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On the state of pure shear

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Abstract

The algebraic proof of the fundamental theorem concerning pure shear, by making use only of the notion of orthogonal projector, is presented. It has been shown that the state of pure shear is the same for all singular symmetric traceless tensors in E_3, up to the rotation.

Keywords: continuum mechanics, pure shear, orthogonal projector.

1 Introduction

It is known that in classical continuum mechanics the Cauchy stress tensor $\mathbf{T}$ is symmetric. By definition, a state of stress is said to be one of pure shear if there is an orthogonal basis $\mathbf{p}_i$ $(i = 1, 2, 3)$ for which

$$\mathbf{p}_i \cdot \mathbf{T}\mathbf{p}_i = 0, \quad \text{no sum over indices } i = 1, 2, 3. \tag{1}$$

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Note that, if $\mathbf{p}_i$ satisfies (1), so does $-\mathbf{p}_i$. Therefore, these vectors $\mathbf{p}_i$ $(i = 1, 2, 3)$ are unique up to their orientation. In what follows, we shall use the term *unique* in this sense.

Theorem 1 *A necessary and sufficient condition for* $\mathbf{T}$ *to be a state of pure shear is*

$$\operatorname{tr}\mathbf{T} = 0. \tag{2}$$

It is almost obvious that (1) is necessary condition for (2). Indeed, from (1) and

$$\sum_{i=1}^{3} \mathbf{p}_i \otimes \mathbf{p}_i = \mathbf{I}, \tag{3}$$

where $\mathbf{I}$ is identity tensor, we have

$$0 = \sum_{i=1}^{3} \mathbf{p}_i \cdot \mathbf{T}\mathbf{p}_i = \sum_{i=1}^{3} \operatorname{tr}\mathbf{T}(\mathbf{p}_i \otimes \mathbf{p}_i) = \operatorname{tr}\mathbf{T}. \tag{4}$$

As for the second part of the theorem it suffices for the proof to exhibit just one orthonormal basis for which (2) $\Rightarrow$ (1).

Belik and Fosdick [1], in order to exhibit all such bases, prove this fundamental theorem from both the geometric and algebraic points of view in three dimensional Euclidean space, E_3.

Recently, Boulanger and Hayes [2] presented what they called an even more elementary proof and gave an insightful geometrical approach in terms of elliptical sections of the stress ellipsoid.

They also stated that "it may be shown that $\mathbf{n} \cdot \mathbf{T}\mathbf{n} = 0$ for all $\mathbf{n}$ lying in a plane, if and only if one of the eigenvalues is zero (say $\sigma_2 = 0$), and all the $\mathbf{n}$ lie in either one or other of the planes of central circular section of ellipsoid E..." No proof was given.

Ting [3] provided a characterization of directions $\mathbf{n}$ such that $\sigma_{nn} = 0$ in terms of the total shear in the plane normal to $\mathbf{n}$, $\tau = |\mathbf{T}\mathbf{n}|$.

Norris [4] discussed the pure shear basis vectors independent of the values of principal stresses.

Here, making use only of the notion of orthogonal projector, we present the proof of the theorem in E_3 . It distinguishes the present discussion from the recent notes ([1]-[4]). The proof includes also the case mentioned by Boulanger and Hayes [2]. In our approach, we do not refer to any ellipsoid, nor do we determine the principal axes of an elliptical section. In this sense our approach is direct and general. This is shown in Section 3, where we analyze the n-dimensional Euclidean space E_n.

Of course, 3-dimensional case is a special one. But in presenting the paper in this order we wanted to emphasize two things some specific feature of the 3-dimensional case as well as the role of the orthogonal projector in solving the general n-dimensional case.

The organization of this paper is as follows. In Section 1, we establish the notation. In Section 2, we discuss only the algebraic approach for the 3-dimensional case, since the geometrical approach is given in [1].

Moreover, we show that the state of pure shear is the same for all singular symmetric traceless tensors in E_3, up to a rotation.

In Section 3 detailed analysis of the n-dimensional case is given. The procedure consists of several steps. Each step is based on corresponding Lemma. Although these lemmas are identical in form, we stated them separately in order to clarified each step.

In Section 4, we extend our results to nonsymmetric tensors in E_n.

Finally, in Section 5, the summary and a brief discussion are given.

2 Notation

We use the following notations:

- π: $\mathbf{p}\cdot\mathbf{x} = 0$ for the plane defined by unit outward normal vector $\mathbf{p}$, where $\mathbf{x} \in E_n$,

- $\mathbf{P} = \mathbf{I} - \mathbf{p}\otimes\mathbf{p} \in \mathrm{Sym}$, $\mathbf{P}^2 = \mathbf{P}$, for the orthogonal projector along $\mathbf{p}$ onto π,

- $\mathbf{v}^\star = \mathbf{P}\mathbf{v}$ for the projection of a vector,

- $\mathbf{S}^\star = \mathbf{PSP}$ for the projection of a second order tensor $\mathbf{S}$.

$\mathbf{S}^\star$ is singular for any $\mathbf{P} \neq \mathbf{I}$. Particularly, $\mathbf{Pp} = \mathbf{0}$ and $\mathbf{S}^\star\mathbf{p} = \mathbf{0}$. Therefore, $\mathbf{p}$ is eigenvector of $\mathbf{P}$ and $\mathbf{S}^\star$, corresponding to zero eigenvalue.

Also,

$$\mathbf{P}\boldsymbol{\xi} = \boldsymbol{\xi}\mathbf{P} = \boldsymbol{\xi}, \qquad \boldsymbol{\xi}\cdot \mathbf{S}^\star\boldsymbol{\xi} = \boldsymbol{\xi}\cdot \mathbf{S}\boldsymbol{\xi}, \tag{5}$$

for any vector $\boldsymbol{\xi} \in \pi$.

For symmetric $\mathbf{S}$:

$$\mathbf{S}^\star = \mathbf{PSP} = \mathbf{S} - \mathbf{Sp} \otimes \mathbf{p} - \mathbf{p} \otimes \mathbf{Sp} + (\mathbf{p}\cdot \mathbf{Sp})\mathbf{p} \otimes \mathbf{p} \tag{6}$$

$$\operatorname{tr} \mathbf{S}^\star = \operatorname{tr} \mathbf{S} - \mathbf{p}\cdot \mathbf{Sp}. \tag{7}$$

As a preliminary to the general definition of a projector, we remind the reader of the definition of the decomposition of E_n into direct sum of subspaces U and V, symbolized by

$$E_n = U \oplus V. \tag{8}$$

Then $\mathbf{P}$ is orthogonal projector, of E_n along V onto U, if $\mathbf{P} \in \mathrm{Sym}$ and U and V are orthogonal.

3 Algebraic Approach in E_3

First, we confine our investigation to a tensor $\mathbf{T} \in \mathrm{Sym}$ in E_3. For further reference we write its spectral form

$$\mathbf{T} = \sigma_1\mathbf{n}_1 \otimes \mathbf{n}_1 + \sigma_2\mathbf{n}_2 \otimes \mathbf{n}_2 + \sigma_3\mathbf{n}_3 \otimes \mathbf{n}_3. \tag{9}$$

Then, in view of (2)

$$\sigma_1 + \sigma_2 + \sigma_3 = 0. \tag{10}$$

We assume that

$$\sigma_1 > 0, \quad \sigma_2 \geq 0, \quad \sigma_3 < 0. \tag{11}$$

We seek $\mathbf{p}(= \mathbf{p}_1)$, such that the component of $\mathbf{T}$ in the direction of $\mathbf{p}$ is zero, i.e.

$$\mathbf{p}\cdot \mathbf{Tp} = 0. \tag{12}$$

Accordingly, $\mathbf{p}$ lies along a generator of the elliptical cone

$$\mathfrak{c}: \ \mathbf{x}\cdot(\mathbf{Tx}) = \sigma_1 x_1^2 + \sigma_2 x_2^2 + \sigma_3 x_3^2 = 0 \tag{13}$$

through its tip. A solution to (13) certainly exists, i.e. $x_i = \pm 1$, $i = 1, 2, 3$. For any particular $\mathbf{p}$, which satisfies (12), $\mathbf{p}_2$ and $\mathbf{p}_3$ must lie in π. Therefore, $\pi \cap \mathfrak{c}$ defines directions of $\mathbf{p}_2$ and $\mathbf{p}_3$, if they are orthogonal. To show that, we make use of $(5)_1$ and write

$$0 = \mathbf{p}_\alpha\cdot \mathbf{T}\mathbf{p}_\alpha = \mathbf{p}_\alpha\cdot \mathbf{PTP}\mathbf{p}_\alpha = \mathbf{p}_\alpha\cdot \mathbf{T}^\star\mathbf{p}_\alpha, \quad \alpha = 2, 3, \tag{14}$$

where

$$\mathbf{T}^\star = \mathbf{PTP} \in \mathrm{Sym} \tag{15}$$

is the projection of $\mathbf{T}$ along $\mathbf{p}$ onto π.

Moreover, in view of (6), (7) and (12) we have

$$\mathbf{T}^\star = \mathbf{PTP} = \mathbf{T} - \mathbf{Tp}\otimes\mathbf{p} - \mathbf{p}\otimes\mathbf{Tp}, \tag{16}$$

$$\mathrm{tr}\,\mathbf{T}^\star = \mathrm{tr}\,\mathbf{T}, \tag{17}$$

and

$$\mathbf{T}^\star = \lambda_1 \boldsymbol{\nu}_1\otimes\boldsymbol{\nu}_1 + \lambda_2\boldsymbol{\nu}_2\otimes\boldsymbol{\nu}_2. \tag{18}$$

But

$$\mathrm{tr}\,\mathbf{T} = 0 \Leftrightarrow \mathrm{tr}\,\mathbf{T}^\star = 0, \tag{19}$$

so that $\lambda_1 = -\lambda_2 (= \lambda)$ and accordingly we write (18) as

$$\mathbf{T}^\star = \lambda(\boldsymbol{\nu}_1\otimes\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2\otimes\boldsymbol{\nu}_2). \tag{20}$$

In view of (14) and (20), the corresponding $\mathbf{p}_\alpha$ must lie on the "cone"

$$\mathfrak{c}^\star: \quad \boldsymbol{\xi}\cdot\mathbf{T}^\star\boldsymbol{\xi} = \lambda(\xi_1^2 - \xi_2^2) = 0, \tag{21}$$

where $\boldsymbol{\xi} = \xi_\alpha\boldsymbol{\nu}_\alpha$.

We shall discuss all possible solutions of (21), which satisfy (1).

I) $\lambda \neq 0$. In this case

$$\xi_2 = \pm\xi_1,$$

i.e.

$$\boldsymbol{\xi} = \xi_1(\boldsymbol{\nu}_1 \pm \boldsymbol{\nu}_2).$$

But, $\mathbf{p} = \dfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ and thus

$$\begin{aligned} \mathbf{p}_2 &= \frac{1}{\sqrt{2}}(\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2), \\ \mathbf{p}_3 &= \frac{1}{\sqrt{2}}(\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2). \end{aligned} \tag{22}$$

Thus obtained $\mathbf{p}_i$ $(i = 1, 2, 3)$ are orthonormal and represent the *unique* solution to our problem.

It is easy to calculate λ. For instance, from (20) we have

$$\mathbf{T}^{\star 2} = \lambda^2(\boldsymbol{\nu}_1 \otimes \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 \otimes \boldsymbol{\nu}_2), \tag{23}$$

and from this

$$2\lambda^2 = \operatorname{tr} \mathbf{T}^{\star 2}. \tag{24}$$

Next, in view of (15) and (12), we obtain

$$2\lambda^2 = \operatorname{tr}(\mathbf{PTP})^2 = \operatorname{tr}(\mathbf{PT})^2 = \operatorname{tr}\mathbf{T}^2 - 2\mathbf{p}\cdot \mathbf{T}^2\mathbf{p} \tag{25}$$

(see (3.6) in [2]).

II) $\lambda = 0$. Then $\mathbf{T}^\star = 0$, and from (16) we have

$$\mathbf{T} = \mathbf{Tp} \otimes \mathbf{p} + \mathbf{p} \otimes \mathbf{Tp},$$

from which we conclude that either

$$\det \mathbf{T} = 0, \tag{26}$$

(see for instance [5]), or

$$\sigma_2 = 0.$$

In view of this and $(19)_1$ we have

$$\sigma_1 = |\sigma_3| (= \sigma),$$

Thus, from (9) we have

$$\mathbf{T} = \sigma\left(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_3 \otimes \mathbf{n}_3\right). \tag{27}$$

Since (21) is identity for any $\boldsymbol{\xi} \in \pi$, we are looking for $\mathbf{p} \in \mathfrak{c}$, which defines such plane π. But in view of (13)

$$\mathfrak{c}: \quad \mathbf{x}\cdot \mathbf{T}\mathbf{x} = \sigma(x_1^2 - x_3^2) = 0, \tag{28}$$

so that

$$\begin{aligned} \tau_1 &: \ x_1 + x_3 = 0 \ \Rightarrow \ \mathbf{t}_1\cdot \mathbf{x} = 0, \\ \tau_2 &: \ x_1 - x_3 = 0 \ \Rightarrow \ \mathbf{t}_2\cdot \mathbf{x} = 0, \end{aligned} \tag{29}$$

are the solutions of (28), where

$$\begin{aligned} \mathbf{t}_1 &= \frac{1}{\sqrt{2}}\left(\mathbf{n}_1 + \mathbf{n}_3\right), \\ \mathbf{t}_2 &= \frac{1}{\sqrt{2}}\left(\mathbf{n}_1 - \mathbf{n}_3\right). \end{aligned} \tag{30}$$

It is clear that $\mathfrak{c} = \tau_1 \cup \tau_2$, where τ_1 and τ_2 represent two perpendicular planes, which intersect along direction defined by $\mathbf{n}_2$.

In order to complete our discussion we consider the following possible cases.

a) Let $\mathbf{p}_2$, $\mathbf{p}_3 \in \tau_1$ be any two orthonormal vectors. Then $\mathbf{p}_1 = \mathbf{t}_1 \in \tau_2$ is the only unit vector perpendicular to the vectors $\mathbf{p}_2$ and $\mathbf{p}_3$. More precisely, any such set of orthonormal vectors satisfying (1) is given by

$$\begin{aligned} \mathbf{p}_1 &= \mathbf{t}_1, \\ \mathbf{p}_2 &= \mathbf{n}_2 \cos\alpha + \mathbf{t}_2 \sin\alpha, \\ \mathbf{p}_3 &= -\mathbf{n}_2 \sin\alpha + \mathbf{t}_2 \cos\alpha, \quad 0 \le \alpha \le \pi. \end{aligned} \tag{31}$$

b) In the same way, we conclude that any two orthonormal vectors $\mathbf{p}_2$, $\mathbf{p}_3 \in \tau_2$, and $\mathbf{p}_1 = \mathbf{t}_1 \in \tau_1$, given by

$$\begin{aligned} \mathbf{p}_1 &= \mathbf{t}_2, \\ \mathbf{p}_2 &= \mathbf{n}_2 \cos\alpha + \mathbf{t}_1 \sin\alpha, \\ \mathbf{p}_3 &= -\mathbf{n}_2 \sin\alpha + \mathbf{t}_1 \cos\alpha, \quad 0 \le \alpha \le \pi, \end{aligned} \tag{32}$$

form a set of orthonormal vectors satisfying (1).

The sets of orthonormal vectors given by (31) and (32) include all possible solutions to (1). Particularly, if $\cos\alpha = 1$, then the set of vectors

$$\mathbf{n}_2, \quad \mathbf{t}_2, \quad \mathbf{t}_1 \tag{33}$$

satisfies (1).

Finally, we go one step further and state

Lemma 1 *Let* $\mathbf{T}, \mathbf{S}$ *be any two symmetric singular traceless tensors. Then the "cones" that define their states of pure shear, differ by the rotation of their eigenvectors.*

Proof. According to the supposition of Lemma 1 $\mathbf{T}\mathbf{n}_2 = \mathbf{0}$ and $\mathbf{S}\mathbf{f}_2 = 0$. Then (27) holds and

$$\mathbf{S} = \mu\left(\mathbf{f}_1 \otimes \mathbf{f}_1 - \mathbf{f}_3 \otimes \mathbf{f}_3\right), \tag{34}$$

where $\mathbf{f}_i$ $(i = 1, 2, 3)$ are orthonormal. Now, there is unique orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{R}\mathbf{f}_i = \mathbf{n}_i.$$

Hence,

$$\begin{aligned} \mathbf{R}\mathbf{S}\mathbf{R}^T &= \mu\left(\mathbf{R}\mathbf{f}_1 \otimes \mathbf{R}\mathbf{f}_1 - \mathbf{R}\mathbf{f}_3 \otimes \mathbf{R}\mathbf{f}_3\right) = \\ &= \mu\left(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_3 \otimes \mathbf{n}_3\right) = \\ &= \frac{\mu}{\sigma}\mathbf{T} \end{aligned}$$

where $\mu/\sigma \neq 0$.

In view of the above relation, the "cone"

$$\mathfrak{c}_{\mathbf{T}} : \quad \mathbf{x} \cdot \mathbf{T}\mathbf{x} = 0,$$

transforms to the "cone"

$$\mathfrak{c}_{\mathbf{S}} : \quad \mathbf{y} \cdot \mathbf{S}\mathbf{y} = 0$$

where

$$\mathbf{y} = \mathbf{R}^T\mathbf{x}.$$

Geometrically these "cones" differ for rotation represented by $\mathbf{R}$. ■

Corollary 1 *Given two singular, symmetric and traceless tensors* $\mathbf{T}$ *and* $\mathbf{S}$*, define an orthogonal tensor* $\mathbf{R} = \mathbf{n}_i \otimes \mathbf{f}_i$*, where* $\mathbf{n}_i$ *and* $\mathbf{f}_i$*,* $i = 1, 2, 3$*, are eigenvectors of* $\mathbf{T}$ *and* $\mathbf{S}$*, respectively.*

Let $\mathbf{p}_i$*,* $i = 1, 2, 3$*, be an orthonormal set of vectors satisfying (1). Then the set of orthonormal vectors* $\mathbf{q}_i = \mathbf{R}^T \mathbf{p}_i$ *satisfies* $\mathbf{q}_i \cdot \mathbf{S}\mathbf{q}_i = 0$*.*

This simply follows from the last three expressions. We shall illustrate it by the following

Example

Let,

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} -3 & 6 & 0 \\ 6 & 0 & -6 \\ 0 & -6 & 3 \end{pmatrix}$$

with respect to the basis $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. Obviously,

$$\operatorname{tr} \mathbf{T} = \det \mathbf{T} = 0 \quad \text{and} \quad \operatorname{tr} \mathbf{S} = \det \mathbf{S} = 0.$$

It easy to show that is

$$s_1 = 9, \quad s_2 = -9, \quad s_3 = 0$$

are the eigenvalues of $\mathbf{S}$. Corresponding eigenvectors are

$$\mathbf{f}_1 = \frac{1}{3}(1, 2, -2), \quad \mathbf{f}_2 = \frac{1}{3}(-2, 2, 1), \quad \mathbf{f}_3 = \frac{1}{3}(2, 1, 2).$$

Then

$$\mathbf{S} = 9(\mathbf{f}_1 \otimes \mathbf{f}_1 - \mathbf{f}_2 \otimes \mathbf{f}_2).$$

The orthogonal tensor

$$\mathbf{R} = \mathbf{n}_i \otimes \mathbf{f}_i,$$

which is represented by

$$\frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix},$$

on the basis $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$, transforms $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ into $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ so that

$$\mathbf{R}\mathbf{f}_i = \mathbf{n}_i.$$

It follows that $\mathbf{R}\mathbf{S}\mathbf{R}^T = 9\mathbf{T}$. Then, for all $\mathbf{x}$, for which

$$\mathbf{x}\cdot \mathbf{T}\mathbf{x} = 0$$

transforms into

$$\mathbf{y} = \mathbf{R}\mathbf{x},$$

so that

$$\mathbf{y}\cdot \mathbf{S}\mathbf{y} = 0.$$

If $\mathbf{p}_i\cdot \mathbf{p}_j = \delta_{ij}$ and $\mathbf{p}_i\cdot \mathbf{T}\mathbf{p}_i = 0$, $i = 1, 2, 3$, no sum with respect to i, then, for all $\mathbf{q}_i = \mathbf{R}\mathbf{p}_i$, $i = 1, 2, 3$, $\mathbf{q}_i\cdot \mathbf{q}_j = \delta_{ij}$ and $\mathbf{q}_i\cdot \mathbf{S}\mathbf{q}_i = 0$, no sum with respect to i.

Further,

$$\frac{1}{\sqrt{2}}\left(\mathbf{n}_1 \pm \mathbf{n}_2\right) = \mathbf{R}\frac{1}{\sqrt{2}}\left(\mathbf{f}_1 \pm \mathbf{f}_2\right).$$

Also, (31) and (32) are mapped by $\mathbf{R}$ into corresponding solutions of $\mathbf{S}$.

4 n-dimensional case

In investigating n-dimensional case, we shall make use of the following notation

$${}_r x_{\alpha_r}, \quad {}_r\sigma_{\alpha_r}, \quad {}_r\mathbf{n}_{\alpha_r}, \quad \alpha_r = 1, \ldots, k - r$$

their meaning will be clear from the context.

Let a tensor $\mathbf{T} \in \mathrm{Sym}$ is given in Euclidean n-dimensional space E_n. Then **Theorem 1** holds, having in mind that now $i = 1, 2, \ldots, n$, in (1) and (3).

In order to include all possible cases, we write its spectral form as

$$\mathbf{T} = \sum_{\alpha=1}^{k} \sigma_\alpha \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha, \qquad k \le n, \tag{35}$$

where

$$\mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, n. \tag{36}$$

Then from (2) and (35), we have

$$\sum_{\alpha=1}^{k} \sigma_\alpha = 0.$$

If $k < n$, then there is a set of $n-k$ orthonormal vectors $\mathbf{n}_\sigma$ such that

$$\mathbf{T}\mathbf{n}_\sigma = 0, \quad \sigma = k+1, \ldots, n. \tag{37}$$

This set of vectors $\mathbf{n}_\sigma$ spans $n-k$ dimensional vector space V, so that any unit vector $\mathbf{v} \in V$ satisfies (1). Hence, there is an infinity of sets of $n-k$ orthonormal vectors which satisfy (1). Remaining k orthonormal vectors are in k-dimensional spaces U, spanned by k orthonormal vectors $\mathbf{n}_\alpha$, $\alpha = 1, \ldots, k$. Thus, $E_n = U \oplus V$, where U and V are mutually orthogonal spaces.

In order to complete the set of n orthonormal unit vectors satisfying (1), we proceed in several steps.

$\mathbf{A}_1$. Let

$$\mathbf{x} = x_i \mathbf{n}_i.$$

Then $\mathbf{x}$ must lay on the cone

$$C: \quad \mathbf{x} \cdot \mathbf{T}\mathbf{x} = \sum_{\alpha=1}^{k} \sigma_\alpha x_\alpha^2 = 0. \tag{38}$$

Obviously (38) does not impose any restrictions upon x_σ, $\sigma = k + 1, \ldots, n$. Therefore, it is sufficient to consider only $\mathbf{x} \in U$, i.e. when $\mathbf{x} = x_\alpha \mathbf{n}_\alpha$, $\alpha = 1, \ldots, k$ or equivalently when $\mathbf{x} \cdot \mathbf{n}_\sigma = 0$.

Let $\mathbf{x} = x_\alpha \mathbf{n}_\alpha \in U$ be any solution of (38), and $\mathbf{p}_1$ its unit vector. Then, the set of orthonormal vectors $\mathbf{p}_1$ and $\mathbf{n}_\sigma$, $\sigma = k+1, \ldots, n$, spans the $(n-k+1)$ dimensional linear vector space $V_1^{(1)}$. Accordingly, any $\mathbf{x} \in V_1^{(1)}$ satisfies (38). Therefore, there is infinite of sets of orthonormal vectors in $V_1^{(1)}$ satisfying (1).

$\mathbf{A}_2$. Now, let $E_n = V_1^{(1)} \oplus U_1$. Then $V_1^{(1)} \in V$, $U_1 \subset U$, and $\dim U_1 = k-1$. Therefore, the orthogonal projection of E_n along $V_1^{(1)}$ onto U_1 is defined by the orthogonal projector

$$\mathbf{P}_1 = \mathbf{I} - \mathbf{p}_1 \otimes \mathbf{p}_1 - \sum_{\sigma=k+1}^{n} \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma. \tag{39}$$

Obviously, $\operatorname{rank} \mathbf{P}_1 = k-1$.

Lemma 2 $\mathbf{x} \in U_1$ *iff* $\mathbf{P}_1\mathbf{x} = \mathbf{x}$.

Proof. If $\mathbf{x} \in U_1$, then $\mathbf{x}\cdot \mathbf{p}_1 = 0$ and $\mathbf{x}\cdot \mathbf{n}_\sigma = 0$. Hence $\mathbf{P}_1\mathbf{x} = \mathbf{x}$. Conversely, from $\mathbf{x} = \mathbf{P}_1\mathbf{x}$ we have $\mathbf{x}\cdot \mathbf{p}_1 = 0$ and $\mathbf{x}\cdot \mathbf{n}_\sigma = 0$, i.e. $\mathbf{x} \in U_1$.

Thus, the remaining $k-1$ orthonormal vectors satisfying (39) must lay in the intersection of U_1 and the cone (38). Making use of Lemma 2, their intersection can be put in the following form:

$$\mathbf{x}\cdot \mathbf{T}\mathbf{x} = \mathbf{P}_1\mathbf{x}\cdot \mathbf{T}\mathbf{P}_1\mathbf{x} = \mathbf{x}\cdot \mathbf{P}_1\mathbf{T}\mathbf{P}_1\mathbf{x} = \mathbf{x}\cdot \mathbf{T}_1\mathbf{x} = 0, \tag{40}$$

where

$$\mathbf{T}_1 = \mathbf{P}_1\mathbf{T}\mathbf{P}_1, \qquad \mathbf{T}_1 \in \mathrm{Sym} \tag{41}$$

is the orthogonal projection of $\mathbf{T}$ by $\mathbf{P}_1$. According to the modified forms of (6) and (7) we conclude that

$$\operatorname{tr} \mathbf{T}_1 = \operatorname{tr} \mathbf{T}, \quad \text{and} \quad \operatorname{tr} \mathbf{T} \Leftrightarrow \operatorname{tr} \mathbf{T}_1 = 0. \tag{42}$$

But $\operatorname{rank} \mathbf{T}_1 \leq k-1$. In general, its spectral form reads

$$\mathbf{T}_1 = \sum_{\alpha_1=1}^{k-1} {}_1\sigma_{\alpha_1} {}_1\mathbf{n}_{\alpha_1} \otimes {}_1\mathbf{n}_{\alpha_1} \tag{43}$$

The set of n vectors ${}_1\mathbf{n}_{\alpha_1}$, $\mathbf{p}_1$, $\mathbf{n}_\sigma$, $\alpha_1 = 1, \ldots, k-1$; $\sigma = k+1, \ldots, n$, are orthonormal and thus linearly independent. They may be taken as the basis of E_n. Then, we may write

$$\mathbf{x} = \sum_{\alpha_1=1}^{k-1} {}_1x_{\alpha_1} + p_1\mathbf{p}_1 + \sum_{\sigma=1}^{n-k} x_\sigma \mathbf{n}_\sigma,$$

for any $\mathbf{x} \in E_n$. But, we need only those $\mathbf{x}$ which satisfy $\mathbf{x} = \mathbf{P}\mathbf{x} = \sum_{\alpha_1=1}^{k-1} {}_1x_{\alpha_1} \in U_1$.

Hence, in view of properties of $\mathbf{T}_1$, we have that

$$C_1 : \quad \mathbf{x}\cdot \mathbf{T}\mathbf{x} = \mathbf{x}\cdot \mathbf{T}_1\mathbf{x} = \sum_{\alpha_1=1}^{k-1} {}_1\sigma_{\alpha_1 1} x_{\alpha_1}^2 = 0. \tag{44}$$

Accordingly, any unit vector $\mathbf{p}_2 \in U_1$ along the generator of the above cone (44) is the solution of (1). Hence, we have $n - k + 2$ orthonormal vectors satisfying (1). They are $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{n}_\sigma$, $\sigma = k + 1, \ldots, n$.

Moreover, in addition to $V_1^{(1)}$, we have space $V_1^{(2)}$ of $n - k + 1$ dimensions, spanned by $\mathbf{p}_2$, $\mathbf{n}_\sigma$, $\sigma = k + 1, \ldots, n$, such that any vector in this space satisfies (1).

The space V_2 of $n - k + 2$ dimension spanned by vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{n}_\sigma$, $\sigma = k + 1, \ldots, n$, does not have this property. For instance, let $\mathbf{x} = p_1\mathbf{p}_1 + p_2\mathbf{p}_2$. Then, in general, $\mathbf{x}\cdot \mathbf{T}\mathbf{x} = 2p_1p_2\mathbf{p}_1\cdot \mathbf{T}\mathbf{p}_2 \neq 0$, i.e. $\mathbf{x}$ is not on the cone (44). Therefore, any set of orthonormal vectors in $V_1^{(1)}$, together with $\mathbf{p}_2$ form a set of $n-k+2$ orthonormal vectors satisfying (1). Likewise, any set of orthonormal vectors in $V_1^{(2)}$, together with $\mathbf{p}_1$ form a set of $n - k + 2$ orthonormal vectors satisfying (1).

$\mathbf{A}_3$. Let $E_n = V_2 \oplus U_2$. Then

$$\mathbf{P}_2 = \mathbf{I} - \sum_{a=1}^{2} \mathbf{p}_a \otimes \mathbf{p}_a - \sum_{\sigma=k+1}^{n} \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma, \tag{45}$$

where $\mathbf{P}_2 \in \mathrm{Sym}$, $\operatorname{rank} \mathbf{P}_2 = k-2$, represents an orthogonal projection of E_n along V_2 onto U_2. Obviously, the remaining $k - 2$ orthonormal vectors satisfying (1) must lay in the intersection of U_2 and the cone (38). But, vectors $\mathbf{x} \in U_2$ are subjected to the following restrictions: $\mathbf{x}\cdot \mathbf{p}_a = 0$, and $\mathbf{x}\cdot \mathbf{n}_\sigma = 0$, $a = 1, 2$; $\sigma = k + 1, \ldots, n$. Therefore, we restate Lemma 2 in this case as:

Lemma 3 $\mathbf{x} \in U_2 \quad iff \quad \mathbf{P}_2\mathbf{x} = \mathbf{x}$.

The proof of this lemma is the same as for **Lemma 2**.

Making use of the **Lemma 3**, we write the intersection of U_2 and (38) as

$$\mathbf{x}\cdot\mathbf{T}\mathbf{x} = \mathbf{P}_2\mathbf{x}\cdot\mathbf{T}\mathbf{P}_2\mathbf{x} = \mathbf{x}\cdot\mathbf{P}_2\mathbf{T}\mathbf{P}_2\mathbf{x} = \mathbf{x}\cdot\mathbf{T}_2\mathbf{x} = 0,$$

where

$$\begin{gathered}\mathbf{T}_2 = \mathbf{P}_2\mathbf{T}\mathbf{P}_2, \quad \mathbf{T}_2 \in \mathrm{Sym}, \quad \operatorname{rank}\mathbf{T}_2 \le k-2 \\ \operatorname{tr}\mathbf{T}_2 = \operatorname{tr}\mathbf{T}, \quad \operatorname{tr}\mathbf{T} = 0 \quad \Rightarrow \quad \operatorname{tr}\mathbf{T}_2 = 0.\end{gathered} \tag{46}$$

The spectral form of $\mathbf{T}_2$, in general, now reads

$$\mathbf{T}_2 = \sum_{\alpha_2=1}^{k-2} {}_2\sigma_{\alpha_2}\,{}_2\mathbf{n}_{\alpha_2} \otimes {}_2\mathbf{n}_{\alpha_2} \tag{47}$$

The set of n orthonormal vectors ${}_2\mathbf{n}_{\alpha_2}$; $\mathbf{p}_a$; $\mathbf{n}_\sigma$; $\alpha_2 = 1, \ldots, k-2$; $a = 1, 2$; $\sigma = k+1, \ldots, n$; spans E_n.

Thus, any $\mathbf{x} \in E_n$ has the following representation

$$\mathbf{x} = \sum_{\alpha_2=1}^{k-2} {}_2x_{\alpha_2}\,{}_2\mathbf{n}_{\alpha_2} + \sum_{a=1}^{2} p_a\mathbf{p}_a + \sum_{\sigma=1}^{n-k} x_\sigma\mathbf{n}_\sigma.$$

Then, in view of **Lemma 3**, we write for the intersection of U_2 and (38)

$$C_2: \quad \mathbf{x}\cdot\mathbf{T}\mathbf{x} = \mathbf{x}\cdot\mathbf{T}_2\mathbf{x} = \sum_{\alpha_2=1}^{k-2} {}_2\sigma_{\alpha_2}\,{}_2x_{\alpha_2}^2 = 0. \tag{48}$$

Then, any unit vector $\mathbf{p}_3 \in U_2$ of $\mathbf{x} = \sum_{\alpha_2=1}^{k-2} {}_2x_{\alpha_2}\,{}_2\mathbf{n}_{\alpha_2}$ along the generator of the cone (48) is the solution of (1). Hence we have $n-k+3$ orthonormal vectors satisfying (1). They are $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{n}_\sigma$, $\sigma = k+1, \ldots, n$.

Moreover, in addition to $V_1^{(1)}$ and $V_1^{(2)}$ of $n-k+1$ dimensions, we have the space $V_1^{(3)}$, also of $n-k+1$ dimension, spanned by $\mathbf{p}_3$, $\mathbf{n}_\sigma$, $\sigma = k+1, \ldots, n$, such that any vector in this space satisfies (1). Again, no space of $n-k+2$ dimension spanned by two of vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{n}_\sigma$, $\sigma = k+1, \ldots, n$, has this property.

Therefore, any set of orthonormal vectors in $V_1^{(1)}$ together with $\mathbf{p}_2$ and $\mathbf{p}_3$ form a set of $n-k+3$ orthonormal vectors satisfying (1).

Likewise, any set of orthonormal vectors in $V_1^{(2)}$ together with $\mathbf{p}_1$ and $\mathbf{p}_3$ form a set of $n-k+3$ orthonormal vectors satisfying (1). Also, any set of orthonormal vectors in $V_1^{(3)}$ together with $\mathbf{p}_1$ and $\mathbf{p}_2$ form a set of $n-k+3$ orthonormal vectors satisfying (1). More concisely, any set of orthonormal vectors in $V_1^{(k)}$, together with the set $\{\mathbf{p}_e\}$ of two vectors, $e, k = 1, 2, 3$, $e \neq k$, form a set of $n-k+3$ orthonormal vectors satisfying (1).

$\mathbf{A}_4$. We may proceed further in the same way until

$$\mathbf{P}_r = \mathbf{I} - \sum_{a=1}^{r} \mathbf{p}_a \otimes \mathbf{p}_a - \sum_{\sigma=k+1}^{n} \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma, \tag{49}$$

where $\operatorname{rank} \mathbf{P}_r = k-r > 0$; $\mathbf{p}_a$, $\mathbf{n}_\sigma$; $a = 1, \ldots, r$; $\sigma = k+1, \ldots, n$. From the way we obtain $\mathbf{p}_a$, vectors $\mathbf{p}_a$; $\mathbf{n}_\sigma$; $a = 1, \ldots, r$; $\sigma = k+1, \ldots, n$, are orthonormal and satisfy (1). We shall denote the $(n-k+r)$ - dimensional space they span, by V_r. Then, we may write $E_n = V_r \oplus U_r$, where $\dim U_r = k - r > 0$, and thus $U_r \subset \cdots \subset U_1 \subset U$. As above, we conclude that any $\mathbf{x} \in U_r$ must satisfy the following conditions: $\mathbf{x} \cdot \mathbf{p}_a = 0$ and $\mathbf{x} \cdot \mathbf{n}_\sigma = 0$, $a = 1, \ldots, r$; $\sigma = k+1, \ldots, n$. Next, as above, we write

Lemma 4 $\mathbf{x} \in U_r \quad iff \quad \mathbf{P}_r \mathbf{x} = \mathbf{x}$.

Then the intersection of U_r and (38) reads as

$$\mathbf{x} \cdot \mathbf{T}\mathbf{x} = \mathbf{P}_r\mathbf{x} \cdot \mathbf{T}\mathbf{P}_r\mathbf{x} = \mathbf{x} \cdot \mathbf{P}_r\mathbf{T}\mathbf{P}_r\mathbf{x} = \mathbf{x} \cdot \mathbf{T}_r\mathbf{x} = 0,$$

where

$$\begin{aligned} \mathbf{T}_r = \mathbf{P}_r\mathbf{T}\mathbf{P}_r, \quad \mathbf{T}_r \in \mathrm{Sym}, \quad \operatorname{rank} \mathbf{T}_r \leq k - r \\ \operatorname{tr} \mathbf{T}_r = \operatorname{tr} \mathbf{T}, \quad \operatorname{tr} \mathbf{T} = 0 \quad \Rightarrow \quad \operatorname{tr} \mathbf{T}_r = 0. \end{aligned} \tag{50}$$

The spectral form of $\mathbf{T}_r$, in general, now reads

$$\mathbf{T}_r = \sum_{\alpha_r=1}^{k-r} {}_r\sigma_{\alpha_r} {}_r\mathbf{n}_{\alpha_r} \otimes {}_r\mathbf{n}_{\alpha_r}. \tag{51}$$

The set of n orthonormal vectors ${}_r\mathbf{n}_{\alpha_r}$; $\mathbf{p}_a$; $\mathbf{n}_\sigma$; $\alpha_r = 1, \ldots, k-r$; $a = 1, \ldots, r$; $\sigma = k+1, \ldots, n$; spans E_n.

Thus any $\mathbf{x} \in E_n$ has the following representation

$$\mathbf{x} = \sum_{\alpha_r=1}^{k-r} {}_r x_{\alpha_r} {}_r\mathbf{n}_{\alpha_r} + \sum_{a=1}^{r} p_a \mathbf{p}_a + \sum_{\sigma=1}^{n-k} x_\sigma \mathbf{n}_\sigma. \tag{52}$$

Then, in view of Lemma r+1, we write for the intersection of U_r and (38) as

$$C_r: \quad \mathbf{x}\cdot \mathbf{T}\mathbf{x} = \mathbf{x}\cdot \mathbf{T}_r\mathbf{x} = \sum_{\alpha_r=1}^{k-r} {}_r\sigma_{\alpha_r} {}_r x_{\alpha_r}^2 = 0. \tag{53}$$

Then any unit vector $\mathbf{p}_{r+1} \in U_r$ along the generator of the cone (48) is the solution of (1). Hence we have $n-k+r+1$ orthonormal vectors satisfying (1). They are $\mathbf{p}_b$; $\mathbf{n}_\sigma$; $b = 1, \dots, r+1$; $\sigma = k+1, \dots, n$.

But now we have $V_1^{(b)}$, $b = 1, \dots, r+1$, linear vector spaces of $n-k+1$ dimensions, spanned by each vector $\mathbf{p}_b$, $b = 1, \dots, r+1$, and set of vectors $\mathbf{n}_\sigma$, $\sigma = k+1, \dots, n$, such that any vector in this spaces satisfies (1). Again, no space of $n-k+2$ or higher dimension spanned by vectors $\mathbf{p}_b$, $b = 1, \dots, r+1$ and set of vectors $\mathbf{n}_\sigma$, $\sigma = k+1, \dots, n$ has this property.

Therefore, any set of orthonormal vectors in $V_1^{(b)}$, $b = 1, \dots, r+1$ and set $\{\mathbf{p}_c\}$ of r vectors, $b, c = 1, \dots, r+1$, $b \neq c$, represents the set of $n-k+r+1$ orthonormal vectors satisfying (1).

It is clear that (51), (52) and (53) represent recurrent formulas. Indeed, for $r = 0, 1, \dots, k-2$ we obtain all possible cases. Particularly, for $r = 0$, with following identification: $C_0 = C$, ${}_0x_{\alpha_0} = x_\alpha$, $p_0 = 0$ we obtain (38).

$\mathbf{A}_5$.

The final step is obtained for $r = k-2$. Then, from (51), (52) and

(53) we have

$$\mathbf{T}_{k-2} = \sum_{\alpha_{k-2}=1}^{2} {}_{k-2}\sigma_{\alpha_{k-2}}\, {}_{k-2}\mathbf{n}_{\alpha_{k-2}} \otimes {}_{k-2}\mathbf{n}_{\alpha_{k-2}}$$

$$\mathbf{x} = \sum_{\alpha_{k-2}=1}^{2} {}_{k-2}x_{\alpha_{k-2}}\, {}_{k-2}\mathbf{n}_{\alpha_{k-2}} + \sum_{a=1}^{k-2} p_a \mathbf{p}_a + \sum_{\sigma=1}^{n-k} x_\sigma \mathbf{n}_\sigma$$

$$C_{k-2}: \quad \mathbf{x}\cdot \mathbf{T}\mathbf{x} = \mathbf{x}\cdot \mathbf{T}_{k-2}\mathbf{x} = \sum_{\alpha_{k-2}=1}^{2} {}_{k-2}\sigma_{\alpha_{k-2}}\, {}_{k-2}x^2_{\alpha_{k-2}} = 0.$$

But since $\operatorname{tr} \mathbf{T}_{k-2} = 0$, then

$${}_{k-2}\sigma_1 + {}_{k-2}\sigma_2 = 0 \quad \text{or} \quad {}_{k-2}\sigma_1 = -{}_{k-2}\sigma_2 = {}_{k-2}\sigma, \tag{54}$$

and hence

$$\mathbf{T}_{k-2} = {}_{k-2}\sigma \left({}_{k-2}\mathbf{n}_1 \otimes {}_{k-2}\mathbf{n}_1 - {}_{k-2}\mathbf{n}_2 \otimes {}_{k-2}\mathbf{n}_2\right), \tag{55}$$

$$C_{k-2}: \quad \mathbf{x}\cdot \mathbf{T}\mathbf{x} = \mathbf{x}\cdot \mathbf{T}_{k-2}\mathbf{x} = {}_{k-2}\sigma \left({}_{k-2}x_1^2 - {}_{k-2}x_2^2\right) = 0. \tag{56}$$

¿From (55) we obtain that

$${}_{k-2}x_2 = \pm {}_{k-2}x_1. \tag{57}$$

Then the last two vectors, which completes the set of n orthonormal set vectors satisfying (1), are obtained as the unit vectors of set of vectors

$$\mathbf{x}_\pm = {}_{k-2}x_2 \left({}_{k-2}\mathbf{n}_1 \pm {}_{k-2}\mathbf{n}_2\right) + \sum_{a=1}^{k-2} p_a \mathbf{p}_a + \sum_{\sigma=1}^{n-k} x_\sigma \mathbf{n}_\sigma. \tag{58}$$

Clearly, $U_{k-2} \subset \cdots \subset U_1 \subset U$. In U_{k-2} we have that

$$\mathbf{x}_\pm = {}_{k-2}x_2 \left({}_{k-2}\mathbf{n}_1 \pm {}_{k-2}\mathbf{n}_2\right),$$

so that their unit vectors are given by very simple expressions

$$\begin{aligned} \mathbf{p}_{k-1} &= \frac{\sqrt{2}}{2}\left({}_{k-2}\mathbf{n}_1 + {}_{k-2}\mathbf{n}_2\right), \\ \mathbf{p}_k &= \frac{\sqrt{2}}{2}\left({}_{k-2}\mathbf{n}_1 - {}_{k-2}\mathbf{n}_2\right) \end{aligned} \tag{59}$$

Obviously, $\mathbf{p}_{k-1}\cdot \mathbf{p}_k = 0$.

Then we conclude that there are k linear spaces $V_1^{(\alpha)}$ of $n-k+1$ dimensions spanned by the set of $\{\mathbf{p}_\beta, \mathbf{n}_\sigma\}$; $\alpha, \beta = 1, \ldots, k$, $\alpha \neq \beta$; $\sigma = k+1, \ldots, n$. Thus, any other set of orthogonal vectors is formed of any set of $n-k+1$ vectors in $V_1^{(\alpha)}$, and the set of $k-1$ vectors $\{\mathbf{p}_\beta\}$, $\alpha \neq \beta$.

In particular, when $n = 3$ and $k = 2$, we have the Case **II**.

The case $k = n$ is included as the special one. Then $x_\sigma = 0$. As a consequence, there is no space of two and higher dimension in which any vector will satisfies (1).

The orthonormal set of vectors satisfying (1) is just the set of vectors $\{\mathbf{p}_\alpha\}$, $\alpha = 1, \ldots, k$. Particularly, when $n = 3$ $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, are orthonormal vectors, the Case **I**.

5 Non-symmetric tensor

It has long been known that non-symmetric stress tensors $\mathbb{T}$ may occur in mechanics. Then, its unique decomposition in symmetric and skew-symmetric tensors, denoted by $\mathbf{T}$ and $\mathbf{T}^A$, respectively, is given by

$$\mathbb{T} = \mathbf{T} + \mathbf{T}^A. \tag{60}$$

But this decomposition holds for any second order tensor $\mathbb{T}$ in E_n. Moreover, from (60), we have

$$\mathbf{p}\cdot \mathbb{T}\mathbf{p} = \mathbf{p}\cdot \mathbf{T}\mathbf{p}, \tag{61}$$

where $\mathbf{p}$ is any unit vector.

Accordingly, we state the following

Theorem 2 *A necessary and sufficient condition for $\mathbb{T}$ to be a state of pure shear is*

$$\operatorname{tr} \mathbf{T} = 0. \tag{62}$$

Obviously, in that case only symmetric part of any tensor of second order matters. Therefore, the proof of the **Theorem 1** holds generally for any tensor $\mathbb{T}$ of second order in E_n.

6 Discussion

Recently Hayes and Laffey in their paper [6], in *Remark 2* stated: another formulation of the Basic Result (in our paper equation (1)) in matrix theory language (valid in all dimensions) is as follows: Let T be an $n \times n$ matrix with $\operatorname{tr} T = 0$. Then there exists a real orthogonal matrix Q such that $Q^T T Q$ has all its diagonal entries zero.

To find this orthogonal matrix Q they proceed as follows. Since $\operatorname{tr} T = 0$, T is not a scalar multiple of the identity matrix, so they can choose a vector $\mathbf{w}$ such that $\mathbf{w}$ and $T\mathbf{w}$ are linearly independent. Let $\mathbf{z} = S\mathbf{w} - r\mathbf{w}$ and note that $\mathbf{w}\cdot \mathbf{z} = 0$. Let $\mathbf{w}_1 = \mathbf{w}/\|\mathbf{w}\|$ and $\mathbf{w}_2 = \mathbf{z}/\|\mathbf{z}\|$, and extend these to an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ of real n-space and let W be the corresponding orthogonal matrix $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$. Then

$$W^T T W = \begin{pmatrix} 0 & T_{12} \\ T_{12}^T & T_{22} \end{pmatrix} \tag{63}$$

where T_{22} is a symmetric $(n-1) \times (n-1)$ matrix with $\operatorname{tr} S_{22} = 0$.

Using induction on n, they can find an orthogonal $(n-1) \times (n-1)$ matrix Y with $Y_2^T T_{22} Y_2$ having zero entries on its diagonal. Let

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & Y_2 \end{pmatrix} \quad \text{and} \quad Q = WY. \tag{64}$$

Then Q is orthogonal and

$$Q^T T Q = \begin{pmatrix} 0 & T_{12} Y_2 \\ Y_2^T T_{12}^T & Y_2^T T_{22} Y_2 \end{pmatrix} \tag{65}$$

has zero diagonal, as desired.

The proof of this statement, in our opinion, is at least incomplete. Besides missprint where instead $r = \mathbf{z}\cdot S\mathbf{w}/\mathbf{w}\cdot \mathbf{w}$ should bee written that $r = \mathbf{w}\cdot S\mathbf{w}/\mathbf{w}\cdot \mathbf{w}$, the expression for $\mathbf{z}$ is missleading.

For instance, zero element in (63) means that $\mathbf{w}_1\cdot T\mathbf{w}_1 = 0$. Hence, this is the only condition which has to be satisfied by $\mathbf{w}_1$. There is no any need for $\mathbf{z}$ at this stage to be define by T and $\mathbf{w}$.

In the next step, of induction, we have to choose $\mathbf{w}_2$ such that $\mathbf{w}_2 \cdot T\mathbf{w}_2 = 0$ and $\mathbf{w}_1 \cdot \mathbf{w}_2 = 0$ and so on.

In fact, the problem of finding Q reduces to the process of finding q_i which has to satisfy the condition that $q_i T q_i = 0$. One of the procedure for this problem is given in our Chapter 3.

7 Conclusion

We prove **Theorem 1** in general form for any tensor $\mathbb{T}$ of second order in E_n, making use of the notation of orthogonal projector. In E_3 we derive some general conclusion concerning singular traceless tensors of second order. The generalization of this problem to the decomposition of a large incompressible deformation in E_3 has been done by He and Zheng [7].

We did not discuss some special cases, such as $\mathbf{T}$ has some eigenvalues of the multiplicity of higher order then one. In these cases, the problem simplifies a lot, but the procedure is same.

Also, we did not discus the applications of these representation for such a state of pure shear in continuum mechanics. This has been investigated in several papers, among them we refer the reader to Boulanger and Hayes [8], and Norris [4].

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O stanju čistog smicanja

U radu se dokazuje fundamentalna teorema, koja se odnosi na čisto smicanje, koristeći samo pojam ortogonalnog operatora. Pokazano je da je stanje čistog smicanja isto, do na rotaciju, za sve singularne simetrične tenzore u E_3, čiji je trag jednak nuli.

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