$

This equation has the form $\frac{1}{\pi} \int_{-a}^{a} \frac{p(\xi)}{\xi - t} d\xi = \frac{2\mu}{\kappa+1} f'(t)$ which solution (using the Cauchy integral formula and the Mikhlin development [5, 6] is

$$p(t) = -\frac{2\mu}{\pi (\kappa+1) (a^2 - t^2)^{1/2}} \int_{-1}^{1} \frac{(a^2 - \xi^2)^{1/2}}{\xi - t} f'(\xi)\, d\xi + \frac{A}{(a^2 - t^2)^{1/2}} \tag{13}$$

where

$$A = \frac{1}{\pi} \int_{-a}^{a} p(t) dt = \frac{P}{\pi} \tag{14}$$

Now, if in Equation (13) we replace t by x,ξ by s, $\frac{2\mu}{\kappa+1} = \frac{E}{2(1-v^2)}$ and we divide by a, we get

$$p(t) = -\frac{2\mu}{\pi (\kappa+1) (a^2 - t^2)^{1/2}} \int_{-1}^{1} \frac{(a^2 - \xi^2)^{1/2}}{\xi - t} f'(\xi)\, d\xi + \frac{A}{(a^2 - t^2)^{1/2}} \tag{15}$$

where

$$f'(S) = \frac{\partial u_z}{\partial S} = \frac{S}{R}$$

Then, applying the Cauchy integral formula to Equation (15), we find

$$p(X) = -\frac{E*}{2\pi aR\left(1-X^2\right)^{1/2}}\int_{-1}^{1}\frac{\left(1-S^2\right)^{1/2}}{X-S}SdS + \frac{P}{\pi a\left(1-X^2\right)^{1/2}} \quad (16)$$

Now, in order to restrict the action of the pressure distribution into the loaded area we make $p(X) = 0$ at $X=\pm 1$ in Equation (16), in this way

$$P = \frac{\pi a^2 E*}{4R} \quad (17)$$

where

$$a = \sqrt{\frac{4RP}{\pi E*}} \quad (17.1)$$

Taking into account that $p(X)$ reaches its maximum value at $X =$ 0and that $p(X)= p(0)=p_0$, from Equation (16) we can get also

$$p_0 = \frac{2P}{\pi a} \quad (18)$$

Finally, substituting Equations (17) and (18) into Equation (16) and the principal value of the integral, we arrive to the Hertzian pressure distribution

$$p(X) = p_0\left(1-X^2\right)^{\frac{1}{2}} \quad (19)$$

$$p(x) = p_0\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} \quad (20)$$

4 Maximum-shear-sress

Once the expressions for the pressure distribution are derived (Equations (19) or (20)), the next step is to determine the ultimate shear stress before failure. To determine this shear stress we make use of the pressure distribution expressions in the following way. By replacing X by S in Equation (19) and substituting the result in Equation (7); afterwards, by replacing X by x/a and S by s/a and integrating over the loaded region, $-1 < S < 1$, we get the next dimensionless stress equations:

$$\frac{\sigma_x}{p_0} = -\frac{2Z}{\pi}\int_{-1}^{1}\frac{\left(1-S^2\right)^{1/2}(X-S)^2\,dS}{\left[(X-S)^2+Z^2\right]^2}$$

$$\frac{\sigma_z}{p_0} = -\frac{2Z^3}{\pi} \int_{-1}^{1} \frac{\left(1 - S^2\right)^{1/2} dS}{\left[(X - S)^2 + Z^2\right]^2} \tag{21}$$

$$\frac{\tau_{xz}}{p_0} = -\frac{2Z^2}{\pi} \int_{-a}^{a} \frac{\left(1 - S^2\right)^{1/2} (X - S)\, dS}{\left[(X - S)^2 + Z^2\right]^2} \tag{22}$$

$$\frac{\tau}{p_0} = \frac{|\sigma_1 - \sigma_2|}{2} \tag{23}$$

Equations (21) are the equations allowing determining the stresses in each point inside the gear tooth when the pressure distribution $p(s)$ is applied. Maximum dimensionless shear stress values, τ/p_0, at some points of the gear tooth contact surface into the region $-1 < X < 1$ and $0 < Z < 1.5$ are shown in Table 1. Additionally, in order to visualize these values, a map indicating the shear stress level inside the solid is shown in Figure 3. From Table 1, it is clear that $\tau/p_0 = 0.3$ is the maximum dimensionless shear stress and it is located at $Z = 0.8$ beneath the gear tooth work surface. By applying the Tresca yield criterion [7], the maximum pressure to reach this shear stress level is:

$$p_0 = \frac{\tau}{0.3} = \frac{\sigma_{\max}}{(2)\,(0.3)} = 1.66\sigma_{\max} \tag{24}$$

where σ_{max} is the maximum normal stress of material in the uniaxial-tension test.

Equation (24) can be rewritten in terms of load per unit length P using Equation (18), which results in

$$\sigma_{max} = \frac{2P}{1.66\pi a} \tag{25}$$

5 Surface stress equation with the gear parameters

In order to introduce the gear parameters into the surface stress equation we make use of the next relationship

$$P = \frac{W}{F}$$

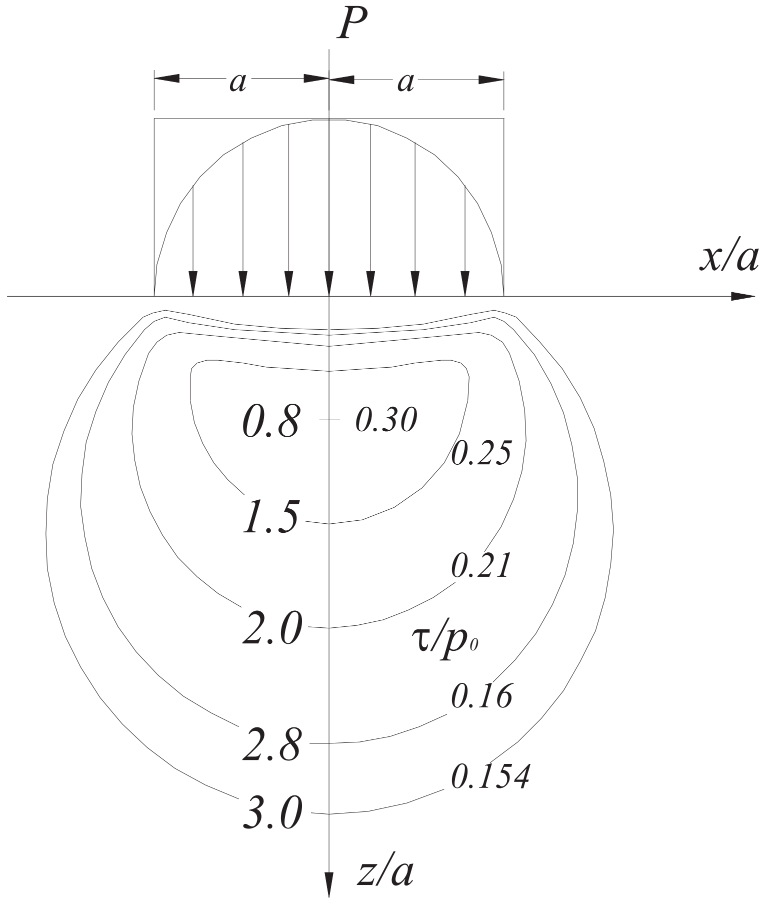

Figure 3: Map that shows the shear stress level τ/p_0 acting in the solid when charged.

where W is the total load applied and F the tooth face width; then, substituting P in Equation (25), we have

$$\frac{\sigma_x}{p_0} = \frac{2Z}{\pi} \int_{-1}^{1} \frac{(1-S^2)^{\frac{1}{2}}(X-S)^2 ds}{\left[(X-S)^2+Z^2\right]^2} \tag{26}$$

On the other hand, the equivalent radius of tooth gear is a function of the pinion and gear tooth curvature radiuses. Then, according with the

	τ/p_0	
z/a	$x/a = 0$	$x/a = 1$
0	0	0
0.1	0.09	0.162
0.2	0.161	0.2
0.3	0.214	0.22
0.4	0.251	0.231
0.5	0.276	0.237
0.6	0.291	0.239
0.7	0.299	0.239
0.8	0.3	0.238
0.9	0.298	0.235
1	0.293	0.231
1.1	0.286	0.227
1.2	0.278	0.223
1.3	0.27	0.218
1.4	0.261	0.213
1.5	0.252	0.208

Table 1: Dimensionless shear stress values τ/p_0 for $x/a = 0$, $x/a = 1$ and $0 < z/a < 1.5$.

AGMA [8]

$$\rho_P = \sqrt{\left(r_P + \frac{1}{p_d}\right)^2 - (r_P \cos\phi)^2} - \frac{\pi}{p_d}\cos\phi = R_1$$

$$\rho_g = C sen\phi + \rho_P = R_2 \tag{27}$$

where r_p is the pinion pitch radius, C is the centre distance and ϕ the pressure angle. Then, substituting Equations (27) into Equation (3), we have

$$\frac{1}{R} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{\rho_p} + \frac{1}{\rho_g} \tag{28}$$

Finally, combining Equations (17.1), (26) and (28), it can be find that

$$\sigma_{\max} = \sqrt{\frac{E * W}{1.66^2 \pi F}\left(\frac{1}{\rho_p} + \frac{1}{\rho_g}\right)} \tag{29}$$

This equation is the so called *Gear Design Equation* used to calculate the surface stresses as function of the gear parameters and the maximum-shear-stress, responsible of material failure. This maximum stress must always be smaller than the material yield stress σ_y, using in gear manufacturing

$$\sigma_{\max} < \sigma_Y \tag{30}$$

6 Discussion

As mentioned before, the stress produced on the contact surface has been routinely determined by using the Buckingham equation [9].

$$S_c = \frac{2W}{\pi F a} \tag{31}$$

where S_c is the maximum normal stress.

Comparing the Buckingham equation with Equation (26) we could observe that these equations are not the same, the difference is the (1/1.66) factor. This difference is the result of considering the failure shear stress criteria (Tresca) into Equation (26), which guarantees its validity. This difference can be explained also by applying the Tresca yield criterion into the Buckingham equation to determine the maximum-shear-stress level, τ/p_0, into the contact surface. The resulting equation is as follow

$$p_0 = \frac{\tau}{M} = \frac{S_c}{2M} = 1S_c \tag{32}$$

Now, to satisfy Equation (32) M must be equal to 0.5, implying that the maximum dimensionless shear stress produced beneath the contact surface is

$$\frac{\tau}{p_0} = 0.5 \tag{33}$$

Returning to Table 1 and Figure 2, it is clear that the value $\tau/p_0 = 0.5$ does not exist. Then, we can conclude that the stress considered by the Buckingham equation is beyond and higher the maximum in the pinion-gear tooth contact case.

7 Conclusions

Considering the Tresca yield criteria, one could arrive to a singular expression, Equation (26), that provides the maximum load W that the gear tooth supports and the location $(z/a = 0.8//at//x/a = 0)$ of maximum stress point.

By using the Buckingham equation, it is not possible to find, in gear tooth, the location of the maximum-shear-stress point.

The stress calculated by means of the Buckingham equation does not take into account any failure criteria, however our equation does.

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Matematički model za odredjivanje površinskog napona koji deluje na zubac zupčanika

Bakingemova jednačina za odredjivanje površinkog napona na kontaktnoj površi zupca zupčanika služi kao osnova formule otpora medjuzublja Američkog udruženja proizvodjača zupčanika (The American Gear Manufacturers Association-AGMA). Ova formula je zasnovana na normalnom naponu koji nije odgoran za otkaz pošto je plastčno tečenje u kontaktnim problemima uzrokovano smičućim naponima. U ovom radu se predlaže jedan alternativni izraz zasnovan na maksimalnom smičućem naponu. Taj novi izraz je dobijen korišćenjem rasporeda maksimalnog smičućeg napona i Treskinog kriterijuma otkaza u cilju odredjivanja vrednosti maksimalnog smičućeg napona i njegovog položaja iza kontaktne površi. Uporedjujući razlike rezultata dobijenih ovom metodom sa rezultatioma AGMA/metoda uočavamo izvanredne razlike.

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