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Conditional stability of Larkin methods with non-uniform grids

Kazuhiro Fukuyo*

Abstract

Stability analysis based on the von Neumann method showed that the Larkin methods for two-dimensional heat conduction with non-uniform grids are conditionally stable while they are known to be unconditionally stable with uniform grids. The stability criteria consisting of the dimensionless time step Δt, the space intervals Δx, Δy, and the ratios of neighboring space intervals α, β were derived from the stability analysis. A subsequent numerical experiment demonstrated that solutions derived by the Larkin methods with non-uniform grids lose stability and accuracy when the criteria are not satisfied.

Keywords: finite difference method; stability, heat conduction; Saul'yev method, Larkin method

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List of notations

f, g	mesh Fourier numbers in the x and y directions
r, s	auxiliary variables
t	dimensionless time
v, w	auxiliary variables
x, y	dimensionless space coordinates
D	denominator of G
G	complex amplification factor
Im	imaginary part of complex numbers
N	numerator of G
Re	real part of complex numbers

Greek symbols

α, β	space interval ratio in the x and y directions
Δt	dimensionless time step
$\Delta x, \Delta y$	dimensionless space intervals in the x and y directions
θ	dimensionless temperature

Subscripts and Superscripts

0	initial condition or location of the point heat source
i, j	indices of grid point
max	maximum number of index
n	index of time steps

1 Introduction

The explicit finite-difference methods proposed by Saul'yev [1], Larkin [2], and Barakat and Clark [3] are generally called the alternating direction explicit (ADE) methods. They are known to be unconditionally stable for solving unsteady diffusion equations with uniform grids. Saul'yev originally proposed his method for one-dimensional parabolic partial differential equations. Larkin, Barakat, and Clark expanded the Saul'yev method to two- and three-dimensional problems. (Here the term "parabolic partial differential equation" generally refers the unsteady diffusion equation, such as the unsteady heat conduction equation. In contrast, the steady diffusion equation, i.e., the Laplace equation is classified as elliptic and numerically solved by the iterative methods such as the Jacobi method, Gauss-Seidel method, etc.).

There are many numerical studies comparing the ADE methods with other methods in terms of accuracy and computational time. Galligani [4] used the Saul'yev method to solve multi-group diffusion equations and compared it with other implicit and explicit methods. Towler and Yang [5] compared the original and modified Saul'yev methods with Crank-Nicolson methods for one-dimensional parabolic partial differential equations. Thibault [6] applied nine finite difference methods, including the Larkin and Barakat-Clark methods, to three-dimensional heat conduction problems and compared them. Darvishi [7] compared the original and modified Saul'yev methods for a three-dimensional heat conduction problem. Dehghan [8] compared six finite difference methods, including two types of Saul'yev methods, for one-dimensional parabolic partial differential equations. Bokahri and Islam [9] applied the Barakat-Clark method to a two-dimensional convective-diffusive equation and compared the solution with that for the other explicit method. These comparative studies showed the superiority of the ADE methods.

However, these studies were carried out only with uniform grid systems, which is problematic because non-uniform grids are often used for practical problems. For example, for a point heat-source problem, the calculation points of grids concentrate around the heat source point to enhance the resolution and computational efficiency. In order to discuss the applicability of ADE methods to practical problems, their stability and accuracy have to be investigated for non-uniform grid systems.

The author of this paper has already discussed the stability of the Saul'yev method for one-dimensional heat conduction with non-uniform grids and presented the criterion for the stability of the Saul'yev method with both non-uniform and uniform grids [10].

This paper discusses the stability of ADE methods for two-dimensional heat conduction, i.e., the Larkin methods, with non-uniform grids. Stability analysis based on the von Neumann methods [11] was carried out and the criteria for the stability of the Larkin methods are presented.

2 Algorithm of the Larkin Methods

For the sake of simplicity, we will treat the two-dimensional unsteady heat conduction equation written in the following dimensionless form.

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \tag{1}$$

Let us now assume the time and space shown in Fig.1 (which has a non-uniform grid system). To solve Eq. (1) numerically, we consider four

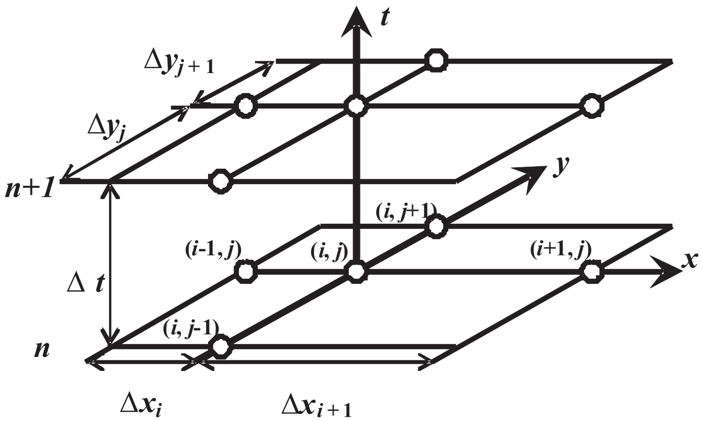


Figure 1: Time and space for two-dimensional calculations

auxiliary variables r, s, v, and w that satisfy Eq. (1). Using Larkin's idea, we can present the following finite difference equations corresponding to Eq. (1).

$$\begin{aligned}\frac{r_{i,j}^{n+1} - r_{i,j}^{n}}{\Delta t} = & \frac{2}{\Delta x_i + \Delta x_{i+1}} \left(\frac{r_{i+1,j}^{n} - r_{i,j}^{n}}{\Delta x_{i+1}} - \frac{r_{i,j}^{n+1} - r_{i-1,j}^{n+1}}{\Delta x_i} \right) + \\ & \frac{2}{\Delta y_j + \Delta y_{j+1}} \left(\frac{r_{i,j+1}^{n} - r_{i,j}^{n}}{\Delta y_{j+1}} - \frac{r_{i,j}^{n+1} - r_{i,j-1}^{n+1}}{\Delta y_j} \right)\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{s_{i,j}^{n+1}-s_{i,j}^{n}}{\Delta t} = & \frac{2}{\Delta x_i+\Delta x_{i+1}}\left(\frac{s_{i+1,j}^{n+1}-s_{i,j}^{n+1}}{\Delta x_{i+1}}-\frac{s_{i,j}^{n}-s_{i-1,j}^{n}}{\Delta x_i}\right)+ \\ & \frac{2}{\Delta y_j+\Delta y_{j+1}}\left(\frac{s_{i,j+1}^{n+1}-s_{i,j}^{n+1}}{\Delta y_{j+1}}-\frac{s_{i,j}^{n}-s_{i,j-1}^{n}}{\Delta y_j}\right)\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta t} = & \frac{2}{\Delta x_i+\Delta x_{i+1}}\left(\frac{v_{i+1,j}^{n}-v_{i,j}^{n}}{\Delta x_{i+1}}-\frac{v_{i,j}^{n+1}-v_{i-1,j}^{n+1}}{\Delta x_i}\right)+ \\ & \frac{2}{\Delta y_j+\Delta y_{j+1}}\left(\frac{v_{i,j+1}^{n+1}-v_{i,j}^{n+1}}{\Delta y_{j+1}}-\frac{v_{i,j}^{n}-v_{i,j-1}^{n}}{\Delta y_j}\right)\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{w_{i,j}^{n+1}-w_{i,j}^{n}}{\Delta t} = & \frac{2}{\Delta x_i+\Delta x_{i+1}}\left(\frac{v_{i+1,j}^{n+1}-v_{i,j}^{n+1}}{\Delta x_{i+1}}-\frac{v_{i,j}^{n}-v_{i-1,j}^{n}}{\Delta x_i}\right)+ \\ & \frac{2}{\Delta y_j+\Delta y_{j+1}}\left(\frac{v_{i,j+1}^{n}-v_{i,j}^{n}}{\Delta y_{j+1}}-\frac{v_{i,j}^{n+1}-v_{i,j-1}^{n+1}}{\Delta y_j}\right)\end{aligned} \tag{5}$$

Now we define the space interval ratios

$$\alpha_i \equiv \frac{\Delta x_{i+1}}{\Delta x_i}, \quad \beta_j \equiv \frac{\Delta y_{j+1}}{\Delta y_j}$$

the mesh Fourier numbers

$$f_i \equiv \frac{\Delta t}{(\Delta x_i)^2}, \quad g_j \equiv \frac{\Delta t}{(\Delta y_j)^2}$$

and the modified Fourier numbers.

$$\hat{f}_i \equiv \frac{2f_i}{\alpha_i\,(1+\alpha_i)}, \quad \hat{g}_j \equiv \frac{2g_j}{\beta_j\,(1+\beta_j)}$$

Using the space interval ratios and modified Fourier numbers, Eqs. (2)-(5) may be arranged in the following form.

$$\begin{aligned}\left(1+\alpha_i\hat{f}_i+\beta_j\hat{g}_j\right) r_{i,j}^{n+1} = & \left(1-\hat{f}_i-\hat{g}_j\right) r_{i,j}^{n} \\ & +\hat{f}_i r_{i+1,j}^{n}+\alpha_i\hat{f}_i r_{i-1,j}^{n+1} \\ & +\hat{g}_j r_{i,j+1}^{n}+\beta_j\hat{g}_j r_{i,j-1}^{n+1}\end{aligned} \tag{6}$$

$$\begin{aligned}\left(1+\hat{f}_i+\hat{g}_j\right)s_{i,j}^{n+1} = &\left(1-\alpha_i\hat{f}_i-\beta_j\hat{g}_j\right)s_{i,j}^{n} \\ &+\hat{f}_i s_{i+1,j}^{n+1}+\alpha_i\hat{f}_i s_{i-1,j}^{n} \\ &+\hat{g}_j s_{i,j+1}^{n+1}+\beta_j\hat{g}_j s_{i,j-1}^{n}\end{aligned} \tag{7}$$

$$\begin{aligned}\left(1+\alpha_i\hat{f}_i+\hat{g}_j\right)v_{i,j}^{n+1} = &\left(1-\hat{f}_i-\beta_j\hat{g}_j\right)v_{i,j}^{n} \\ &+\hat{f}_i v_{i+1,j}^{n}+\alpha_i\hat{f}_i v_{i-1,j}^{n+1} \\ &+\hat{g}_j v_{i,j+1}^{n+1}+\beta_j\hat{g}_j v_{i,j-1}^{n}\end{aligned} \tag{8}$$

$$\begin{aligned}\left(1+\hat{f}_i+\beta_j\hat{g}_j\right)w_{i,j}^{n+1} = &\left(1-\alpha_i\hat{f}_i-\hat{g}_j\right)w_{i,j}^{n} \\ &+\hat{f}_i w_{i+1,j}^{n+1}+\alpha_i\hat{f}_i w_{i-1,j}^{n} \\ &+\hat{g}_j w_{i,j+1}^{n}+\beta_j\hat{g}_j w_{i,j-1}^{n+1}\end{aligned} \tag{9}$$

Larkin [2] showed the five computing schemes as described below.

1. Consider that the variable r is equivalent to the dimensionless temperature θ, use Eq.(6) only, and proceed calculation from the grid point $i=1$ and $j=1$ in a sequence of increasing i and j (see Fig.2).

2. Consider that the variable s is equivalent to the dimensionless temperature θ, use Eq.(7) only, and proceed calculation from the grid point $i=i_{max}$ and $j=j_{max}$ in a sequence of decreasing i and j (see Fig.2).

3. Consider that the variables r and s are equivalent to the dimensionless temperature θ, use Eqs.(6) and (7) alternately as follows. Proceed calculation from the grid point $i=1$ and $j=1$ in a sequence of increasing i and j by using Eq.(6) at the time level $n+1$,

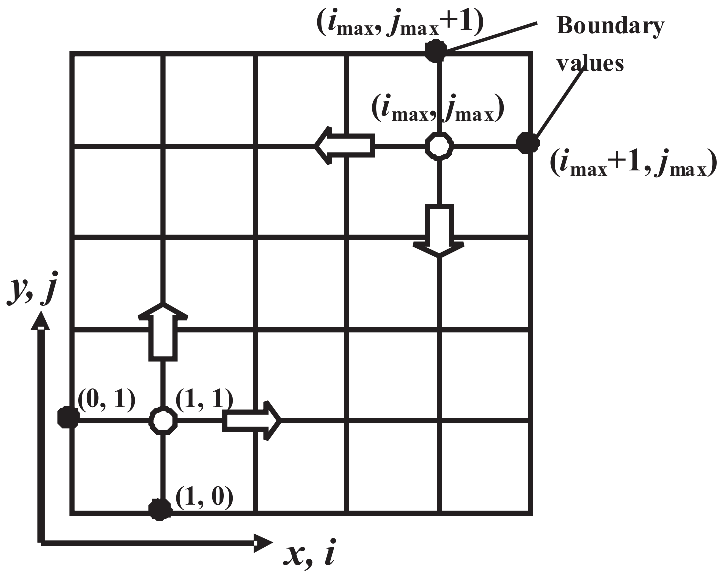


Figure 2: Schematic of a grid system, starting points for calculation, and directions of calculation

substitute the value of $r_{i,j}^{n+1}$ into the variables $s_{i,j}^{n+1}$, and proceed calculation from the grid point $i = i_{max}$ and $j = j_{max}$ in a sequence of decreasing i and j by using Eq.(7) at the time level $n + 2$.

4. Consider that the variables r and s are equivalent to the dimensionless temperature θ, use Eqs.(6) and (7) at every time level, and average the results:

$$\theta_{i,j}^{n+1} = \frac{r_{i,j}^{n+1} + s_{i,j}^{n+1}}{2} \tag{10}$$

Next, substitute the value of $\theta_{i,j}^{n+1}$ into the variables $r_{i,j}^{n+1}$ and $s_{i,j}^{n+1}$. The Barakat-Clark method is similar to this scheme, but the variables r, s, and θ are calculated separately [3]. In the Barakat-Clark method, the variables $r_{i,j}^{n+1}$ and $s_{i,j}^{n+1}$ are not replaced by $\theta_{i,j}^{n+1}$.

5. Consider that the variables v and w are equivalent to the dimensionless temperature θ and use Eqs. (8) and (9) alternately as follows. Proceed calculation from the grid point $i = 1$ and $j = j_{max}$ in a

sequence of increasing i and decreasing j by using Eq. (8) at the time level n+1, substitute the value of $r_{i,j}^{n+1}$ into the variables $s_{i,j}^{n+1}$, and proceed calculation from the grid point $i = i_{max}$ and $j = 1$ in a sequence of decreasing i and increasing j by using Eq. (9) at the time level $n + 2$.

It is clear that the fifth scheme is a variation of the third scheme. We can make more schemes with combinations of Eqs.(6)-(9). These equations are symmetric, and we therefore primarily discuss Eq.(6) in the following sections.

3 Stability Analyses

3.1 Larkin's first scheme

The stability of Larkin's first scheme is examined using the von Neumann method [11]. This method assumes that the solution of the discretized equations can be represented by Fourier expansion. The general term of the expansion can be written as

$$\theta_g\,(t, x, y) = \exp\,(\gamma_0\, t)\exp\,(i\,\gamma_1\, x)\exp\,(i\,\gamma_2\, y) \tag{11}$$

Substituting this into Eq. (6) and arranging it, the complex amplification factor G for Larkin's first scheme is derived as

$$\begin{aligned} G &= \frac{\theta_g\,(t+\Delta t,\ x, y)}{\theta_g\,(t,\ x, y)} = \exp\,(\gamma_0\,\Delta t) \\ &= \frac{1 - \hat{f}_i\,\{\,1 - \exp\,(i\,\gamma_1 \alpha_i \Delta x_i)\,\} - \hat{g}_j\,\{\,1 - \exp\,(i\,\gamma_2 \beta_j \Delta y_j)\,\}}{1 + \alpha_i \hat{f}_i\,\{\,1 - \exp\,(-i\,\gamma_1 \Delta x_i)\,\} + \beta_j \hat{g}_j\,\{\,1 - \exp\,(-i\,\gamma_2 \Delta y_j)\,\}} \end{aligned} \tag{12}$$

If we call the numerator and denominator at the far right side of this equation N and D, respectively, the absolute value of G is derived by

$$|G| = \frac{|N|}{|D|} = \frac{\sqrt{\{Re\,(N)\}^2 + \{Im\,(N)\}^2}}{\sqrt{\{Re\,(D)\}^2 + \{Im\,(D)\}^2}} \tag{13}$$

The finite-difference methods are stable if the absolute value of G is less than or equal to unity:

$$|G| \le 1 \tag{14}$$

Substituting Eq. (13) into inequality (14), we obtain the following inequality:

$$|D|^2 - |N|^2 = \left[\{Re\,(D)\}^2 + \{Im\,(D)\}^2\right] - \left[\{Re\,(N)\}^2 + \{Im\,(N)\}^2\right] \geq 0 \tag{15}$$

By expansion of { Re(D) }^{2}, { Im(D) }^{2}, { Re(N) }^{2}, and $|D|^2 - |N|^2$ is derived as follows.

$$\begin{aligned} |D|^2 - |N|^2 = 2\hat{f}_i \Big\{ \alpha_i \Big(1 + \alpha_i \hat{f}_i - \beta_j \hat{g}_j\Big) & (1 - \cos\gamma_1 \Delta x_i) \\ & + \Big(1 - \hat{f}_i - \hat{g}_j\Big)(1 - \cos\gamma_1 \alpha_i \Delta x_i)\Big\} \\ & +2\hat{g}_j \Big\{ \beta_j \Big(1 - \alpha_i \hat{f}_i + \beta_j \hat{g}_j\Big)(1 - \cos\gamma_2 \Delta y_j) \\ & \times\Big(1 - \hat{f}_i - \hat{g}_j\Big)(1 - \cos\gamma_2 \beta_j \Delta y_j)\Big\} \\ & +2\hat{f}_i \hat{g}_j \left[\alpha_i \beta_j \{3 - 2\cos\gamma_1 \Delta x_i - 2\cos\gamma_2 \Delta y_j \right. \\ & \left. + \cos(\gamma_1 \Delta x_i - \gamma_2 \Delta y_j)\} + \{1 - \cos(\gamma_1 \alpha_i \Delta x_i - \gamma_2 \beta_j \Delta y_j)\}\right] \end{aligned} \tag{16}$$

The values of $(1 - \cos\gamma_1\Delta x_i)$, $(1 - \cos\gamma_1\alpha_i\Delta x_i)$, $(1 - \cos\gamma_2\Delta y_j)$, $(1 - \cos\gamma_2\beta_j\Delta y_j)$ and $\{1 - \cos(\gamma_1\alpha_i\Delta x_i - \gamma_2\beta_j\Delta y_j)\}$ are positive or equal to zero for all γ_1 and γ_2.

The value of $\{3 - 2\cos\gamma_1\Delta x_i - 2\cos\gamma_2\Delta y_j + \cos(\gamma_1\Delta x_i - \gamma_2\Delta y_j)\}$ is also positive or equal to zero for all γ_1 and γ_2 (see appendix). Therefore, the following three inequalities have a sufficient condition to satisfy inequality (15).

$$1 + \alpha_i \hat{f}_i - \beta_j \hat{g}_j \geq 0$$
$$1 - \alpha_i \hat{f}_i + \beta_j \hat{g}_j \geq 0$$
$$1 - \hat{f}_i - \hat{g}_j \geq 0$$

These are rewritten as

$$1 + \frac{2f_i}{1 + \alpha_i} - \frac{2g_j}{1 + \beta_j} \geq 0 \tag{17}$$

$$1 - \frac{2f_i}{1+\alpha_i} + \frac{2g_j}{1+\beta_j} \geq 0 \tag{18}$$

$$1 - \frac{2f_i}{\alpha_i\,(1+\alpha_i)} - \frac{2g_j}{\beta_j\,(1+\beta_j)} \geq 0 \tag{19}$$

Solving these inequalities for g_i, we obtain the following criteria for the stability of Larkin's first scheme.

$$g_j \leq \frac{1+\beta_j}{2} + \frac{1+\beta_j}{1+\alpha_i} f_i \tag{20}$$

$$g_j \geq -\frac{1+\beta_j}{2} + \frac{1+\beta_j}{1+\alpha_i} f_i \tag{21}$$

$$g_j \leq \frac{\beta_j\,(1+\beta_j)}{2} - \frac{\beta_j\,(1+\beta_j)}{\alpha_i\,(1+\alpha_i)} f_i \tag{22}$$

The solution of this system of equalities for the space interval ratios α_i, $\beta_j \neq 1$(i.e., the grid system is non-uniform) is shown in Fig. 3. When the space interval ratios α_i and β_j are equal to unity (i.e., the grid system is uniform), we obtain

$$|D|^2 - |N|^2 = 4f_i\,(1 - \cos\gamma_1\Delta x_i) + 4g_j\,(1 - \cos\gamma_2\Delta y_j)$$

It obviously satisfies inequality (15) for all γ_1 and γ_2. This means that the Larkin methods with a uniform grid system are unconditionally stable.

3.2 Larkin's second scheme

By substituting Eq. (11) into Eq. (7) and arranging it, the complex amplification factor G for Larkin's second scheme is derived as

$$G = \frac{1 - \alpha_i\hat{f}_i\left\{1 - \exp\left(-\sqrt{-1}\,\gamma_1\Delta x_i\right)\right\} - \beta_j\hat{g}_j\left\{1 - \exp\left(-\sqrt{-1}\,\gamma_2\Delta y_j\right)\right\}}{1 + \hat{f}_i\left\{1 - \exp\left(\sqrt{-1}\,\gamma_1\alpha_i\Delta x_i\right)\right\} + \hat{g}_j\left\{1 - \exp\left(\sqrt{-1}\,\gamma_2\beta_j\Delta y_j\right)\right\}} \tag{23}$$

Let the numerator and denominator on the right side of this equation be N and D, respectively; then we obtain $|D|^2 - |N|^2$ as follows.

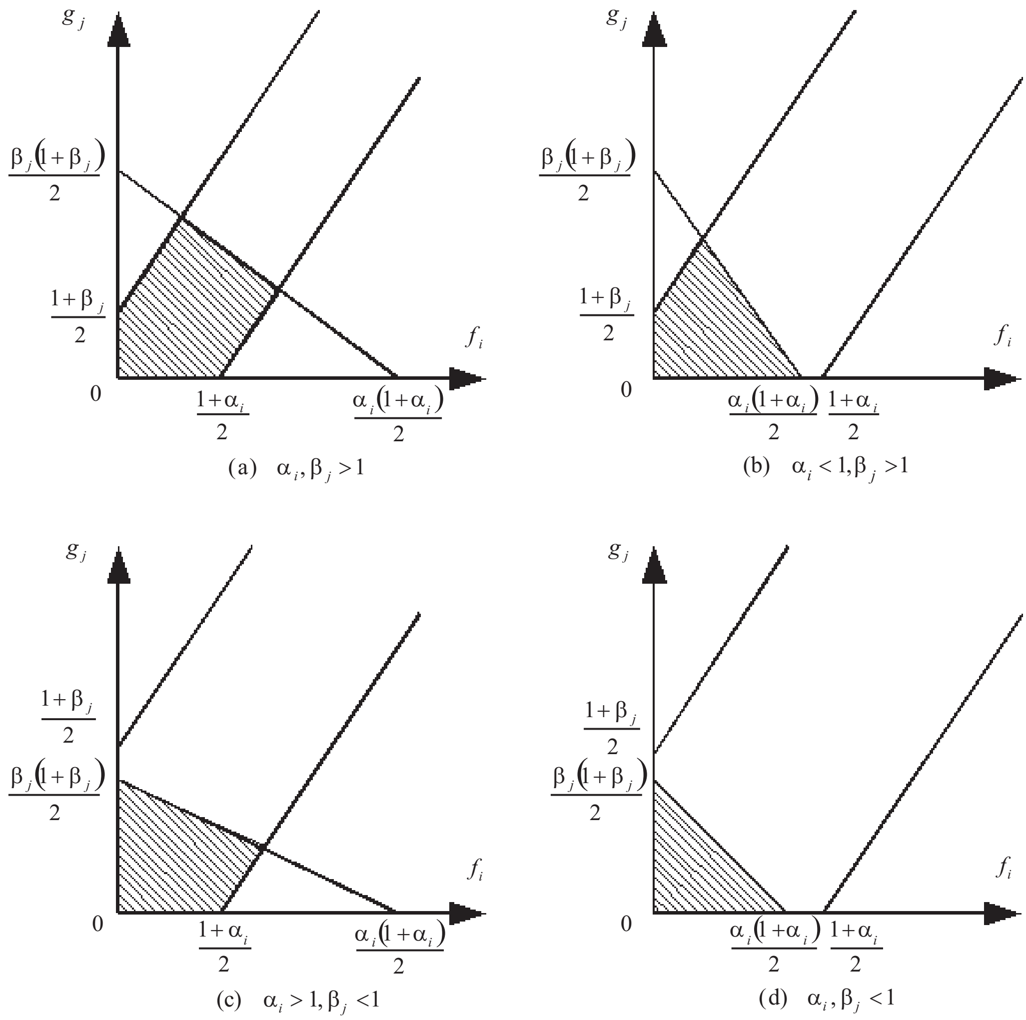


Figure 3: Solution of the stability criteria for Larkin's first scheme

$$
\begin{aligned}
|D|^2 - |N|^2 = 2\hat{f}_i & \left\{ \left(1 + \hat{f}_i - \hat{g}_j\right)(1 - \cos\gamma_1\alpha_i\Delta x_i) \right. \\
& \left. +\alpha_i \left(1 - \alpha_i\hat{f}_i - \beta_j\hat{g}_j\right)(1 - \cos\gamma_1\Delta x_i) \right\} \\
& +2\hat{g}_j \left\{ \left(1 - \hat{f}_i + \hat{g}_j\right)(1 - \cos\gamma_2\beta_j\Delta y_j) \right. \\
& \left. +\beta_j \left(1 - \alpha_i\hat{f}_i - \beta_j\hat{g}_j\right)(1 - \cos\gamma_2\Delta y_j) \right\} \\
& +2\hat{f}_i\hat{g}_j \left[\{3 - 2\cos\gamma_1\alpha_i\Delta x_i - 2\cos\gamma_2\beta_j\Delta y_j + \cos(\gamma_1\alpha_i\Delta x_i - \gamma_2\beta_j\Delta y_j)\} \right. \\
& \left. +\alpha_i\beta_j \{1 - \cos(\gamma_1\Delta x_i - \gamma_2\Delta y_j)\} \right]
\end{aligned}
\tag{24}
$$

In order to satisfy the inequality (15), the following inequalities are required.

$$1+\alpha_i\hat{f}_i-\beta_j\hat{g}_j\geq 0$$

$$1-\alpha_i\hat{f}_i+\beta_j\hat{g}_j\geq 0$$

$$1-\hat{f}_i-\hat{g}_j\geq 0$$

Solving these inequalities for g_i, we obtain the following criteria for the stability of Larkin's second scheme.

$$g_j\leq\frac{\beta_j\left(1+\beta_j\right)}{2}+\frac{\beta_j\left(1+\beta_j\right)}{\alpha_i\left(1+\alpha_i\right)}f_i \tag{25}$$

$$g_j\geq-\frac{\beta_j\left(1+\beta_j\right)}{2}+\frac{\beta_j\left(1+\beta_j\right)}{\alpha_i\left(1+\alpha_i\right)}f_i \tag{26}$$

$$g_j\leq\frac{1+\beta_j}{2}-\frac{1+\beta_j}{1+\alpha_i}f_i \tag{27}$$

Now, we define the new variables $\alpha_i'\equiv\Delta x_i/\Delta x_{i+1}=1/\alpha_i$, $\beta_j'\equiv\Delta y_j/\Delta y_{j+1}=1/\beta_j$, $f_i'\equiv\Delta t/(\Delta x_{i+1})^2=\alpha_i'^2 f_i$ and $g_j'\equiv\Delta t/(\Delta y_{j+1})^2=\beta_j'^2 g_j$. By substituting them into the inequalities (20), (21), and (22), we obtain the following inequalities.

$$g_j'\leq\frac{\beta_j'\left(1+\beta_j'\right)}{2}+\frac{\beta_j'\left(1+\beta_j'\right)}{\alpha_i'\left(1+\alpha_i'\right)}f_i'$$

$$g_j'\geq-\frac{\beta_j'\left(1+\beta_j'\right)}{2}+\frac{\beta_j'\left(1+\beta_j'\right)}{\alpha_i'\left(1+\alpha_i'\right)}f_i'$$

$$g_j'\leq\frac{1+\beta_j'}{2}-\frac{1+\beta_j'}{1+\alpha_i'}f_i'$$

These inequalities have the same forms as inequalities (25), (26), and (27).

3.3 Larkin's other schemes

Larkin's third and forth schemes use both Eqs. (6) and (7). Therefore, the stability criteria of these schemes are the six inequalities (20), (21), (22), (25), (26), and (27). The ranges of the mesh Fourier numbers f_i and g_j which satisfy the stability condition for the third and forth schemes may be more limited than for the first and second ones due to increasing number of the inequalities. By carrying out stability analyses for Eqs (8) and (9), we can obtain the stability criteria for the fifth scheme.

4 Numerical Experiment

To verify the derived criteria, we will carry out a numerical experiment on Larkin's first scheme with non-uniform grids.

Now we seek the solution of Eq. (1) in the region bounded by $0 \leq x \leq 1$ and $0 \leq y \leq 1$, subject to the initial condition:

$$\theta\,(x,\,y) = \exp\left(-\frac{(x-0.5)^2+(y-0.5)^2}{4t_0}\right). \tag{28}$$

The boundary conditions are given by:

$$\theta\,(t,\,y) = \frac{t_0}{t}\exp\left(-\frac{0.25+(y-0.5)^2}{4t}\right) \quad \text{at} \quad (x=0,\,1) \tag{29}$$

and

$$\theta\,(t,\,x) = \frac{t_0}{t}\exp\left(-\frac{(x-0.5)^2+0.25}{4t}\right) \quad \text{at} \quad (y=0,\,1) \tag{30}$$

where $t_0 > 0$.

The analytical solution of this problem is

$$\theta\,(t,\,x,\,y) = \frac{t_0}{t}\exp\left(-\frac{(x-0.5)^2+(y-0.5)^2}{4t}\right) \tag{31}$$

This is the instantaneous point heat source problem [12], and the solution describes a two-dimensional Gaussian pulse.

We will solve the two-dimensional Gaussian-pulse problem numerically for $t_0 = 0.0001$ and $t = 0.0011$ by using Larkin's first scheme with the non-uniform grid shown in Fig.4. The non-uniform grid has 841 calculating points and 116 boundary-condition points. This non-uniform grid is generated by the rule that states α_i and $\beta_j = 0.8$ for i and $j = 1, 2, \ldots, 14$, α_i and $\beta_j = 1.0$ for i and $j = 15$, and α_i and $\beta_j = 1.25$ for i and $j = 16, 17, \ldots, 29$ to concentrate the calculation points around the point heat source.

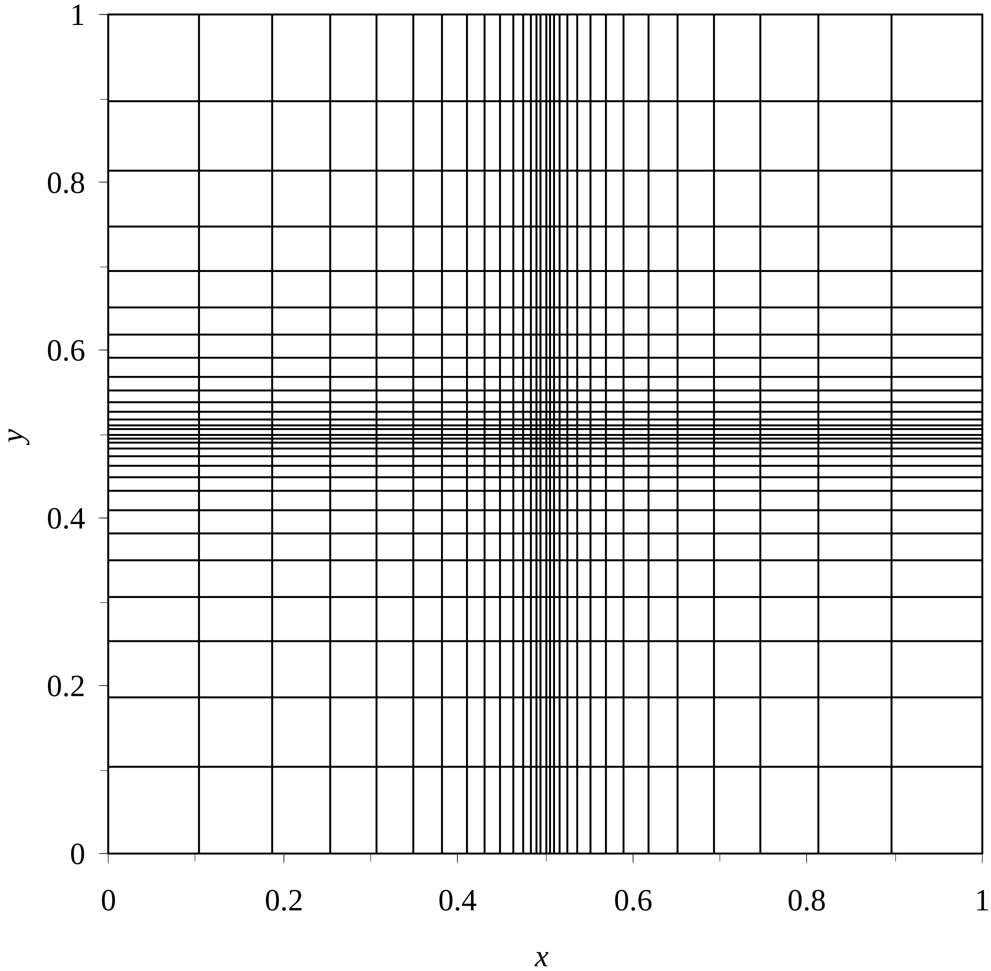


Figure 4: Grid system for the numerical experiment

The analytical and numerical solutions for $\Delta t = 0.0002, 0.0001$, and 0.000002 are shown in Figs.5 and 6. Figure 5 shows the three-dimensional view of the dimensionless temperature distribution at $t = 0.0011$. The numerical solutions for $\Delta t = 0.0002$ and 0.0001 differ from the analytical one while the numerical solution for $\Delta t = 0.000002$ agrees well with the analytical one. To discuss the accuracy of each numerical solution, we investigate the temperature distribution on a line across the center of the grid. Figure 6 shows the cross-sectional temperature distribution along

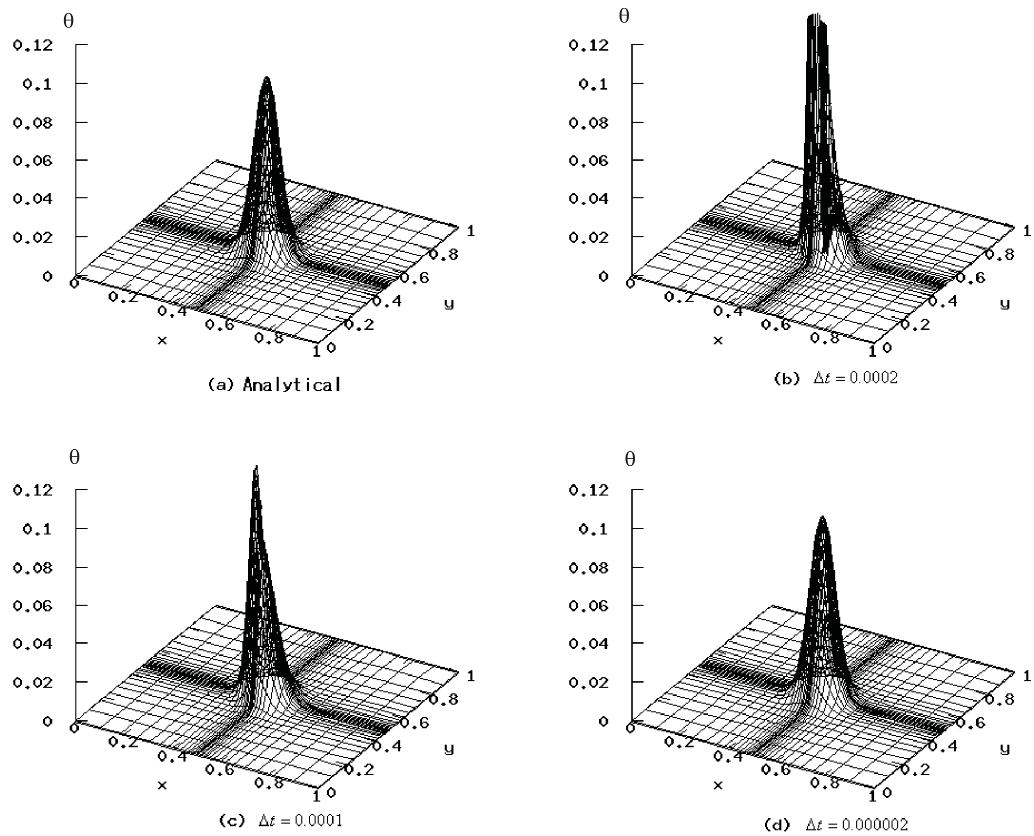


Figure 5: Three-dimensional view of the analytical and Larkin's solutions for the two-dimensional Gaussian pulse with the non-uniform grid at $t = 0.0011$

the line $y = 0.5$ at $t = 0.0011$. This figure shows the difference between the analytical and Larkin's solutions. Now we define the relative error by

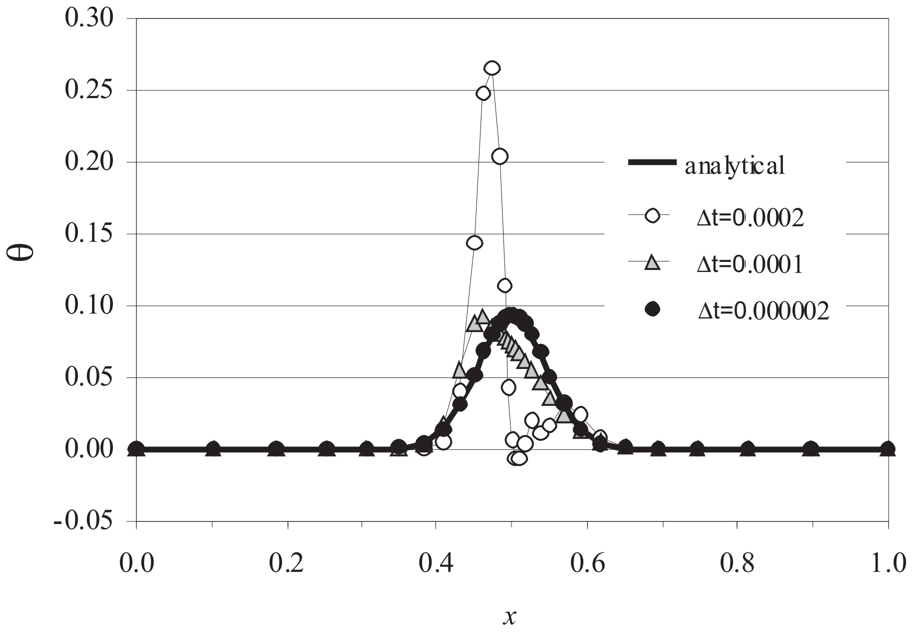


Figure 6: Sectional view of analytical and Larkin's solutions along the line $y = 0.5$ at $t = 0.0011$ for two-dimensional Gaussian pulse with the non-uniform grid

the following equation:

$$relative\,error = \left| \frac{\theta_{numerical} - \theta_{analytical}}{\theta_{analytical}} \right| \times 100[\%] \tag{32}$$

The maximum difference between the analytical and Larkin's solution for $\Delta t = 0.0002$ is 0.1877 at $x = 0.4737$ and the relative error is 241.6%. As to the Larkin's solution for $t = 0.0001$, the maximum difference is 0.0370 at $x = 0.4487$ and the relative error is 74.0%.

The numerical solution for $\Delta t = 0.0002$ especially loses the physical reality, that is, the dimensionless temperature at the calculating points around the center of the grid shows a negative value. This is because the stability criteria are violated at those calculating points. In contrast, Fig. 6 shows that the numerical solution for $\Delta t = 0.000002$ agrees well with the analytical one. The maximum difference between the analytical and Larkin's solution for $\Delta t = 0.000002$ is only 0.0027 at $x = 0.5$ and the relative error is 3.0%.

Stability and accuracy of a numerical solution with Larkin's first scheme depend on whether the criteria (inequalities) (20), (21), and (22) are satisfied. In the calculations based on Larkin's first scheme with $t = 0.0002$ and 0.0001, the criteria are violated at 657 calculating points (78.1% of all the calculating points) and 550 calculating points (65.4%), respectively. By contrast, in the calculation with $\Delta t = 0.000002$, the criteria are satisfied at all calculating points. The result of the numerical experiment showed that the dimensionless time step Δt should be sufficiently small to satisfy the criteria.

In this chapter, we solve the two-dimensional Gaussian-pulse problem numerically to demonstrate that solutions of the Larkin methods with non-uniform grids lose stability and accuracy when the criteria are not satisfied. The stability criteria can be applied to any other two-dimensional heat conduction problem described by the equation (1) such as a finite rectangle with non-homogeneous boundary, etc.

5 Discussion

Non-uniform grids are often used for practical problems. However, the stability analyses and numerical experiment show that the Larkin methods with non-uniform grids are conditionally stable. This is expected by the result of the stability analysis of the one-dimensional ADE, i.e., Saul'yev methods [10]. The stability criteria of Larkin's first and second schemes are more complicated than those of Saul'yev methods.

The primary focus of this paper was Larkin's first scheme, but those who investigate Larkin's other schemes with non-uniform grids can confirm their conditional stability and derive similar stability criteria by the same procedure as described in section 3.1, because the other schemes are symmetrical to the first scheme or are combinations of these symmetrical schemes.

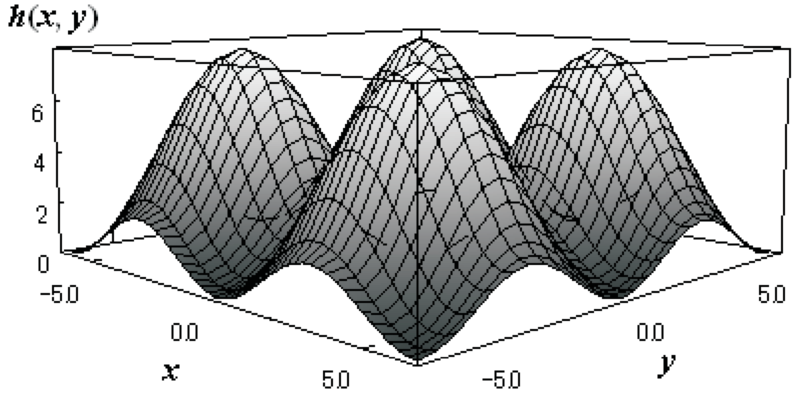
6 Conclusion

This paper has shown that the Larkin methods for two-dimensional heat conduction with non-uniform grids are conditionally stable while they are known to be unconditionally stable with uniform grids. The stability of the Larkin methods depends on the dimensionless time step Δt, the space intervals Δx_i, Δy_j, and the ratios of neighboring space intervals $\alpha_i = \Delta x_{i+1}/\Delta x_i$, $\beta_j = \Delta y_{j+1}/\Delta y_j$. For Larkin's first scheme, the stability criteria are described as $g_j \leq (1+\beta_j)/2 + (1+\beta_j)\, f_i/\,(1+\alpha_i)$, $g_j \geq -(1+\beta_j)/2 + (1+\beta_j)\, f_i/\,(1+\alpha_i)$ and $g_j \leq \beta_j\,(1+\beta_j)/2-\ \beta_j\,(1+\beta_j)\, f_i/ \{\alpha_i\,(1+\alpha_i)\}$, where the mesh Fourier numbers $f_i = 2\Delta t/(\Delta x_i)^2$ and $g_j = 2\Delta t/(\Delta y_j)^2$. The numerical experiment for two-dimensional Gaussian pulse showed that the solutions by Larkin's first scheme with non-uniform grids lose stability and accuracy when the criteria are not satisfied.

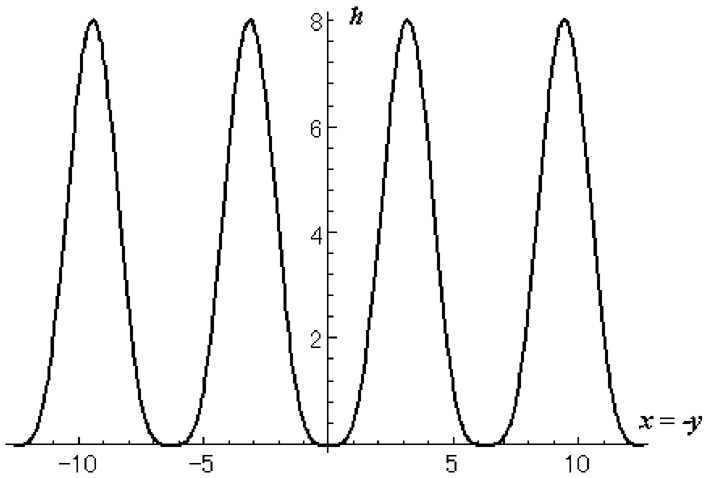
Appendix

Figure 7 (a) shows the three-dimensional view of the value of $h\,(x,\,y) = 3 - 2\cos x - 2\cos y + \cos\,(x-y)$ for $-2\pi\ \leq x\ \leq 2\pi$ and $-2\pi\ \leq y\ \leq 2\pi$. Figure 7 (a) shows the cross-sectional view along the diagonal line x= -y. The value of h changes periodically. It takes maximum value (= 8) or minimum value (= 0) at the points where x and y take the integral multiple of π.

Substituting $\gamma_1\Delta x_i$ and $\gamma_2\Delta y_j$ into x and y, respectively, it is clear that the value of $\{3 - 2\cos\gamma_1\Delta x_i - 2\cos\gamma_2\Delta y_j + \cos\,(\gamma_1\Delta x_i - \gamma_2\Delta y_j)\}$ is positive or equal to zero for all γ_1 and γ_2.



(a) three-dimensional view



(b) cross-sectional view along the diagonal line (x=-y)

Figure 7: Value of $h\,(x,\,y) = 3 - 2\cos x - 2\cos y + \cos\,(x-y)$

References

[1] V. K. Saul'yev, Integration of Equations of Parabolic Type by the Method of Nets, Macmillan, New York, 1964.

[2] B. K. Larkin, Some Stable Explicit Difference Approximation to the Diffusion Equation, Math. Comp., 18, (1964) 196 - 202.

[3] H. Z. Barakat, J. A. Clark, On the solution f the diffusion equations by numerical methods, Transaction of the ASME, J. Heat Trans. (1966) 421-427.

[4] I. Galligani, Numerical solution of the time-dependent diffusion equations using the alternating method of Saul'yev, Calcolo, 2, supplement 1, (1966) 111-127.

[5] B. F. Towler, R. Y. K. Yang, On comparing the accuracy of some finite difference methods for parabolic partial differential equations, Int. J. for Num. Methods in Eng. 14, (1979) 1021–1035.

[6] J. Thibault, Comparison of nine three-dimensional numerical methods for the solution of the heat diffusion equation, Num. Heat Trans., 8, (1985) 281–298.

[7] M. T. Darvishi, Forward-backward Saul'yev method to solve PDEs, WSEAS Transactions on Mathematics 2, issues 1 and 2, (2003) 21–25.

[8] M. Dehghan, Numerical schemes for one-dimensional parabolic equations with nonstandard initial condition, Applied Math. and Comput. 147, (2004) 321–331.

[9] K. Bokahri and M. R. Islam, Improvement in the time accuracy of numerical methods in petroleum engineering problems - a new combination, Energy Sources 27, (2005) 45–60.

[10] K. Fukuyo, Stability of Saul'yev's methods with nonuniform grids, Num. Heat Trans., Part B, 52, (2007) 341–352.

[11] M. N. Ozisik, Boundary Value Problems of Heat Conduction, pp. 392 - 394, Dover, New York, 2002.

[12] *ibid.*, pp 98–99.

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Uslovna stabilnost Larkinovih metoda sa neuniformnim mrežama

Analiza stabilnosti zasnovana na Nojmanovom metodu je pokazala da su Larkinove metode za dvodimenziono provodjenje toplote sa neuniformnim mrežama uslovno stabilne, što je suprotno poznatoj činjenici da su bezuslovne stabilne za uniformne mreže. Kriterijumi stabilnosti koji se sastoje od bezdimenzionog vremenskog koraka Δt, prostornih opsega Δx, Δy, kao i razlomaka susednih prostornih intervala α, β su izvedeni u svrhu analize stabilnosti. Jedan numerički eksperiment je pokazao da rešenja izvedena Larkinovim metodama sa neuniformnim mrežama gube stabilnost i tačnost kada ovi kriterijumi nisu zadovoljeni.

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