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# Bimodal optimization with constraints: critical value of the constraint and post-critical configurations

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**Abstract**

By using a method based on Pontryagin's principle, formulated in [13], and [14] we study optimal shape of an elastic column with constraints on the minimal value of the cross-sectional area. We determine the critical value of the minimal cross-sectional area separating bi from unimodal optimization. Also we study the post-critical shape of optimally shaped rod and find the preferred configuration of the bifurcating solutions from the point of view of minimal total energy.

**Keywords**: Bimodal optimization, Stability of rods, Post-buckling behavior

## 1 Introduction

Lagrange [1] in 1773 formulated the problem of determining the shape of a rod of given volume that is the strongest against buckling. Correct solution of the problem, with the simply supported boundary conditions, leading to the optimally shaped column, was obtained by Clausen [2] in 1851.

We recall the problem formulated by Tadjbakhsh and Keller [3] in which Lagrange type of problem was treated but for the column with clamped-clamped boundary conditions. Such boundary conditions lead, to *bimodal* optimization. This means that the optimal structure may possess two linearly independent buckling modes. It was shown by Olhoff and Rasmussen [4], in

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their seminal paper, that the *unimodal* solution presented in [3] for clamped-clamped boundary conditions is not correct, and the bimodal formulation of the optimization problem is required.

First bimodal optimal solutions for elastic columns were found in Olhoff and Rasmussen [4]. Later the problem was treated by Seyranian [5],[6], and Masur [7]. In [5],[6] the problem of an axially loaded column with clamped-clamped boundary conditions was solved for three particular cases corresponding to the second moment of inertia being proportional to the first ($\alpha = 1$), square ($\alpha = 2$) and cubic power ($\alpha = 3$) of the cross–sectional area.

The paper [4] contains another important result. For the column with prescribed minimal cross–sectional area (constrained optimization problem) it was found that optimization may be bimodal or unimodal, depending on the value of the prescribed minimal cross-sectional area. In [4] the value of the constraint that separates bimodal from the unimodal optimal shape, was determined for a column with moment of inertia proportional to square power of the cross-sectional area, i.e., $\alpha = 2$. This result was later confirmed by Tada and Wang [8]

Another important problem treated recently is the problem of post-buckling analysis of bimodal optimum columns. It was analyzed by Seyranian and Privalova [9], Seyranian [10] and Olhoff and Seyranian [11]. By using asymptotic expansion in the vicinity of trivial configuration (in which the rod axis is straight) the authors concluded that for non-extensible column with clamped ends in the general case the post-buckling behavior is governed by a fourth order polynomial equation, i.e., near the bifurcation point there may be up to four post-buckling equilibrium states emanating from the trivial equilibrium state. Two of those solutions (corresponding to the symmetric and anti-symmetric buckling modes) are stable while the other two (corresponding to asymmetric buckling modes) are unstable. Let $y\,(t)$ be a bifurcating solution starting from the trivial configuration in which column axis is straight. Let the normalized first buckling mode be denoted by $\overline{x}$, and the second buckling mode by $\widehat{x}$. Due to the symmetry conditions we have $\overline{x}\,(t) = \overline{x}\,(1-t)$ and $\widehat{x}\,(t) = -\widehat{x}\,(1-t)$, where $t$ is the dimensionless arc-length of the column axis. The results presented in [11] show that $y = \gamma_1\overline{x} + \gamma_2\widehat{x}$, with $\gamma_1$ and $\gamma_2$ satisfying certain system of algebraic equations (see [11] equations (4.9),(4.15)). The stable bifurcating solutions correspond to values of $(\gamma_1, \gamma_2)$ given by $(\gamma_1^*, 0)$ and $(0, \gamma_2^*)$. It is important to note that in [9], [10] and [11] it was not possible to state is any of two stable solutions preferred from the point of view of minimum of the total potential energy.

Our intention in this work is to generalize the optimization procedure based on Ponrtyagin's principle, see [12],[13] and [14] to the case when minimal value of the cross-sectional area is prescribed. Then, we shall solve the following two problems: a) determine the critical value of the restriction on cross-sectional area that separates bi from unimodal solution of the optimization problem for moment of inertia proportional to the first ($\alpha = 1$), and cubic power ($\alpha = 3$) of the cross–sectional area; b) to study numerically, the post-critical behavior of the rod. We shall confirm the validity of the findings presented in [11]. However, we shall introduce another criteria that will give possibility to select one of the two stable solutions as a preferred one. Our notation will follow work [14].

## 2 Formulation of the problem

Consider a column of length $L$ shown in Figure 1. The column is clamped at both ends, with end $C$ having the possibility of sliding along the axis $x$. At the end $C$ the column is loaded by a compressive force $F$. Equilibrium equations for the column are, see [15]

$$\frac{dH}{dS} = 0, \quad \frac{dV}{dS} = -q_y, \quad \frac{dM}{dS} = -V\cos\theta + H\sin\theta, \tag{1}$$

where $H$ and $V$ are components of the contact force (i.e. the resultant force in an arbitrary cross–section) along $x$ and $y$ axes, respectively, $M$ is the bending moment, $\theta$ is the angle between the tangent to the column axis and the $x$ axis of a rectangular Cartesian coordinate system $x - B - y$, and $S$ is the arc–length of the column axis measured from the origin of the coordinate system $B$.

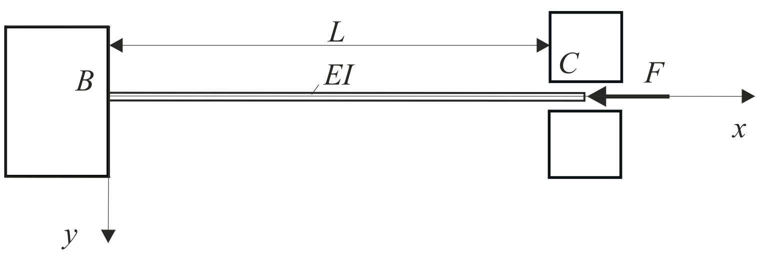


Figure 1: Coordinate system and load configuration

We adjoin to (1) the geometrical equations

$$\frac{d\bar{x}}{dS} = \cos\theta, \quad \frac{d\bar{y}}{dS} = \sin\theta, \tag{2}$$

and the constitutive equation

$$M = EI\frac{d\theta}{dS}. \tag{3}$$

In (2), (3) we used $\bar{x}$ and $\bar{y}$ to denote coordinates of an arbitrary point on the rod axis in the coordinate system $x-B-y$, $E$ is the modulus of elasticity and $I$ is the moment of inertia of the cross–section. Equations (2), (3) correspond to the classical Bernoulli–Euler rod theory. The boundary conditions for the column shown in Figure 1 are

$$\bar{y}\,(0) = \bar{y}\,(L) = 0, \quad \theta\,(0) = \theta\,(L) = 0, \quad H\,(L) = -F. \tag{4}$$

Solving $(1)_{1,2}$ and by using $(4)_3$ we obtain

$$H = -F. \tag{5}$$

The volume of the column is

$$W = \int_0^L A\,(S)\,dS, \tag{6}$$

where $A\,(S)$ is the cross–sectional area. We assume that

$$I = kA^{\alpha}, \tag{7}$$

where $k$ is a constant and $\alpha = 1, 2, 3$. By introducing the dimensionless quantities

$$\begin{gathered} t = \frac{S}{L}, \quad a = \frac{A}{L^2}, \quad \zeta = \frac{\bar{x}}{L}, \quad \eta = \frac{\bar{y}}{L}, \quad v = \frac{V}{kEL^2}, \\ w = \frac{W}{L^3}, \quad \lambda = \frac{F}{kEL^2}, \quad m = \frac{M}{kEL^3}, \end{gathered} \tag{8}$$

we obtain from (1)–(7) the following system of differential equations that describe small deformations of the column

$$\begin{gathered} \dot{v} = 0, \quad \dot{m} = -v - \lambda\theta, \\ \dot{\zeta} = 1, \quad \dot{\eta} = \theta, \quad \dot{\theta} = \frac{m}{a^{\alpha}}, \end{gathered} \tag{9}$$

subject to

$$\eta(0) = 0, \quad \eta(1) = 0, \quad \theta(0) = 0, \quad \theta(1) = 0. \tag{10}$$

where $\dot{(\cdot)} = \frac{d}{dt}(\cdot)$. The dimensionless volume becomes

$$w = \int_0^1 a(\xi)\, d\xi. \tag{11}$$

We assume that the cross sectional area $a(t)$ belongs to the set $\mathbf{U}$ called the set of *admissible* cross-sectional area functions. In what follows we assume that $\mathbf{U}$ is the set of continuous functions on the interval $[0,1]$, i.e., $\mathbf{U} = C(0,1)$ that satisfy restriction $0 < a_{\min} \leq a(t)$, where $a_{\min} \geq 0$ is prescribed.

Suppose now that $\lambda \in \mathbb{R}$ is given. We define *the optimal compressed column* as the column so shaped that any other column of the same length (in our case equal to one) and smaller volume will buckle under load characterized by $\lambda$. Thus, the problem of determining the shape of the optimal column may be, mathematically, stated as follows:

Given $\lambda$, find $a^*(t) \in \mathbf{U}$ such that the integral (11) is a minimum for all those $a(t) \in \mathbf{U}$ under the equations and boundary conditions (9), (10). Note that this optimization problem is equivalent to the problem of maximization of the critical load $\lambda$ with the given volume constraint $w = 1$, see [16].

## 3 Solution to the problem

We introduce new dependent variables as

$$x_1 = \eta, \quad x_2 = \theta, \quad x_3 = v, \quad x_4 = m. \tag{12}$$

Then, the system (9), (10) becomes

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{x_4}{a^\alpha}, \quad \dot{x}_3 = 0, \quad \dot{x}_4 = -x_3 - \lambda x_2, \tag{13}$$

and

$$x_1(0) = 0, \quad x_1(1) = 0, \quad x_2(0) = 0, \quad x_2(1) = 0. \tag{14}$$

In terms of the optimal control, the Problem now becomes: Given $\lambda$ find the control $a^*(t) \in \mathbf{U}$ such that

$$\min_{a \in \mathbf{U}} I = \min_{a \in \mathbf{U}} \int_0^1 a(t)\, dt = \int_0^1 a^*(t)\, dt. \tag{15}$$

under differential constraints (13), (14).

Suppose that for given $\lambda, p, t_0$ and for the optimal $a(t) = a^*(t)$ the linear boundary value problem (13), (14) has *two* linearly independent solutions, $(\overline{x}_1, ...\overline{x}_4)$ and $(\widehat{x}_1, ...\widehat{x}_4)$ corresponding to two buckling modes. Since both solutions correspond to the same $\lambda$ and $a(t) = a^*(t)$ we have, see [13]

$$\begin{aligned} &\dot{\overline{x}}_1 = \overline{x}_2, \quad \dot{\overline{x}}_2 = \frac{\overline{x}_4}{a^\alpha}, \quad \dot{\overline{x}}_3 = 0, \quad \dot{\overline{x}}_4 = -\overline{x}_3 - \lambda \overline{x}_2, \\ &\dot{\widehat{x}}_1 = \widehat{x}_2, \quad \dot{\widehat{x}}_2 = \frac{\widehat{x}_4}{a^\alpha}, \quad \dot{\widehat{x}}_3 = 0, \quad \dot{\widehat{x}}_4 = -\widehat{x}_3 - \lambda \widehat{x}_2, \end{aligned} \tag{16}$$

satisfying

$$\begin{aligned} &\overline{x}_1(0) = 0, \quad \overline{x}_1(1) = 0, \quad \overline{x}_2(0) = 0, \quad \overline{x}_2(1) = 0, \\ &\widehat{x}_1(0) = 0, \quad \widehat{x}_1(1) = 0, \quad \widehat{x}_2(0) = 0, \quad \widehat{x}_2(1) = 0, \end{aligned} \tag{17}$$

To determine $a^*(t)$ we use the standard procedure of Optimal control theory, [17], [18]. The Pontryagin's function $\mathcal{H}$, taking into account that differential constraints are given by (16), reads

$$\mathcal{H} = a + \overline{p}_1 \overline{x}_2 + \overline{p}_2 \frac{\overline{x}_4}{a^\alpha} + \overline{p}_4 (-\overline{x}_3 - \lambda \overline{x}_2) + \widehat{p}_1 \widehat{x}_2 + \widehat{p}_2 \frac{\widehat{x}_4}{a^\alpha} + \widehat{p}_4 (-\widehat{x}_3 - \lambda \widehat{x}_2), \tag{18}$$

where the co–state variables $\overline{p}_i, \widehat{p}_i, i = 1, ..., 4$ satisfy

$$\begin{aligned} &\dot{\overline{p}}_1 = -\frac{\partial \mathcal{H}}{\partial \overline{x}_1} = 0, \quad \dot{\overline{p}}_2 = -\frac{\partial \mathcal{H}}{\partial \overline{x}_2} = -\overline{p}_1 + \lambda \overline{p}_4, \\ &\dot{\overline{p}}_3 = -\frac{\partial \mathcal{H}}{\partial \overline{x}_3} = \overline{p}_4, \quad \dot{\overline{p}}_4 = -\frac{\partial \mathcal{H}}{\partial \overline{x}_4} = -\frac{\overline{p}_2}{a^\alpha}, \\ &\dot{\widehat{p}}_1 = -\frac{\partial \mathcal{H}}{\partial \widehat{x}_1} = 0, \quad \widehat{p}_2 = -\frac{\partial \mathcal{H}}{\partial \widehat{x}_2} = -\widehat{p}_1 + \lambda \widehat{p}_4, \\ &\dot{\widehat{p}}_3 = -\frac{\partial \mathcal{H}}{\partial \widehat{x}_3} = \widehat{p}_4, \quad \dot{\widehat{p}}_4 = -\frac{\partial \mathcal{H}}{\partial \widehat{x}_4} = -\frac{\widehat{p}_2}{a^\alpha}, \end{aligned} \tag{19}$$

subject to

$$\begin{aligned} &\overline{p}_3(0) = 0, \quad \overline{p}_3(1) = 0, \quad \overline{p}_4(0) = 0, \quad \overline{p}_4(1) = 0, \\ &\widehat{p}_3(0) = 0, \quad \widehat{p}_3(1) = 0, \quad \widehat{p}_4(0) = 0, \quad \widehat{p}_4(1) = 0. \end{aligned} \tag{20}$$

Note that the systems (13),(14) and (19),(20) are the same (to see this note that differential equations and boundary conditions for the variables $x_1, x_2, x_3$

and $x_4$ are the same as differential equations and boundary conditions for $-p_3, -p_4, p_1$ and $p_2$, respectively). Since we assumed that (13),(14) has two linearly independent solutions (in [14] this fact is proved) it follows that solution of (19),(20) must be of the form

$$\begin{aligned} \overline{p}_1 &= \beta_{11}\overline{x}_3 + \beta_{12}\widehat{x}_3, & \widehat{p}_1 &= \beta_{21}\overline{x}_3 + \beta_{22}\widehat{x}_3, \\ \overline{p}_2 &= \beta_{11}\overline{x}_4 + \beta_{12}\widehat{x}_4, & \widehat{p}_2 &= \beta_{21}\overline{x}_4 + \beta_{22}\widehat{x}_4, \\ \overline{p}_3 &= -\beta_{11}\overline{x}_1 - \beta_{12}\widehat{x}_1, & \widehat{p}_3 &= -\beta_{21}\overline{x}_1 - \beta_{22}\widehat{x}_1, \\ \overline{p}_4 &= -\beta_{11}\overline{x}_2 - \beta_{12}\widehat{x}_2, & \widehat{p}_4 &= -\beta_{21}\overline{x}_2 - \beta_{22}\widehat{x}_2, \end{aligned} \tag{21}$$

where $\beta_{ij}, i = 1, 2$ are constants. The optimality condition $\min\limits_{a\in\mathbf{U}}\mathcal{H}$ leads to $\frac{\partial\mathcal{H}}{\partial a} = 1 - \alpha\overline{p}_2\frac{\overline{x}_4}{a^{\alpha+1}} - \alpha\widehat{p}_2\frac{\widehat{x}_4}{a^{\alpha+1}} = 0$ if $a(t) \geq a_{\min}$ and $a = a_{\min}$ otherwise. Therefore

$$a^* = \begin{cases} \left[\alpha\left(\overline{p}_2\overline{x}_4 + \widehat{p}_2\widehat{x}_4\right)\right]^{1/(\alpha+1)}, & \text{if } \left[\alpha\left(\overline{p}_2\overline{x}_4 + \widehat{p}_2\widehat{x}_4\right)\right]^{1/(\alpha+1)} \geq a_{\min} \\ a_{\min}, & \text{if } \left[\alpha\left(\overline{p}_2\overline{x}_4 + \widehat{p}_2\widehat{x}_4\right)\right]^{1/(\alpha+1)} \leq a_{\min} \end{cases} \tag{22}$$

Relations (21) when used in (22) lead to a "feedback" control. To see this, we use (21) in (22) to obtain

$$a^* = \begin{cases} \left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{1/(\alpha+1)}, \\ \quad \text{if } \left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{1/(\alpha+1)} \geq a_{\min} \\ a_{\min}, \quad \text{if } \left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{1/(\alpha+1)} \leq a_{\min} \end{cases} \tag{23}$$

where $\gamma_{11} = \beta_{11}, \gamma_{12} = (\beta_{12} + \beta_{21})/2, \gamma_{22} = \beta_{22}$. From $(23)_1$ and the condition $a(t) \geq a_{\min} \geq 0$ it follows

$$\gamma_{11}\gamma_{22} \geq (\gamma_{12})^2. \tag{24}$$

To show that (23) leads to minimum of $\mathcal{H}$ we note that (18) may be written as

$$\mathcal{H} = a + \frac{\left(\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2\right)}{a^\alpha} + \text{ terms independent of } a$$

Thus, we conclude that (23) minimizes $\mathcal{H}$. Note also that $\mathcal{H}$ does not depend on $t$ explicitly. Therefore on the solution of (16), (17) we have $\mathcal{H} = const.$ Therefore, on the optimal solution $\mathcal{H}$ has the value

$$\begin{aligned}\mathcal{H} = &\left[\alpha\left(\left(\overline{x}_4\right)^2 + \left(\widehat{x}_4\right)^2\right)\right]^{1/(\alpha+1)} + \frac{\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2}{\left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{\alpha/(\alpha+1)}} \\ &+ \overline{x}_2\overline{x}_3 + \overline{x}_2\left(\overline{x}_3 + \lambda\overline{x}_2\right) + \widehat{x}_2\widehat{x}_3 + \widehat{x}_2\left(\widehat{x}_3 + \lambda\widehat{x}_2\right) = const. \\ &\text{if } \left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{1/(\alpha+1)} \geq a_{\min}, \end{aligned} \tag{25}$$

or

$$\begin{aligned}\mathcal{H} = a_{\min} &+ \frac{\left(\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2\right)}{\left(a_{\min}\right)^{\alpha}} \\ &+ \overline{x}_2\overline{x}_3 + \overline{x}_2\left(\overline{x}_3 + \lambda\overline{x}_2\right) + \widehat{x}_2\widehat{x}_3 + \widehat{x}_2\left(\widehat{x}_3 + \lambda\widehat{x}_2\right) \; = \; const. \\ &\text{if } \left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2 + 2\gamma_{12}\overline{x}_4\widehat{x}_4 + \gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{1/(\alpha+1)} \leq a_{\min} \end{aligned} \tag{26}$$

This can be easily verified by differentiation.

Therefore, when determining optimal shape of the rod, the system to be solved is obtained when (23) is substituted in (16), that is

$$\begin{aligned} \dot{\overline{x}}_1 &= \overline{x}_2, \\ \dot{\overline{x}}_2 &= \frac{\overline{x}_4}{\left(a^*\right)^{\alpha}} \\ \dot{\overline{x}}_3 &= 0, \\ \dot{\overline{x}}_4 &= -\overline{x}_3 - \lambda\overline{x}_2, \\ \\ \dot{\widehat{x}}_1 &= \widehat{x}_2, \\ \dot{\widehat{x}}_2 &= \frac{\widehat{x}_4}{\left(a^*\right)^{\alpha}} \\ \dot{\widehat{x}}_3 &= 0, \\ \dot{\widehat{x}}_4 &= -\widehat{x}_3 - \lambda\widehat{x}_2, \end{aligned} \tag{27}$$

with

$$\left(a^{*}\right)^{\alpha}=\begin{cases}\left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2+2\gamma_{12}\overline{x}_4\widehat{x}_4+\gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{\alpha/(\alpha+1)}, \\ \text{if } \left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2+2\gamma_{12}\overline{x}_4\widehat{x}_4+\gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{1/(\alpha+1)}\geq a_{\min}, \\ \left(a_{\min}\right)^{\alpha}, \text{ if } \left[\alpha\left(\gamma_{11}\left(\overline{x}_4\right)^2+2\gamma_{12}\overline{x}_4\widehat{x}_4+\gamma_{22}\left(\widehat{x}_4\right)^2\right)\right]^{1/(\alpha+1)}\leq a_{\min}.\end{cases}$$

The system (27) is subject to (17). The constants $\gamma_{ij}, i,j=1,2$ must be chosen so to satisfy (24). In what follows, due to the symmetry we take $\gamma_{11}=\gamma_{22}=1, \gamma_{12}=0$. On the solution (27),(17) the first integral (25),(26) holds.

Next we formulate nonlinear problem. Suppose that the optimal cross-sectional area $a^*$ is determined from (23). The differential equations describing large deformation of the rod are, see [15] and [11]

$$\dot{\mathbb{Y}}=\sin\Theta,\quad \dot{\mathbb{X}}=\cos\Theta,$$
$$\dot{\Theta}=\frac{\mathbb{M}}{(a^*)^{\alpha}},\quad \dot{\mathbb{V}}=0,\quad \dot{\mathbb{M}}=-\mathbb{V}\cos\Theta-\Lambda\sin\Theta, \tag{28}$$

subject to

$$\mathbb{X}(0)=0,\quad \mathbb{Y}(0)=0,\quad \mathbb{Y}(1)=0,\ \ \Theta(0)=0,\quad \Theta(1)=0, \tag{29}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are coordinates of an arbitrary point on the rod axis, $\Theta$ is the slope of the rod axis and $\mathbb{M}$, $\mathbb{V}$ and $\Lambda$ are bending moment, vertical and axial force at an arbitrary cross-section of the rod. The equation (28) may be written in the form

$$\left(\frac{1}{\sqrt{1-\dot{\mathbb{Y}}^2}}\left((a^*)^{\alpha}\frac{\ddot{\mathbb{Y}}}{\sqrt{1-\dot{\mathbb{Y}}^2}}\right)^{\cdot}\right)^{\cdot}+\lambda\left(\frac{\dot{\mathbb{Y}}}{\sqrt{1-\dot{\mathbb{Y}}^2}}\right)^{\cdot}=0.$$

In [11] by using asymptotic expansion, it was shown that for $\Lambda$ close to $\lambda$ (the value for which $a^*$ is determined) the system may have four solutions $\mathbb{Y}_i, i=1,...4$. A solution to (28),(29) is termed *stable* if the total potential energy of inner and outer forces has a weak local minimum. Also in [11] it was shown that two solutions to (28),(29) are stable.

Let $\mathbb{U}_i, i=1,...,n$ be the total internal energy corresponding to configuration $\mathbb{Y}_i, i=1,...n$, i.e.,

$$\mathbb{U}_i\left(\mathbb{Y}_i\right)=\frac{1}{2}\int_0^1\left(a^*\left(t\right)\right)^{\alpha}\frac{\left(\ddot{\mathbb{Y}}_i\left(t\right)\right)^2}{1-\dot{\mathbb{Y}}_i\left(t\right)^2}dt=\frac{1}{2}\int_0^1\frac{\mathbb{M}^2\left(t\right)}{\left(a^*\left(t\right)\right)^{\alpha}}dt.$$

Further, let $\mathbb{L}_i, i = 1, ...n$ be the position of the end $C$ in the deformed state, i.e., $\mathbb{L}_i = \mathbb{X}_i(1)$. Then the work of the external force $\Lambda$ in the configuration described by $\mathbb{Y}_i, i = 1, ...n$, is

$$W_i(\mathbb{Y}_i) = \Lambda \int_0^1 \sqrt{1 - \dot{\mathbb{Y}}_i(t)^2} dt = -\Lambda(1 - \mathbb{L}_i)$$

We note that the *total potential energy* of the rod in the configuration $\mathbb{Y}_i, i = 1, ...n$, is, see [19], [11]

$$E_i(\mathbb{Y}_i) = \mathbb{U}_i - W_i = \frac{1}{2}\int_0^1 (a^*(t))^\alpha \frac{\left(\ddot{\mathbb{Y}}_i(t)\right)^2}{1 - \dot{\mathbb{Y}}_i(t)^2} dt - \Lambda \int_0^1 \sqrt{1 - \dot{\mathbb{Y}}_i(t)^2} dt. \quad (30)$$

We call $\mathbb{Y}_i$ the *preferred* equilibrium configuration if the total potential energy is minimum with respect to the other equilibrium configurations, i.e., $E_i(\mathbb{Y}_i) < E_j(\mathbb{Y}_j), j = 1, ..., n, i \neq j$.

# 4 Numerical results

## 4.1 The transition values between uni and bi modal optimization

We start with $\alpha = 2$. This is the classical case treated by many researchers. It will serve to test our procedure. In [14] we obtained $\lambda_{a_{\min}=0} = 52.3562542669$ for $w_{\min} = 1$ and for $a_{\min} = 0$. The smallest value of cross-sectional area is found to be $a^*_{\min} = 0.22582372$ at $t_1 = 0.24658$ and $t_2 = 1 - 0.24658 = 0.75342$. We solved (27),(17) for the imposed restriction on the minimal cross-sectional area $a_{\min} = 0.25$. The optimization is bimodal with $\lambda_{a_{\min}=0.25} = 52.3495443$ and the buckling modes are shown in Fig. 2. Note that the buckling load for $a_{\min} = 0.25$ is only slightly smaller than the value corresponding to $a_{\min} = 0$.

Since for uniform cross-section we have $\lambda_{const} = 4\pi^2$ we conclude that the optimum buckling load factor $\lambda_{a_{\min}=0.25}/\lambda_{const} = 1.32602945$. In Fig. 3a we show cross-sectional area of the rod, while in Fig. 3b we show the cross-sectional area near the part where area is constant.

The characteristic values of the cross-sectional area are $a(0) = a(1) = 1.3321372590$. Also $a(t) = 0.25, t \in [0.241063333, 0.2537025]$ and $t \in [1 - 0.2537025, 1 - 0.241063333]$. Our results compare well with the those presented in [4] where buckling load factor is obtained as $(\lambda_{a_{\min}=0.25}/\lambda_{const})_{O-R} = 1.3260$.

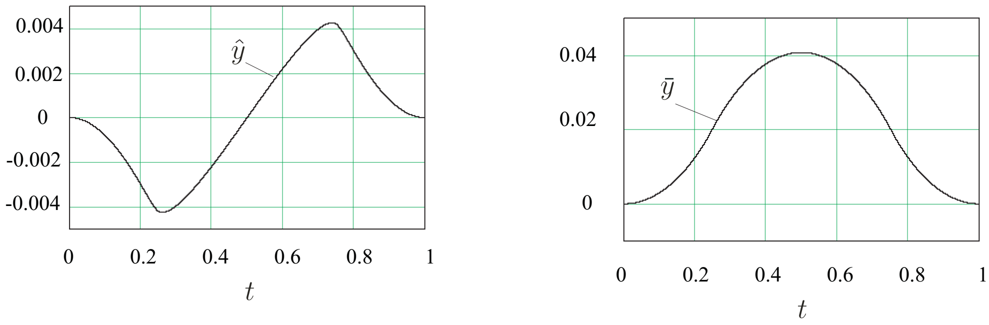


Figure 2: Buckling modes correspondig to $\lambda_{a_{\min}=0.25} = 52.3495443, a_{\min} = 0.25$

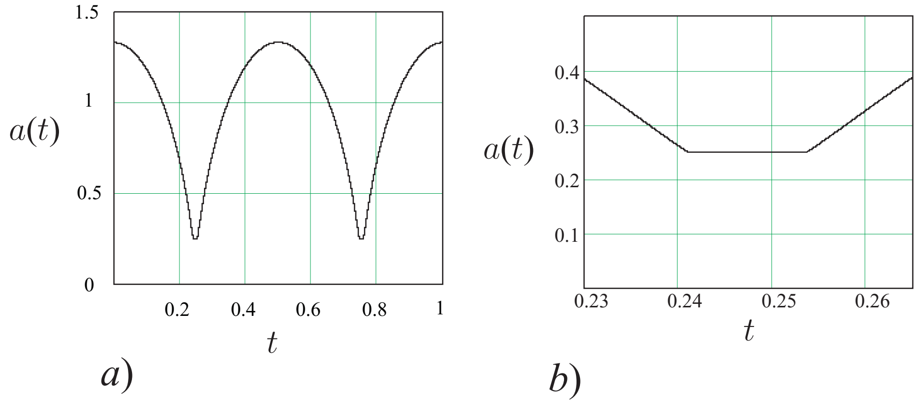


Figure 3: Cross-sectional area correspondig to $\lambda_{a_{\min}=0.25} = 52.3495443$, $a_{\min} = 0.25$

Now we increase the value of $a_{\min}$ to the value $a_{\min} = a_{tr} = 0.28171$ representing the transition value between single and bi modal optimization. The corresponding dimensionless force is $\lambda_{a_{tr}} = 52.290719$. The load factor $\lambda_{a_{tr}}/\lambda_{const} = 1.32454$. These values compare well with those obtained by numerical solution of nonlinear integral equations in [11] where the value $(a_{tr})_{O-R} = 0.280$ was obtained. In [8] the values $(a_{tr})_{T-W} = 0.2817$ and $(\lambda_{a_{tr}})_{T-W} = 52.2908$.

For the case $\alpha = 1$ we obtain $a_{tr} = 0.03985$ and $\lambda_{tr} = 47.99145$. In [14] the value $\lambda_{a_{\min}=0} = 47.99305032$ was obtained. Finally for $\alpha = 3$ the transition value and dimensionless force are $a_{tr} = 0.45125$ and $\lambda_{tr} = 54.6251$. Also

Table 1:

| $\alpha$ | 1 | 2 | 3 |
|---|---|---|---|
| $\lambda_{a_{\min}=0}$ | 47.99305032 | 52.3562542669 | 54.82543305 |
| $a_{tr}$ | 0.03985 | 0.28171 | 0.45125 |
| $\lambda_{tr}$ | 47.99145 | 52.290719 | 54.6251 |

Table 2:

| Configuration | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Energy $E_i \times 10^5$ | $-3.21366$ | $-2.77572$ | $-1.767195$ | $-1.767195$ |
| $\mathbb{X}(1)$ | 0.99853 | 0.99873 | 0.99919 | 0.99919 |

from [14] we quote $\lambda_{a_{\min}=0} = 54.82543305$. The results are summarized in the Table 1.

We note that the values of $a_{tr}$ and $\lambda_{tr}$ for $\alpha = 1$ and $\alpha = 3$ are obtained by the authors for the first time.

## 4.2 Post-critical behavior

In what follows we shall study the optimally shaped rod in the post-critical regime. We restrict our analysis to the case $a_{\min} = 0$. In [11] the problem was treated for $\alpha = 2$ analytically for the case when the load parameter $\lambda$ is close to $\lambda_{a_{\min}=0}$. We shall study the post-critical behavior of the optimally shaped rod for $\lambda$ not necessarily close to $\lambda_{a_{\min}=0}$. Also we shall study the energy $E_i$ for each branch bifurcating from the trivial state $\mathbb{Y}_i$=0, $i = 1, ...4$, see (28).

In Figure 4 we show solution to (28),(29) for $\Lambda = 52.4$. There are four bifurcating branches having the shape obtained in [11] and [9] by using perturbation analysis. We note that the stable configurations 1 and 2 evolve from the anti-symmetric and symmetric buckling modes shown in Fig. 2, while unstable configurations 3 and 4 evolve from the asymmetric buckling modes.

The relevant numerical values are shown in Table 2.

Next we increase load parameter to $\Lambda = 55$. The possible deformed configurations of the rod are shown in Fig. 5. We note that there are no new equilibrium configurations appearing with the increase of load. Only configurations coming from the bifurcation point, corresponding to $\lambda = 52.3562542669$, increase in amplitude.

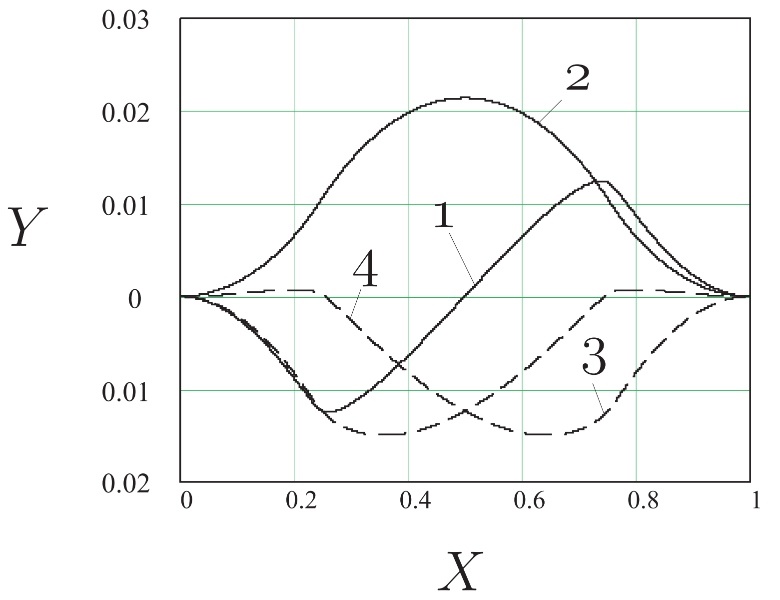


Figure 4: The deformed configuration of the rod for $\Lambda = 52.4$

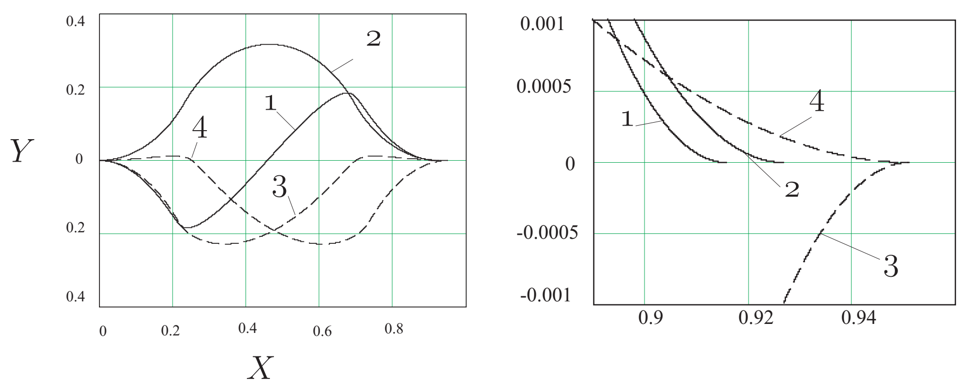


Figure 5: The deformed configuration of the rod for $\Lambda = 55$

Here the right end of the rod moved significantly and we showed on the right part of the Figure the positions of the right end of the rod in different configurations. Also in Table 3 we present the relevant numerical values for $\Lambda = 55$.

Finally, in Fig 6 we present the shape of the rod for $\Lambda = 60$ and $\Lambda = 65$

Table 3:

| Configuration | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Energy $E_i$ | $-0.11337$ | $-0.09825$ | $-0.06454$ | $-0.06454$ |
| $\mathbb{X}(1)$ | 0.915687 | 0.926833 | 0.951142 | 0.951142 |

Table 4:

| Configuration | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Energy $E_i$ | $-0.89406$ | $-0.77586$ | $-0.54200$ | $-0.54200$ |
| $\mathbb{X}(1)$ | 0.77535 | 0.80551 | 0.85769 | 0.8769 |

respectively.

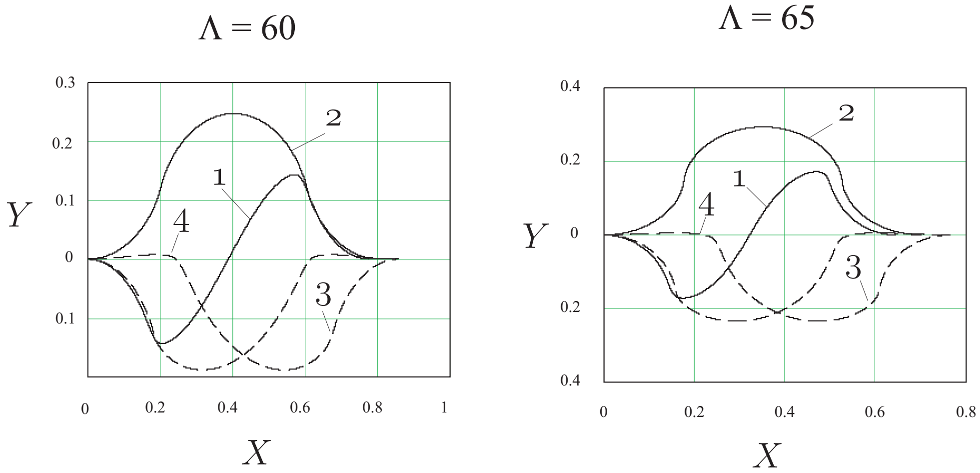


Figure 6: The deformed configuration of the rod for $\Lambda = 60$ and $\Lambda = 65$

In Table 4 we summarize the numerical results, corresponding to $\Lambda = 60$, and in Table 5 results corresponding to $\Lambda = 65$.

In calculating the energy it was assumed that the potential energy in the undeformed state is equal to zero.

Schematically, the evolution of configurations 1-4 according to total potential energy is shown in Fig. 7.

Table 5:

| Configuration | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Energy $E_i$ | $-2.3435$ | $-2.0136$ | $-1.48530$ | $-1.48530$ |
| $\mathbb{X}(1)$ | 0.6438 | 0.7022 | 0.7657 | 0.7657 |

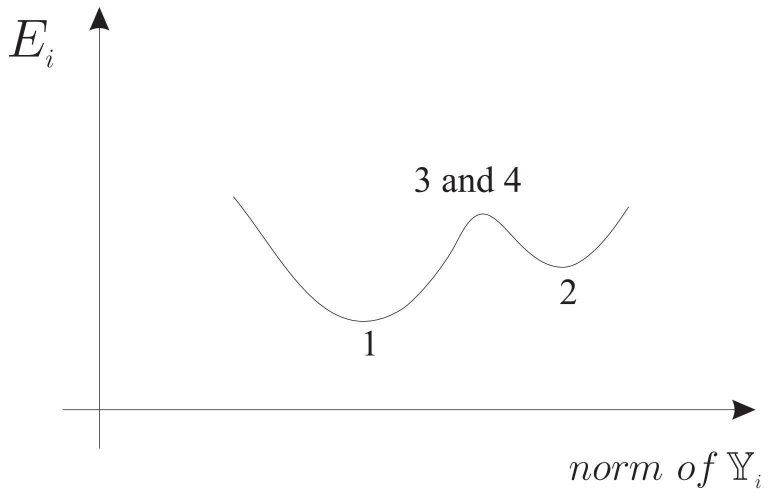


Figure 7: Schematic view of total potential energy of configurations 1-4

## 5 Conclusions

We use the procedure formulated in [13] and [14] to study two problems concerning the optimally shaped rod. In the first problem we determined the values of the restrictions on the minimal cross-sectional area that separate bimodal from unimodal optimal shapes. This value was known for $\alpha = 2$ in equation (7). We confirmed the value for $\alpha = 2$ by our procedure and determined the corresponding values for $\alpha = 1$ and $\alpha = 3$.

In the second problem we studied the post-critical behavior of the optimally shaped rod. We found that all four configurations existing at the bifurcation point continue to exist for larger loads (compared with the bifurcation load) increasing in amplitude. In all examples we determined the total potential energy according to (30) and obtained that the anti-symmetric configuration is preferred one (see Fig. 7), since the total potential energy in this configuration is minimal compared with other configurations. This is in qualitative agreement with the initial post-buckling behavior obtained by [9].

Note also that in all examples treated here the minimum of the total potential energy corresponds to the configuration with the largest displacement of the right support of the rod.

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## Bimodalna optimizacija sa vezama: kritična vrednost veze i posle-kritične konfiguracije

Koristeći metod baziran na Pontrjaginovom principu maksimuma, u obliku datom u radovima [13] i [14], odredili smo optimalni oblik elastičnog štapa sa ograničenjima na površinu poprečnog preseka. Odredili smo i minimalnu vrednost površine poprečnog preseka koja razdvaja unimodalnu od bimodalne optimizacije. Osim toga, proučavali smo i posle-kritični oblik (izvijeni oblik) optimalno oblikovanog štapa u najpovoljnijoj konfiguraciji, to jest u konfiguraciji u kojoj je ukupna energija štapa u minimumu.

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