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On some nonlinear inverse problems in elasticity

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Abstract

In this paper, we make a review of some inverse problems in elasticity, in statics and dynamics, in acoustics, thermoelasticity and viscoelasticity. Crack inverse problems have been solved in closed form, by considering a nonlinear variational equation provided by the reciprocity gap functional. This equation involves the unknown geometry of the crack and the boundary data. It results from the symmetry lost between current fields and adjoint fields which is related to their support. The nonlinear equation is solved step by step by considering linear inverse problems. The normal to the crack plane, then the crack plane and finally the geometry of the crack, defined by the support of the crack displacement discontinuity, are determined explicitly. We also consider the problem of a volumetric defect viewed as the perturbation of a material constant in elastic solids which satisfies the nonlinear Calderon's equation. The nonlinear problem reduces to two successive ones: a source inverse problem and a Volterra integral equation of the first kind. The first problem provides information on the inclusion geometry. The second one provides the magnitude of the perturbation. The geometry of the defect in the nonlinear case is obtained in closed form and compared to the linearized Calderon's solution. Both geometries, in linearized and nonlinear cases, are found to be the same.

Keywords. Nonlinear fracture mechanics, symmetry loss, material constants perturbation, defect geometry.

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1 Introduction

Inverse problems for crack and defect identification have been widely studied in the last decades. The first papers on this topics dealt with mathematical aspects of inverse problems such as the uniqueness of the solution, the number of data required for the inversion, the stability of numerical scheme, Ang et al [8], [9], Alessandrini [5], [6], [7], Colton and Monk [32], [34], Kohn and Vogelius [39], Kubo [40], Rondi [44] *etc.* Applications of inverse problems to crack and defect detections in Solids and Materials are important in Engineering Mechanics. An overlook of this topic can be found in Langenberg [41], Achenbach [1], Aki and Richards [4], Adler and Achenbach [3]. There are many applications in Medicine and in the mechanics of materials. In medicine, tomography techniques using mechanical loads such as an antiplane shear loading on life tissue, are worked out in Catheline et al [29]. Cancer tumors are expected to have a higher density and higher stiffness or shear modulus than sound tissues so that the difference of material property between sound and malicious tissues are detected by mechanical loads and responses. In the mechanics of materials, damage is known to result from micro-cracks which lower locally the elastic constants. New topics in mechanical tomography have been then the subjects of several works. For example, exact solutions to crack inverse problems in 2D and 3D are recently known in elasticity using mechanical loads, see Andrieux and Ben Abda [13], Andrieux et al [16], Bui et al [23], in acoustics in frequency domain Ben Abda et al [18], as well as in time domain [22], and in viscoelasticity Bui et al [26], in statics as well as in dynamics under the assumption of small frequency. In elastodynamics, solutions of inverse crack problems are obtained in [23] where the solution to an earthquake inverse problem to recover the faulting process was proposed. A review of several exact solutions to inverse problems is found in [24]. For example, the first explicit solution to an inverse acoustic scattering in an unbounded medium was given by Bojarski [20], the solution for a small perturbation of elastic constant in a bounded solid was discovered by Calderon [28].

Traditionally, numerical methods for solving inverse problems are based on the best fitting method, with the L^2-norm. One of the weaknesses of the best fitting method, particularly in the space-time domain, is that, according to Das and Suhadolc [33], there is no clear criterion or relationship between the *smallness* of the residual norm and the *goodness* of the numerical solution. They wrote in their paper "even if the fitting of data seems to be quite good, it would be difficult to know when one has obtained the correct solution". Here the new method deals with the so-called "reciprocity gap functional" which is shown in this paper to be a loss of symmetry in the equations. By exploiting these properties, a *variational* equation involving the unknown geometry of defects is worked out and solved step by step by considering suitable *sub-spaces* of adjoint functions.

The aim of this paper is to make a review of some closed form solutions to nonlinear inverse problems for a *bounded* solid in elasticity, acoustics, elastodynamic scatterings, thermoelasticity and viscoelasticity.

2 Symmetry lost and nonlinear variational equation

We first show how a variational equation involving the defect geometry can be derived in elastostatics and elastodynamics.

Consider an elastic solid Ω having a defect (crack, volumetric defect). The sound solid without defect is denoted by Ω^0. Both solids have the same external boundary denoted by S^{ext}. We assume the usual symmetry of elastic moduli. More precisely, we assume linear isotropic elasticity with Young modulus E and Poisson ratio ν. The symmetry between two systems of solutions in Ω^0 is known as the Betti-Somigliana theorem which states that

$$R := \int_{S^{ext}} (\boldsymbol{u}^1.\boldsymbol{T}(\boldsymbol{u}^2) - \boldsymbol{u}^2.\boldsymbol{T}(\boldsymbol{u}^1))dS = 0 \tag{1}$$

where $\boldsymbol{T}(\boldsymbol{u})$ is the stress vector on S^{ext}, $\boldsymbol{T}(\boldsymbol{u})=\sigma(\boldsymbol{u}).\boldsymbol{n}$. In the case where the actual displacement field $\boldsymbol{u}^1$ is discontinuous across the crack Σ, R is no longer equal to zero

$$R(\boldsymbol{u},\boldsymbol{v}) := \int_{S^{ext}} (\boldsymbol{u}.\boldsymbol{T}(\boldsymbol{v}) - \boldsymbol{v}.\boldsymbol{T}(\boldsymbol{u}))dS \neq 0 \tag{2}$$

for any adjoint field $\boldsymbol{v}$ continuous in Ω^0. The reciprocity gap R becomes a defect indicator: if the linear form R vanishes on every adjoint field v (i.e. if R is the null linear form) then there is no defect, conversely if the linear form R is non zero, it takes non zero values on some adjoint fields then there is certainly a defect inside Ω^0.

The property of R as a defect indicator allows the nonlinear inverse problem to be solved by exploiting the transition from non zero to zero values of the functional of $\boldsymbol{v}$. Better still, if subspaces of adjoint functions $\boldsymbol{v}$ depending on parameters of *finite* dimension can be used to determine the defect geometry, one then has a zero crossing method for a *function* of these parameters. In crack inverse problems, the non zero value of R is related to the crack geometry and crack displacement jump $[\![\boldsymbol{u}]\!]$

$$\int_{\Sigma} [\![\boldsymbol{u}]\!].\sigma(\boldsymbol{v}).\boldsymbol{n}dS = \int_{S^{ext}} (\boldsymbol{u}.\boldsymbol{T}(\boldsymbol{v}) - \boldsymbol{v}.\boldsymbol{T}(\boldsymbol{u}))dS := R(\boldsymbol{u}^d, \boldsymbol{T}^d, \boldsymbol{v}), \quad \forall \boldsymbol{v}, \tag{3}$$

In elastodynamics, the variational equation becomes

$$\begin{aligned}\int_0^\infty \int_{\Sigma} [\![\boldsymbol{u}]\!].\sigma(\boldsymbol{v}).\boldsymbol{n}dSdt &= \int_0^\infty \int_{S^{ext}} (\boldsymbol{u}^d.\boldsymbol{T}(\boldsymbol{v}) - \boldsymbol{v}.\boldsymbol{T}^d(\boldsymbol{u}))dSdt \\ &:= R(\boldsymbol{u}^d, \boldsymbol{T}^d, \boldsymbol{v}), \quad \forall \boldsymbol{v},\end{aligned} \tag{4}$$

where the boundary data $\boldsymbol{u}^d, \boldsymbol{T}^d$ are introduced in R and $\boldsymbol{v}$ is an adjoint field defined below.

Eq. (4) can be proved under various conditions, for example the following ones. Instead of the wave equation, we consider the regularized one, with vanishing positive number $\epsilon \to 0^+$ introduced for convergence purpose of the solution in Fourier's space (For example, the Heaviside function $Y(x) = 0$ for $x < 0$, $Y(x) = 1$ for $x > 0$ is replaced by the regularized one $Y^\epsilon(x) = exp(-\epsilon x)$ for $x > 0$ and $\epsilon = 0^+$)

$$\text{div}\sigma[\boldsymbol{u}] - \rho\partial_t\partial_t\boldsymbol{u} + \epsilon\partial_t\boldsymbol{u} = 0, \quad \text{in } (\Omega - \Sigma) \times [0, \infty], \tag{5}$$

$$\sigma[\boldsymbol{u}].\boldsymbol{n} = \boldsymbol{T}^d \quad \text{in } S^{ext}, \quad \sigma[\boldsymbol{u}].\boldsymbol{n} = 0, \quad \text{on the crack } \Sigma \tag{6}$$

The initial conditions $\boldsymbol{u} = 0$, $\partial_t\boldsymbol{u} = 0$ for $t \leq 0$ and the boundedness of $\|\boldsymbol{u}\|$, $\|\partial_t\boldsymbol{u}\|$ at infinite time are assumed. The adjoint field satisfies

$$\text{div}\sigma[\boldsymbol{v}] - \rho\partial_t\partial_t\boldsymbol{v} - \epsilon\partial_t\boldsymbol{v} = 0, \quad \text{in } \Omega^0 \times [-\infty, +\infty], \tag{7}$$

We shall consider the subspace of adjoint functions of exponential decay at large time, which includes functions vanishing for t greater than some T. More specific functions will be considered for determining the geometry of defects. The key point is that adjoint functions depend on N-dimensional parameters, $N = 1$ or 2. Variational Eqs. (3) and (4) are nonlinear in $\boldsymbol{u}$ and Σ. If the crack geometry Σ is known, these equations become *linear*. Therefore the key method of solution to crack inverse problems consists in determining first the crack plane, then the crack geometry by considering suitable adjoint functions, with different N and by identifying the displacement jump and then its support set.

3 The crack inverse problem in elastostatics

The problem has been solved by Andrieux et al [16]. Let us recall the main results of this paper which illustrate the notion of sub-variations which solves, step by step, a *nonlinear* inverse problem by considering simpler *linear* ones.

<u>The crack normal</u>

To determine the crack normal we consider the left hand side of Eq. (3) which is a linear combination of n_1, n_2, n_3. We choose a constant adjoint stress $\sigma(\boldsymbol{v})$ with the displacement field

$$v_k^{(ij)} = \frac{1}{2}(L^{-1})_{kh}^{mn}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})x_h, \tag{8}$$

where L is the Hooke tensor. Inserting Eq. (8) in Eq. (3) we get a linear system for $\boldsymbol{n}$

$$(\boldsymbol{n} \otimes \int_\Sigma [\![\boldsymbol{u}]\!] dS)_{ij}^{sym} = R^{ij} \equiv R(\boldsymbol{u}^d, \boldsymbol{T}^d; \boldsymbol{v}^{(ij)}) \tag{9}$$

where (sym) stands for the "symmetric part" and where coefficients depending on $[\![\boldsymbol{u}]\!]$ are yet unknown. However we have the following properties for tangential and normal components, by setting $Q = (\boldsymbol{n} \otimes \int_\Sigma [\![\boldsymbol{u}]\!] dS)_{ij}^{sym}$:

1. $\| \int_\Sigma [\![\boldsymbol{u}]\!] dS\| = \sqrt{2Q^2 - (TrQ)^2}$,

2. $\| \int_\Sigma [\![\boldsymbol{u}_t]\!] dS\| = \sqrt{2Q^2 - 2(TrQ)^2}$,

3. $\| \int_\Sigma [\![\boldsymbol{u}_n]\!] dS\| = tr(Q)$.

Assume that $\| \int_\Sigma [\![\boldsymbol{u}]\!] dS\| \neq 0$. Define the unit vector $U = \int_\Sigma [\![\boldsymbol{u}]\!] dS / \| \int_\Sigma [\![\boldsymbol{u}]\!] dS\|$ and consider the normalized $Q' = Q/\sqrt{2Q^2 - (TrQ)^2}$. Since $\boldsymbol{n}$ or $\boldsymbol{U}$ is parallel to one of the following vectors, $[\sqrt{\lambda_1}, \sqrt{-\lambda_2}, 0]$ and $[\sqrt{\lambda_1}, -\sqrt{-\lambda_2}, 0]$, in the basis of eigenvectors $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}, \boldsymbol{q}^{(3)}$ of Q', with eigenvalues $(\lambda_1, \lambda_2, 0)$, we consider two systems of loads (a) and (b). The normal $\boldsymbol{n}$ is then determined by the vector

$$\boldsymbol{n} = \boldsymbol{q}^{(3)(a)} \times \boldsymbol{q}^{(3)(b)} / \|\boldsymbol{q}^{(3)(a)} \times \boldsymbol{q}^{(3)(b)}\|. \tag{10}$$

The crack plane

Once the normal to the crack plane has been determined, we take Ox_3 along the normal and determine the constant C defining the crack plane by $x_3 - C = 0$. For this purpose we consider a quadratic adjoint field such that $\sigma_{31}(\boldsymbol{v}) = x_3 - d$, $\sigma_{32}(\boldsymbol{v}) = \sigma_{33}(\boldsymbol{v}) = 0$. This is a 1-dimensional subspace of adjoint field depending on the scalar d

$$v_1^{(d)} = -x_1^2/2E - \nu x_2^2/2E + (2+\nu)(x_3-d)^2/2E, \tag{11}$$

$$v_2^{(d)} = \nu x_1 x_2/E, \quad v_3^{(d)} = \nu x_1(x_3-d)/E$$

The left hand side of Eq. (3) is proportional to $(x_3 - d) = C - d$ for points in the crack plane and thus the reciprocity gap $R(d)$ considered as a function of d vanishes when it crosses the single zero $d = C$. A plot of function $R(d)$ reveals the constant C as its single zero $R(C) = 0$.

The crack geometry

To recover a planar crack in opening mode and sliding ones, the sub-space of adjoint fields $\boldsymbol{v}^{(k)}$ is necessarily 2-dimensional, parameterized by vector $\boldsymbol{k} = [k_1, k_2, 0]$. We introduce two complex vectors $Z_k = \boldsymbol{k} + i\|\boldsymbol{k}\|\boldsymbol{e}^3$ and $Z_k^* = \boldsymbol{k} - i\|\boldsymbol{k}\|\boldsymbol{e}^3$. The adjoint fields are of Calderon's type, for opening mode and sliding mode respectively:

$$v^+(\boldsymbol{x},\boldsymbol{k}) = \nabla_x exp(-iZ_k.\boldsymbol{x}) + \nabla_x exp(-iZ_k^*.\boldsymbol{x}) \tag{12}$$

$$v^-(\boldsymbol{x},\boldsymbol{k}) = \nabla_x exp(-iZ_k.\boldsymbol{x}) - \nabla_x exp(-iZ_k^*.\boldsymbol{x}) \tag{13}$$

In pure opening mode, with Eq. (3) for the adjoint field (12), we obtain

$$R(\boldsymbol{v}^+(\boldsymbol{k})) = \frac{2E\|k\|^2}{1+\nu} F_x[\![u_3]\!](\boldsymbol{k}) \tag{14}$$

where F_x is the spatial Fourier transform. Therefore, the crack opening displacement as well as the crack geometry Σ defined by the support of $[\![\boldsymbol{u_3}]\!]$ is explicitly determined by the inverse Fourier transform of a known function of $\boldsymbol{k}$

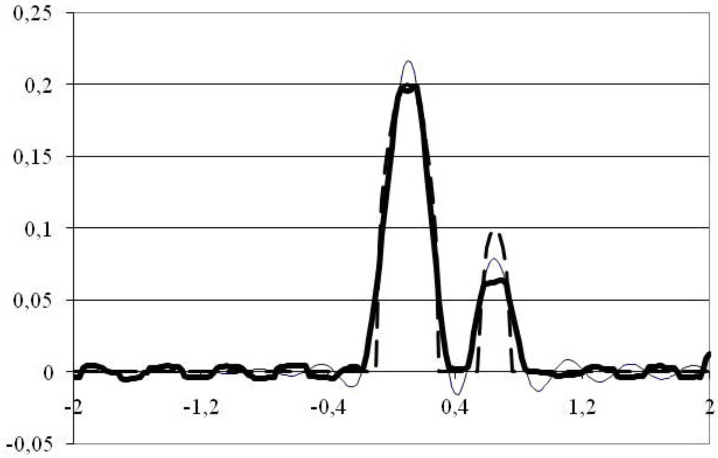

Figure 1: True displacement jump (dotted lines) in $[-0.1, 0.3] \cup [0.55, 0.75]$; 9 terms of Fourier's series (thin solid line); the regularized identified jump (bold solid line)

$$\Sigma = \mathrm{Supp} F_k^{-1}[R(\boldsymbol{v}^+(\boldsymbol{k}))\frac{1+\nu}{2E\|\boldsymbol{k}\|^2}](\boldsymbol{x}) \tag{15}$$

The method presented in this section is valid for a system of cracks lying in the same plane. Its application to the antiplane case is very simple since the only out of plane displacement component is $u_3(x_1, x_2)$ satisfying the harmonic equation in $\Omega - \Sigma$. The extended function $[\![\tilde{u}_3]\!]$ is identified by inverting a formula analogous to (14), with a 9 terms Fourier decomposition. The discontinuity function is finally regularized by the $Total\ Variation$ method [31] in order to smooth out the oscillating behavior, due to N=9 terms used in its representation by a truncated Fourier series. It can be seen that the accuracy of the reconstruction of the cracked domain is quite good, even for two near cracks, Fig. 1.

4 The crack inverse problem in thermoelasticity

An important extension of the previous result has been given in thermoelasticity by Andrieux and Bui [15] by adding thermal effects and including the heat equation in the description of the physics of the system. It also can pave the way to applications in NDT because we shall show that identification results can be derived without any information about the time dependent temperature field or thermal boundary quantities. Indeed, as a consequence of the results presented here, and provided that a thermal sollicitation is prescribed to an elastic solid free of mechanical loading or geometrical constraints, the measurement of surface displacements are sufficient to perform the identification of planar cracks lying inside the solid.

The thermoelastic constitutive equation for the isotropic solids is now, with α the linear dilatation coefficient and θ the temperature (I_2 and I_4 are unit second and forth orders respectively)

$$\sigma = L : (\epsilon - \theta I_2), \quad L = 3K I_2 \otimes I_2 + \frac{E}{1+\nu} I_4 \tag{16}$$

The reciprocity gap being still defined by Eq. (2), it is straightforward to derive the expression similar to Eq. (3) for R at time t and for adjoint fields v satisfying the *elastic* equilibrium equation

$$R(\boldsymbol{u}^d(t), \boldsymbol{T}^d, \boldsymbol{v}) = \int_\Sigma (L : \epsilon(\boldsymbol{v}).\boldsymbol{n}).[\![\boldsymbol{u}(t)]\!] dS + \int_\Omega 3K\alpha\theta(t)\mathrm{div}(\boldsymbol{v}) d\Omega \tag{17}$$

$$\int_\Omega (L : \epsilon(\boldsymbol{v})) : \epsilon(\boldsymbol{w}) d\Omega = \int_{S^{ext}} (L : \epsilon(\boldsymbol{v}).\boldsymbol{n}).[\![\boldsymbol{w}]\!] dS, \quad \forall \boldsymbol{w} \tag{18}$$

The identification of the crack(s) follows the same three steps as for the elastostatics case. The only difference relies on the divergence-free constraint put on the adjoint fields $\mathrm{div}\boldsymbol{v} = 0$ in order to cancel the second term of Eq. (17) which involves the unknown time dependent temperature field inside the whole domain.

<u>The crack normal</u>

Consider the following divergence-free displacement fields (for convenience, both vector $\boldsymbol{v}$ and transposed vector $\boldsymbol{v}^t$ are denoted by $[v_1, v_2, v_3]$)

$$\boldsymbol{v}^1 = [4x_1, -2x_2, -2x_3], \quad \boldsymbol{v}^2 = [-2x_1, 4x_2, -2x_3], \quad \boldsymbol{v}^3 = [-2x_1, -2x_2, 4x_3] \tag{19}$$

$$\boldsymbol{w} = [2x_2x_3, 2x_3x_1, 2x_1x_2]$$

and denote by $\tilde{Q}$ the deviatoric part of tensor Q

$$\tilde{Q} = \mathrm{dev}[(\boldsymbol{n} \otimes \int_{\Sigma} [\![\boldsymbol{u}]\!] dS)^{sym}] = (\boldsymbol{n} \otimes \int_{\Sigma} [\![\boldsymbol{u}]\!] dS)^{sym} - \frac{1}{3}\boldsymbol{n}. \int_{\Sigma} [\![\boldsymbol{u}]\!] dS \tag{20}$$

Then the components of $\tilde{Q}$ are calculated via the reciprocity gap:

$$\begin{aligned} \tilde{Q}_{ii} &= \frac{1}{12\mu} R(\boldsymbol{v}^d(t), \boldsymbol{T}^d(t), \boldsymbol{v}^i) \ \text{ no summation}, \\ \tilde{Q}_{ij} &= \frac{|\epsilon_{ijk}|}{8\mu} R(\boldsymbol{v}^d(t), \boldsymbol{T}^d, \partial_k \boldsymbol{w}), \ i \neq j \end{aligned} \tag{21}$$

Regarding the eigenvalues and eigenvectors of the deviatoric tensorial product $\tilde{Q}$, it can be established that there are only two possible cases. In the first one, there is a double eigenvalue and the associated eigenvector is the common direction of the displacement jump and the normal $\boldsymbol{n}$, the eigenvalue is exactly $\frac{2}{3}\int_{\Sigma} [\![\boldsymbol{u}]\!] dS$. In the second possible case, there are three distinct eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ and the normal vector $\boldsymbol{n}$ and mean displacement jump are given by one of the following formulae

$$\int_{\Sigma} [\![\boldsymbol{u}]\!] dS = -3\lambda_2 \boldsymbol{m}^1 + \frac{1}{2}\sqrt{\lambda_3^2 + \lambda_1^2 - \lambda_2^2 - \lambda_1\lambda_3}\ \boldsymbol{m}^2 \ \text{ and } \ \boldsymbol{n} = \boldsymbol{m}^1 \tag{22}$$

$$\int_{\Sigma} [\![\boldsymbol{u}]\!] dS = -3\lambda_2 \boldsymbol{m}^2 + \frac{1}{2}\sqrt{\lambda_3^2 + \lambda_1^2 - \lambda_2^2 - \lambda_1\lambda_3}\ \boldsymbol{m}^1 \ \text{ and } \ \boldsymbol{n} = \boldsymbol{m}^2 \tag{23}$$

where vectors $\boldsymbol{m}^i$ are calculated with the eigenvalues and eigenvectors $\boldsymbol{\nu}^1, \boldsymbol{\nu}^2, \boldsymbol{\nu}^3$

$$\boldsymbol{m}^1 = \frac{1}{\sqrt{2(\lambda_2 - \lambda_1)}}(\sqrt{\lambda_3 - \lambda_2 - \lambda_1}\ \boldsymbol{\nu}^1 - \sqrt{\lambda_3 + \lambda_2 - \lambda_1}\ \boldsymbol{\nu}^3) \tag{24}$$

$$\boldsymbol{m}^2 = \frac{1}{\sqrt{2(\lambda_2 - \lambda_1)}}(\sqrt{\lambda_3 + \lambda_2 - \lambda_1}\ \boldsymbol{\nu}^1 - \sqrt{\lambda_3 - \lambda_2 - \lambda_1}\ \boldsymbol{\nu}^3) \tag{25}$$

<u>The crack plane</u>

As in the elastostatic case, the scalar constant determining the affine plane $x_3 - C = 0$ containing the crack in the coordinate system with Ox_3 parallel to $\boldsymbol{n}$, is given by the reciprocity gap computed with a particular auxiliary field:

$$C = \frac{1}{6\mu \int_\Sigma [\![u_1]\!] dS} R(\boldsymbol{u}^d(t), \boldsymbol{T}^d(t), \boldsymbol{v}), \quad \boldsymbol{v} = [3(x_3^2 - x_2^2), 0, 0] \tag{26}$$

<ins>The crack geometry</ins>

The last step consists in identifying the normal displacement jump function continued by zero on a rectangle $\Pi = [0, L_1] \times [0, L_2]$ containing the intersection of the plane crack and the solid. It can again be proved that the support of this function is exactly the cracked domain (up to a zero measure set). For that purpose, let us define the following divergence-free adjoint fields family, where the components are harmonic functions ($\lambda_{mn} = \sqrt{m^2 + n^2}$)

$$v_{\alpha\beta}^{mn}(x_1, x_2, x_3) = [f_{\alpha\beta}^{mn}(x_1, x_2) cosh(\lambda_{mn} x_3), 0, \frac{1}{\lambda_{mn}} f_{\alpha\beta,1}^{mn}(x_1, x_2) sinh(\lambda_{mn} x_3)] \tag{27}$$

where partial derivative of $f_{\alpha\beta}^{mn}$ with respect to x_1 is denoted by $f_{\alpha\beta,1}^{mn}$ and functions $f_{\alpha\beta}^{mn}$ are

$$f_{ss}^{mn} = sin\frac{m\pi x_1}{L_1} sin\frac{n\pi x_2}{L_2}, \quad f_{cc}^{mn} = cos\frac{m\pi x_1}{L_1} cos\frac{n\pi x_2}{L_2}$$
$$f_{sc}^{mn} = sin\frac{m\pi x_1}{L_1} cos\frac{n\pi x_2}{L_2}, \quad f_{cs}^{mn} = cos\frac{m\pi x_1}{L_1} sin\frac{n\pi x_2}{L_2}$$

Denoting by $[\![\tilde{u}_3(t)]\!]$ the extension to zero of the normal displacement jump to the rectangle Π, we obtain the reciprocity gap on the fields of this family

$$\begin{aligned} R(\boldsymbol{u}^d(t), \boldsymbol{T}^d(t), v_{\alpha\beta}^{mn}) &= 2\mu \int_\Sigma \epsilon(v_{\alpha\beta}^{mn}) : (\boldsymbol{n} \otimes [\![\boldsymbol{u}(t)]\!])^{sym} dS \\ &= -2\mu \int_\Pi [\![\tilde{u}_3(t)]\!] f_{\alpha\beta,1}^{mn} dS \end{aligned} \tag{28}$$

It is readily seen that the double Fourier series terms of the function can be computed by using the reciprocity gap on fields $v_{\alpha\beta}^{mn}$, except the constant term that is given by the identification of the mean value of $[\![\boldsymbol{u}]\!]$ when determining the normal of the crack.

Finally, let us mention that inverse crack problems for the transient heat equation, with the boundary measurements of the temperature and the normal flux, have been studied in the paper [19].

5 The inverse elastic scattering by a planar crack

We wish to determine the crack by studying the scattering of elastic waves in a *bounded* elastic solid, due to either a stress free crack or the release of stress by a shear slip on a crack, like what is observed in an earthquake. The first case is described by the variational equation (4). In the second case, the reciprocity gap is defined by the integral over the external surface

$$R(\boldsymbol{u}^d, \boldsymbol{T}^d, \boldsymbol{v}) = \int_0^\infty \int_{S^{ext}} (\boldsymbol{u}^d . \boldsymbol{T}(\boldsymbol{v}) - \boldsymbol{v} . \boldsymbol{T}^d(\boldsymbol{u})) dSdt \tag{29}$$

It is equal to the double integral over times and $\Sigma^\pm$ (or Σ^- with displacement jump)

$$R(\boldsymbol{v}) = \int_0^\infty \int_{\Sigma^-} [\![\boldsymbol{u}]\!] . \boldsymbol{T}(\boldsymbol{v}) dSdt - \int_0^\infty \int_{Crack} (\boldsymbol{v} . \boldsymbol{T}^+(\boldsymbol{u}) + \boldsymbol{v} . \boldsymbol{T}^-(\boldsymbol{u})) dSdt \tag{30}$$

The last integral vanishes because stress vectors are opposite together $\boldsymbol{T}^+ + \boldsymbol{T}^- = 0$. Therefore Eq. (4) holds in both cases, Bui *et al.* [21]. In the case of release of stress on the unknown crack, under stress free condition on the external surface, the data are the accelerations of points on S_{ext}, from which $\boldsymbol{u}(t)$ can be calculated on the external boundary. We have $R(\boldsymbol{v}) = \int_0^\infty \int_{\Sigma^-} [\![\boldsymbol{u}]\!] . \boldsymbol{T}(\boldsymbol{v}) dSdt$

To determine the crack plane in the sliding mode (no stress $\boldsymbol{T}(\boldsymbol{u})$ on the external boundary), a zero crossing method is used with an instantaneous reciprocity gap functional [21], defined by the adjoint wave (with c_s being the shear wave velocity), $\boldsymbol{v}(\boldsymbol{x}, t; \tau) = \boldsymbol{k} H(t - \boldsymbol{x}.\boldsymbol{p}/c_s - \tau)$, where $H(y)$ is the *down* step function, $H(y) = 0$, for $y > 0$, $H(y) = 1$, for $y < 0$, τ is a parameter chosen for characterising the initial wave front, $\boldsymbol{p}$ is the propagation vector directed towards the perturbed zone (back propagation). At time $t = 0$ the front S_2 is defined by $\boldsymbol{x}.\boldsymbol{p}/c_s + \tau = 0$. The only non zero adjoint stress is $\sigma(\boldsymbol{v}) = -(\mu/c_s)(\boldsymbol{k} \otimes \boldsymbol{p} + \boldsymbol{p} \otimes \boldsymbol{k}) \delta(t - \boldsymbol{x}.\boldsymbol{p}/c_s - \tau)$. We have in 2D, $\boldsymbol{k} = \boldsymbol{e}^3$, $R(\boldsymbol{v})$ $= \int_0^\infty \int_{S^{ext}} \boldsymbol{u} . \boldsymbol{T}(\boldsymbol{v}) dSdt = -(\mu/c_s)(u_3(A) n_p(A) + u_3(B) n_p(B))$.

As shown in Fig. 2 at time $t \geq 0$ the adjoint wave front propagates backwards and cannot meet the crack. According to the second expression of $R(\boldsymbol{v}) = \int_0^\infty \int_{\Sigma^-} [\![\boldsymbol{u}]\!] . \boldsymbol{T}(\boldsymbol{v}) dSdt$, in terms of the inner boundary Σ, the supports of $[\![\boldsymbol{u}]\!]$ and $\boldsymbol{T}(\boldsymbol{v})$ being disjoint sets for any time $t \geq 0$, the reciprocity gap vanishes identically. By changing τ and $\boldsymbol{p}$ so that the initial front has an intersection with the crack, we obtain a non zero value of R. By this way, we can even determine the geometry of a *convex* planar crack from the *exterior*

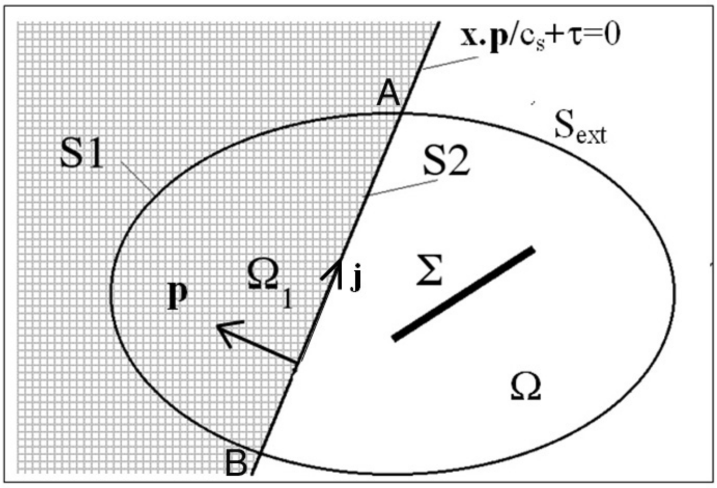

Figure 2: Back propagation of adjoint wave. Constant displacement behind the front Γ_t, null displacement in front of Γ_t. Initial front $\Gamma_0 = S_2$ defined by $\boldsymbol{x}.\boldsymbol{p}/c_s + \tau = 0$

simply by checking the value of R. Fig. 3 shows the numerical result for an antiplane problem, with the convex hull containing the sliding crack obtained by different values of τ and $\boldsymbol{p}$. The transition between zero value and non zero value of R is detected by fixing a threshold value. Remark that if $\boldsymbol{p}$ is parallel to the crack, we have $R = 0$ even when the initial front S_2 has an intersection with Σ. This means that, we quote Alves and Haduong [11], the adjoint wave does not "see" the crack.

The zero crossing method for determining (numerically) the crack plane is not suitable for studying a concave shaped crack. To determine a more general crack (concave shaped crack, moving crack $\Sigma(t)$) we have many possible methods. For example, we consider adjoint waves of the form $\boldsymbol{v} = \mathrm{grad}\phi(\boldsymbol{x}, t) + \mathrm{curl}[\psi(\boldsymbol{x}, t)\boldsymbol{e}^3]$ and determine directly the crack displacement jump which corresponds to the true crack. In what follows, we assume that the crack plane is $x_3 = 0$ and consider only solenoidal adjoint field depending on a 2-dimensional parameter $\boldsymbol{s} = [s_1, s_2, 0]$ for space and a scalar parameter q for time dependence (ϵ being a vanishing positive number $\epsilon = 0^+$). By this

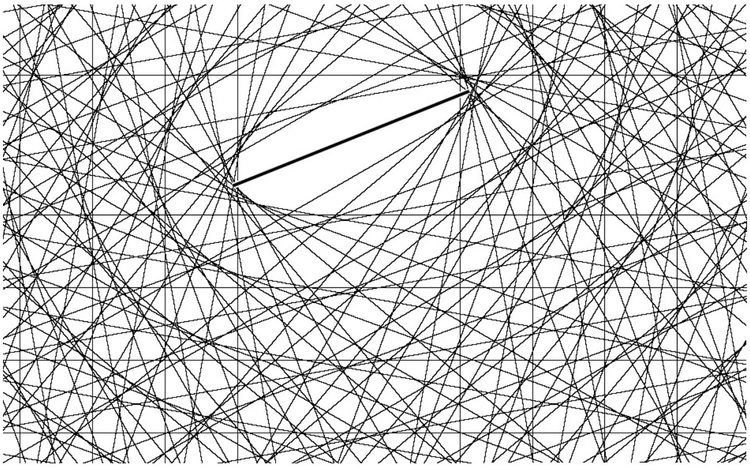

Figure 3: The convex hull of initial fronts $\Gamma_0(\tau, \boldsymbol{p})$ not intersecting the crack

way we determine the crack geometry partially,

$$\boldsymbol{v}^{(\boldsymbol{s},q)}(\boldsymbol{x},t) = \mathrm{curl}[\psi(\boldsymbol{x},t;\boldsymbol{s},q)\boldsymbol{e}^3] \tag{31}$$

$$\psi(\boldsymbol{x},t;\boldsymbol{s},q) = exp(iqt - \epsilon t)exp[x_3(\|s\|^2 + (iq-\epsilon)^2/c_s^2)^{1/2}]exp(i\boldsymbol{s}.\boldsymbol{x}) \tag{32}$$

Eq. (4) can be written as

$$\int_0^\infty \int_{R^2} \mu(is_2[\![u_1]\!] - is_1[\![u_2]\!])exp((iq-\epsilon)t)exp(i\boldsymbol{s}.\boldsymbol{x})dSdt$$
$$= \frac{R(\boldsymbol{s},q)}{[\|s\|^2 + (iq-\epsilon)^2/c_s^2]^{1/2}} \tag{33}$$

where the second integral is taken over the *whole* crack plane $x_3 = 0$ since the displacement jump vanishes outside the crack.

We introduce the vector $[\![\boldsymbol{u}]\!]^\perp = [[\![u_2]\!], -[\![u_1]\!], 0]$ orthogonal to $[\![\boldsymbol{u}]\!]$. We see that the left hand side Eq. (33) is the double time Fourier transform and space Fourier transform of $-\mu div([\![\boldsymbol{u}]\!]^\perp)$. Therefore, owing to $\epsilon = 0^+$ strictly positive, by inverse space and time Fourier transforms of the above equation we obtain:

$$\mathrm{div}([\![\boldsymbol{u}]\!]^\perp)(\boldsymbol{x},t) = -\frac{1}{\mu}(F_t)^{-1}(F_x)^{-1}R(\boldsymbol{v}^{(\boldsymbol{s},q)})[\|s\|^2 + (iq-\epsilon)^2/c_s^2]^{-1/2} \tag{34}$$

We have obtained Supp div($[\![\boldsymbol{u}]\!]^{\perp}$)$\subset \Sigma$. If the supports of function div($[\![\boldsymbol{u}]\!]^{\perp}$)$(\boldsymbol{x},t)$ and function $[\![\boldsymbol{u}_t]\!](\boldsymbol{x},t)$ are the same, we obtain explicitly the geometry of the moving crack by

$$\Sigma(t) = \mathrm{Supp}[-\frac{1}{\mu}(F_t)^{-1}(F_x)^{-1}R(\boldsymbol{v}^{(\boldsymbol{s},q)})[\|s\|^2 + (iq-\epsilon)^2/c_s^2]^{-1/2}] \qquad (35)$$

Actually, we can get explicitly the support of each component of the displacement jump by considering different adjoint fields and then the crack by $\Sigma = \mathrm{Supp}[\![u_1]\!] \cup \mathrm{Supp}[\![u_2]\!]$.

To obtain each component of the crack displacement jump, for example $[\![u_2]\!]$, we consider a 1-dimensional parameter $\boldsymbol{s} = [s_1, 0, 0]$ for the Fourier spatial variable and calculate $\boldsymbol{v}^{(s_1,q)} = \mathrm{curl}[\psi(\boldsymbol{x},t)\boldsymbol{e}^3]$ with $\psi = exp(iqt-\epsilon)exp(x_3(s_1^2 + (iq-\epsilon)^2/c_s^2)^{1/2})exp(is_1x_1)$. We obtain the equation which provides the Fourier transform in space and time of $\partial[\![u_2]\!]/\partial x_1$, and thus the jump $[\![u_2]\!]$ by using the null boundary condition on the crack front

$$\int_0^{\infty}\int_{R^2}\mu(-is_1[\![u_2]\!])exp((iq-\epsilon)t)exp(is_1x_1)dSdt = \frac{R(s_1,q)}{[s_1^2+(iq-\epsilon)^2/c_s^2]^{1/2}} \quad (36)$$

Similarly, with an adjoint function $\boldsymbol{v}^{(s_2,q)}$ parametrized by $\boldsymbol{s} = [0, s_2, 0]$ and $\psi = exp(iqt-\epsilon)exp(x_3(s_2^2 + (iq-\epsilon)^2/c_s^2)^{1/2})exp(is_2x_2)$, we obtain ($\epsilon = 0^+$)

$$\int_0^{\infty}\int_{R^2}\mu(is_2[\![u_1]\!])exp((iq-\epsilon)t)exp(is_2x_2)dSdt = \frac{R(s_2,q)}{[s_2^2+(iq-\epsilon)^2/c_s^2]^{1/2}} \quad (37)$$

which provides the Fourier transform in space and time of $-\partial[\![u_1]\!]/\partial x_2$.
Remark that also $\mathrm{Supp}(\partial[\![u_1]\!]/\partial x_2) = \mathrm{Supp}([\![u_1]\!])$ because of the boundary condition $[\![u_1]\!] = 0$ on the crack front. Thus
Supp div($[\![\boldsymbol{u}]\!]^{\perp}$) $= \mathrm{Supp}[\![u_1]\!] \cup \mathrm{Supp}[\![u_2]\!] = \Sigma$.

To the authors's knowledge, traditional methods of minimization of the residuals to solve crack inverse problems are restricted to a stationary crack. They are unable to provide the solution for a moving crack. The symmetry lost method with the reciprocity gap functional provides us a variational equation to determine the solution for a moving crack analytically, from data defined by the reciprocity gap $R(\mathrm{data}; v)$.

6 Inverse acoustic scattering by a crack in time domain

Most works in this topic have been done in frequency domain and for an unbounded medium, see for example, Bojarski [20], Colton and Monk [11], Alves and Ha Duong [11], Ben Abda et al [18]. The Reciprocity gap functional method provides us a very simple means to study acoustic scattering in time domain for a bounded solid. The notations are similiar to those of the previous Section. The current field satisfies

$$(\partial_t\partial_t - \text{div grad} - \epsilon\partial_t)u = 0 \quad \text{in } \Omega \times [0, \infty[\tag{38}$$

$$u(\boldsymbol{x}, t < 0) = 0, \quad \partial_t u(\boldsymbol{x}, t < 0) = 0 \tag{39}$$

We assume a good behavior of u at infinite time $t^2|u| \to 0$ and $t^2|\partial_t u| \to 0$ and assume that u and $\partial_n u$ are known on the external boundary. The adjoint field satisfies

$$(\partial_t\partial_t - \text{div grad} + \epsilon\partial_t)v = 0 \quad \text{in } \Omega^0 \times [0, \infty[\tag{40}$$

We obtain the variational equation

$$\int_0^\infty \int_\Sigma [\![u]\!] \partial_n dSdt = \int_0^\infty \int_{S^{ext}} (u\partial_n v - v\partial_n u)dSdt := R(v) \quad \text{for any } v \tag{41}$$

Now we take an adjoint plane wave of propagation vector $\boldsymbol{k}$ of the form $v^{(k)}(\boldsymbol{x}, t) = g(\boldsymbol{x}.\boldsymbol{k} + t)$ so that the reciprocity gap depends on $\boldsymbol{k}$. The integral in the left hand side of Eq. (41) is proportional to $\boldsymbol{n}.\boldsymbol{k}$. Therefore the zeros value of the right hand side $R(\boldsymbol{k})$ of Eq. (41) corresponds to vector $\boldsymbol{k}$ parallel to the crack plane, $\boldsymbol{n}.\boldsymbol{k} = 0$. Two independent propagation vectors so that $R(\boldsymbol{k})$ vanishes, gives the normal to the crack plane as $\boldsymbol{n} = \boldsymbol{k}^1 \times \boldsymbol{k}^2 / \|\boldsymbol{k}^1 \times \boldsymbol{k}^2\|$. We then take Ox_3 along the normal direction and determine the crack plane $x_3 - b = 0$ by considering the adjoint wave $v^{(b)}(\boldsymbol{x}, t) = (x_3 - b)^2 + (t - T/2)^2$. The reciprocity gap which depends on b, is proportional to $x_3 - b$ as shown by its integral expression over the crack surface. It vanishes when $x_3 - b = 0$. Finally by studying the zero of $R(b)$ we detect the position b of the crack plane by $R(b) = 0$. This result is similar to the one given in Alves and Ha Duong [11] who considered $v^{(b)}(\boldsymbol{x}, t)$ as an *analysing waves*. When the wave is parallel to the crack plane, it does not *see* the crack. The difference with our work is that we are dealing here with a bounded domain, while Alves and Ha Duong [11] considered an infinite medium.

7 Solution of the Calderon's problem for the geometry of a volumetric defect

In a famous paper, Calderon [28] considered the following inverse problem for determining the perturbation $h(\boldsymbol{x})$ of the material constant from boundary data.

$$\mathrm{div}(1 + h(\boldsymbol{x}))\mathrm{grad}u = 0 \quad \text{in } \Omega \tag{42}$$

$$u(\boldsymbol{x}) = f \quad \text{on } \partial\Omega, \quad \partial_n u = g \quad \text{on } \partial\Omega \tag{43}$$

Eq. (42) can be considered as the elastic equilibrium in antiplane mode. Introduce the adjoint harmonic equation,

$$\mathrm{divgrad}\ v = 0 \quad \text{in } \Omega \tag{44}$$

to obtain the variational equation

$$\int_\Omega h(\boldsymbol{x})\mathrm{grad}\ u(\boldsymbol{x}, h)\boldsymbol{.}\mathrm{grad}\ v(\boldsymbol{x})dV = R(v) \quad \text{for any } v \tag{45}$$

where reference to the boundary data is omitted in the reciprocity gap

$$R(v) = \int_{\partial\Omega} (vg - f\partial_n v)dS \tag{46}$$

Eq. (45) can be solved if the adjoint field is parameterized by a N-dimensional vector $\boldsymbol{\xi}$, N=2 in 2D, N=3 in 3D cases. It becomes a Fredholm integral equation

$$\int_\Omega h(\boldsymbol{x})\mathrm{grad}\ u(\boldsymbol{x}, h)\boldsymbol{.}\mathrm{grad}\ v^{(\xi)}(\boldsymbol{x})dV = R(v^{(\xi)}) \tag{47}$$

Calderon (1980) solved Eq. (47) in the case of small perturbation $h \ll 1$. The linearized equation is obtained from Eq. (47) by the substitution $u(\boldsymbol{x}, h) \rightarrow u(\boldsymbol{x}, 0)$ which is harmonic. By considering a particular loading corresponding to function u and adjoint function v of the Calderon type $exp(-i(\boldsymbol{\xi} + i\boldsymbol{\xi}^\perp))$, in the 2D case, he got the exact solution

$$h^{(0)}(\boldsymbol{x}) = -\frac{1}{4\pi^2}\int_{R^2} \frac{2R(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^2} exp(i\boldsymbol{x}\boldsymbol{.}\boldsymbol{\xi})d^2\boldsymbol{\xi} \tag{48}$$

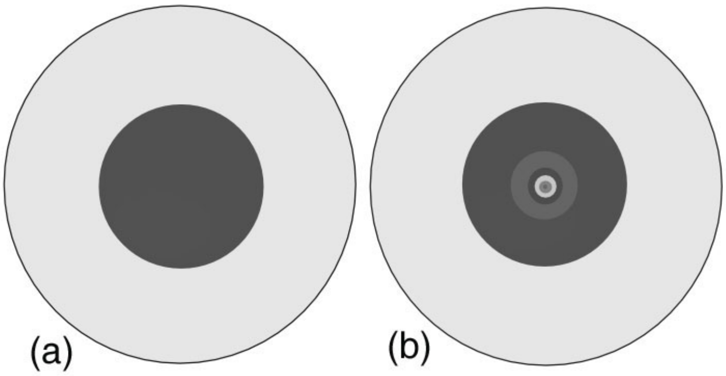

Figure 4: (a) Original image of a constant perturbation, not necessarily small; (b) The image calculated from Calderon's formula. The intensity of the reconstructed perturbation differs noticeably from the original image while the geometries are identical

The question on the validity of a linearized approximation was raised by Isaacson and Isaacson [37]. They solved numerically the nonlinear problem for the axisymmetric case by comparing their solution with Calderon's formula Eq. (48). Surprisingly, they got the same geometry for the defect, while the amplitude of the solutions in both linear and nonlinear cases are different, Fig. 4

The question about the ability of Calderon's formula to predict the geometry of the defect has been considered in [27] for the general case of geometry and loadings. It is very important for applications to know if a linearized theory can be used for determining exactly the geometry of defects, because we have only to solve a *linear* inverse problem to determine the magnitude of the perturbation. Let us make first the following remarks. We set $S(\boldsymbol{x}) = \mathrm{div}(h\mathrm{grad}u)$. Eq. (42) can be written as

$$\mathrm{divgrad}\; u + S(\boldsymbol{x}) = 0 \quad \text{in } \Omega \tag{49}$$

with the same boundary data (f, g).

The support of function $S(\boldsymbol{x}) = \mathrm{div}(h\mathrm{grad}u)$, which is related to h and u can be obtained by solving the source inverse problem and do not require any assumption on the smallness of h. One expects that the supports of h in linearized and nonlinear theories are the same because they are linked to the same source S.

Consider the adjoint function $v^{(\xi)}(\boldsymbol{x}) = exp(-i(x_1\xi_1 + x_2\xi_2))exp(-x_1\xi_2 + x_2\xi_1)$

This adjoint function as well as its gradient grad $v^{(\xi)}(\boldsymbol{x})$ are analytic in the whole x-space and ξ-space (except at infinity) and thus can be expanded into infinite series of x_r and ξ_h. We expand grad $v^{(\xi)}$ as

$$\text{grad } v^{(\xi)}(\boldsymbol{x}) = [\sum_{h,k,r,s=1}^{2} \sum_{n,m,p,q=0}^{\infty} \boldsymbol{a}_{nmpq}^{hkrs}(i\xi_h)^n(i\xi_k)^m x_r^p x_s^q] exp(-i(\boldsymbol{x}.\boldsymbol{\xi})), \quad (50)$$

with constant complex vectors $\boldsymbol{a}_{nmpq}^{hkrs}$. We extend $h(\boldsymbol{x})$ to the infinite plane $\mathbb{R}^2$ by putting $h = 0$ outside C and denote its extension by $\tilde{h}$ and obtain the nonlinear Calderon equation in the form (the dot means scalar product between vectors)

$$\int_{\mathbb{R}^2} \widetilde{h}(\boldsymbol{x})\text{grad}U(\boldsymbol{x}).[\sum_{h,k,r,s=1}^{2} \sum_{n,m,p,q=0}^{\infty} \boldsymbol{a}_{nmpq}^{hkrs}(i\xi_h)^n(i\xi_k)^m x_r^p x_s^q] exp(-i\boldsymbol{x}.\boldsymbol{\xi})d^2x$$
$$= R(\xi), \quad (51)$$

which is equivalent, in the Fourier's transform context, to

$$\int_{\mathbb{R}^2} \sum_{h,k,r,s=1}^{2} \sum_{n,m,p,q=0}^{\infty} \boldsymbol{a}_{nmpq}^{hkrs}. \frac{\partial^n}{\partial x_h^n}\frac{\partial^m}{\partial x_k^m}[x_r^p x_s^q \widetilde{h}(\boldsymbol{x})\text{grad}U(\boldsymbol{x})] exp(-i\boldsymbol{x}.\boldsymbol{\xi})d^2x$$
$$= R(\xi). \quad (52)$$

where $U = u(\boldsymbol{x}; h)$ is yet unknown. Let the function appearing in the above series be

$$F(\boldsymbol{x}) = \sum_{h,k,r,s=1}^{2} \sum_{n,m,p,q=0}^{\infty} \boldsymbol{a}_{nmpq}^{hkrs}. \frac{\partial^n}{\partial x_h^n}\frac{\partial^m}{\partial x_k^m}[x_r^p x_s^q \tilde{h}(\boldsymbol{x})\text{grad}U(\boldsymbol{x})], \quad (53)$$

$$\int_{\mathbb{R}^2} F(\boldsymbol{x})exp(-i\boldsymbol{x}.\boldsymbol{\xi})d^2x = R(\boldsymbol{\xi}). \quad (54)$$

It follows that function $F(\boldsymbol{x})$ is the inverse Fourier transform of $R(\boldsymbol{\xi})$.

$$F(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} R(\boldsymbol{\xi})exp(+i\boldsymbol{x}.\boldsymbol{\xi})d^2\xi. \tag{55}$$

Function F is a linear combination of h and its partial derivatives, denoted hereafter by $F[h]$. Now we compare the solution $\mathrm{Supp}(F)$ with the linearized one given by Calderon [28], Eq.(48), which can be written differently as

$$-\frac{1}{2}(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})h^0(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} R(\boldsymbol{\xi})exp(+i\boldsymbol{x}.\boldsymbol{\xi})d^2\xi \equiv F[h](\boldsymbol{x}) \tag{56}$$

The Laplacian of h^0 is identical to $-2F[h]$. Therefore we have the same support $C^0 = \mathrm{Supp}(h^0) \equiv \mathrm{Supp}(h) = C$, because otherwise, for example in an open set $D \subset C$ but $D \not\subset C^0$, we have $F[h] \neq 0$ and $-\Delta h^0 = 0$. The latter equality conflicts with the identity $-\Delta h^0 \equiv 2F[h] \equiv 0$ in D. The same contradiction exists for $D \subset C^0$ but $D \not\subset C$. We conclude that both linearized and nonlinear theories provide the same geometry of defect $C \equiv C^0$.

8 Inverse problems in viscoelasticity

Tomographies techniques, which avoid X-ray, using mechanical loads such as antiplane shear loading on life tissue, considered as a viscoelastic medium, have been worked out for Kelvin-Voigt's viscoelasticity (Catheline et al [29], Muller et al [43]. In a 1-dimensional model, the rheological Kelvin-Voigt's model is characterized by a block consisting of an elastic spring in *parallel* with a dashpot shown in Fig. 5(b). Let us consider the Zener model which adds another elastic spring in *series* with the Kelvin-Voigt's block Fig. 5(c).

Mathematically, formulations of 3D viscoelasticity by Boltzmann functional of stress and strain with relaxation functions $\lambda(t)$ and $\mu(t)$ or by complex elastic moduli are not suitable for studying inverse crack and defect problems. We consider rather the differential approach of the Zener law which corresponds to exponential relaxation functions

$$\sigma + \beta\dot{\sigma} = L : (\epsilon + \alpha\dot{\epsilon}) \tag{57}$$

Coefficients α and β are characteristic times related to the spring stiffnesses k_0 and k_1 and the dashpot viscosity η by $\alpha = \eta/k_1$ and $\beta = \eta/(k_0 + k_1)$. We

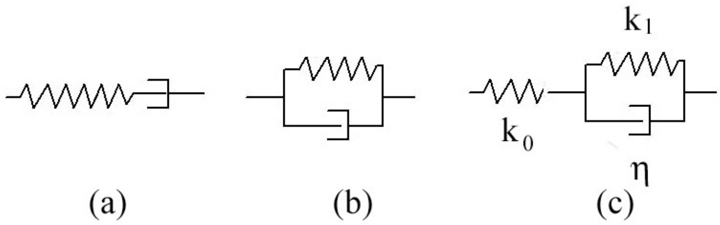

Figure 5: Viscoelastic models: (a) Maxwell; (b) Kelvin-Voigt; (c) Zener

consider transformed displacement, strain and stress variables introduced by Goryacheva [35]

$$\boldsymbol{u}^* = u + \alpha \frac{\partial \boldsymbol{u}}{\partial t}, \quad \epsilon^* = \epsilon + \alpha \frac{\partial \epsilon}{\partial t} \tag{58}$$

$$\sigma^* = \sigma + \beta \frac{\partial \sigma}{\partial t}. \tag{59}$$

The relationship between star fields is the same as in elasticity

$$\sigma^* = L : \epsilon^* \tag{60}$$

Moreover, for small out of phase $\theta = (\alpha - \beta)\omega \ll 1$ between stress and strain and for $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{w}(\boldsymbol{x}) cos(\omega t)$, the equation of motion in the frequency domain can be written as, Chaillat and Bui [30]

$$\mathrm{div}\sigma^* - \rho \partial_t \partial_t \boldsymbol{u} = \rho(\beta - \alpha)\partial_t \partial_t \partial_t \boldsymbol{u} \simeq \rho\omega^3 |\alpha - \beta| \|v\| \tag{61}$$

The latter term can be neglected in comparison with the second one $-\rho \partial_t \partial_t \boldsymbol{u} = \rho\omega^2 \|\boldsymbol{w}\|$ if and only if (this corresponds again to the assumption on small out of phase):

$$\theta = |\alpha - \beta|\omega \ll 1 \tag{62}$$

Finally, under the assumption of small frequency $\omega \ll \frac{1}{|\alpha-\beta|}$, the star fields satisfy the elastodynamic equations in the frequency domain, $\sigma^* = L : \epsilon^*$ and $\mathrm{div}\sigma^* + \rho\omega^2 \boldsymbol{w} \simeq 0$.

Applications of the equivalence between elasticity and viscoelasticity have been exploited in [26] for studying crack inverse problems in viscoelasticity and in [25] for identifying volumic defect.

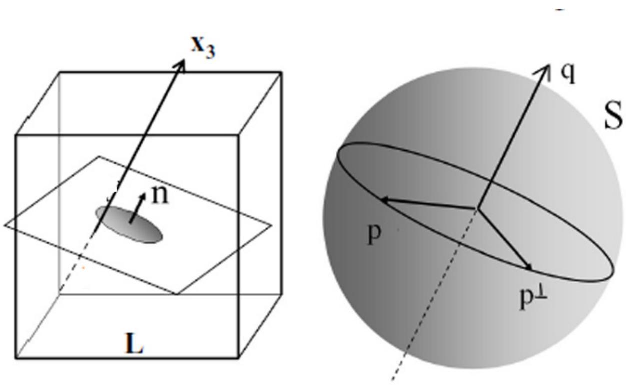

Figure 6: The sphere of radius $k/\sqrt{\mu}$; parameters $\boldsymbol{p}$ and $\boldsymbol{p}^{\perp}$ along the equator and $\boldsymbol{n}$ along the poles axis

8.1 Inverse crack problem

The current field satisfies the equation ($k^2 = \rho\omega^2$)

$$\mu \text{div grad} \boldsymbol{u}^* + (\lambda + \mu) \text{grad div} \boldsymbol{u}^* + k^2 \boldsymbol{u}^* = 0, \quad \text{in } \Omega. \tag{63}$$

The adjoint function satisfies the same equation

$$\mu \text{div grad} \boldsymbol{v}^* + (\lambda + \mu) \text{grad div} \boldsymbol{v}^* + k^2 \boldsymbol{v}^* = 0 \quad \text{in } \Omega^0. \tag{64}$$

The variational equation with the reciprocity gap R has the same form as in elasticity

$$\begin{aligned} \int_{\Sigma} [\![\boldsymbol{u}^*]\!] . \sigma(\boldsymbol{v}^*) . \boldsymbol{n} dS = & \int_{S^{ext}} (\boldsymbol{u}^* . \boldsymbol{T}(\boldsymbol{v}^*) - \boldsymbol{v}^* . \boldsymbol{T}(\boldsymbol{u}^*)) dS \\ & := R(\boldsymbol{u}^{*d}, \boldsymbol{T}^{*d}, \boldsymbol{v}^*), \quad \forall \boldsymbol{v}^*. \end{aligned} \tag{65}$$

We summarise the results of [26].

<u>The crack normal</u>

Consider an adjoint S-wave depending on two orthogonal vectors $\boldsymbol{p}$ and $\boldsymbol{p}^{\perp}$ of equal norm $k/\sqrt{\mu}$ of the form $\boldsymbol{v}^{(p,p^{\perp})} = sin(\boldsymbol{x}.\boldsymbol{p}^{\perp})\boldsymbol{p}$. The variational equation provides

$$\mu[(\boldsymbol{p}^{\perp}.\boldsymbol{n} p_i + \boldsymbol{p}.\boldsymbol{n} p_i^{\perp}] \int_{\Sigma} [\![u_i^*]\!] cos(\boldsymbol{x}.\boldsymbol{p}) dS = R(\boldsymbol{p}, \boldsymbol{p}^{\perp}) \tag{66}$$

The left hand side of Eq. (66) shows that R vanishes when parameters $\boldsymbol{p}, \boldsymbol{p}^{\perp}$ are orthogonal to the normal $\boldsymbol{n}$. Geometrically, R vanishes when these vectors are on the equator of the sphere S of radius $k/\sqrt{\mu}$ while $\boldsymbol{n}$ is along the poles axis.

Finally, the zero crossing method consisting in the search of the unique zero of $R(\boldsymbol{q})$ with $\boldsymbol{q} = \boldsymbol{p} \times \boldsymbol{p}^{\perp}/(k/\sqrt{\mu})$ solves the problem for the crack normal.

The crack plane

Take Ox_3 along the normal. The crack plane is defined by $x_3 - C = 0$ with constant C to be determined. We consider the adjoint wave $\boldsymbol{v}^{(\eta)} = cos[q(x_3 - \eta)]\boldsymbol{e}^3$, with $q = k/\sqrt{\lambda + 2\mu}$. The variational equation yields

$$-q(\lambda + 2\mu) sin[q(C - \eta)] \int_{\Sigma} [\![u_3^*]\!] dS = R(\eta). \tag{67}$$

If we choose the frequency or k so that the wave length $2\pi/q > L$ is greater than the diameter of Ω then the reciprocity gap $R(\eta)$ has a unique zero $\eta = C$ which determines the crack plane. Other zeros of $sin[q(C - \eta)]$ outside the solid are not physical.

The crack geometry

We need a 2-dimensional parameter $\boldsymbol{p} = [p_1, p_2, 0]$ for adjoint fields. We consider two complex vectors

$$Z(\boldsymbol{p})^{\pm} = \boldsymbol{p} \pm i\gamma \|\boldsymbol{p}\| \boldsymbol{e}^3, \quad \gamma^2 = 1 - \frac{k^2}{(\lambda + 2\mu)\|p\|^2}, \tag{68}$$

and two vectors fields

$$\boldsymbol{w}^{\pm}(\boldsymbol{x}, \boldsymbol{p}) = \nabla_x exp(-iZ(\boldsymbol{p})^{\pm} . \boldsymbol{x}). \tag{69}$$

which satisfy the adjoint wave equation. Define the adjoint field $\boldsymbol{v}^{(p)} = \boldsymbol{w}^+ + \boldsymbol{w}^-$ to obtain

$$2[\lambda(\gamma^2 - 1) + 2\mu\gamma^2] \|\boldsymbol{p}\|^2 \int_{\Sigma} [\![u_3(\boldsymbol{x})]\!] exp(-i\boldsymbol{p}.\boldsymbol{x}) dS_x = R(\boldsymbol{v}^{(p)}). \tag{70}$$

which gives the crack opening displacement and the crack geometry:

$$[\![u_3(\boldsymbol{x})]\!] = \frac{1}{4\pi^2} \int_{p_3=0} \frac{exp(i\boldsymbol{p}.\boldsymbol{x})}{2[\lambda(\gamma^2 - 1) + 2\mu\gamma^2 \|p\|^2]} R(\boldsymbol{v}^{(p)}) dp_1 dp_2. \tag{71}$$

8.2 Volumetric defect inverse problem

The Calderon's method can be extended to viscoelasticity in the frequency domain, for small frequency. Let us write the Calderon inverse problem for h

$$\mathrm{div}((1+h)\mathrm{grad}u^*) + k^2u^* = 0 \ \text{ in } \Omega. \tag{72}$$

with usual boundary data u^*, $\partial_n u^*$ in the form

$$\mathrm{div}(\mathrm{grad}u^*) + k^2u^* + S(\boldsymbol{x}) = 0 \ \text{ in } \Omega. \tag{73}$$

where $S = \mathrm{div}(h\mathrm{grad})u^*$ is unknown. Function $h(\boldsymbol{x})$ satisfies the Volterra integral equation,

$$\int_C h(\boldsymbol{x}) \text{ grad } U(\boldsymbol{x}).\text{grad } v(\boldsymbol{x};\boldsymbol{\xi})d^2x = R(v;\boldsymbol{\xi}), \quad \text{for any adjoint function } v(\boldsymbol{x};\boldsymbol{\xi}) \tag{74}$$

where $U(\boldsymbol{x})$ is the solution $u^*(\boldsymbol{x})$ of Eq. (73) and $v(\boldsymbol{x};\boldsymbol{\xi})$ is an adjoint function parameteri- zed by vector $\boldsymbol{\xi}$. Remark that, according to Holmgren's Theorem [36], grad U cannot vanish in a non zero measure set.

Eq. (73) is a source inverse problem for S, which has been widely investigated in the literature, Isakov [38], Alves and Ha-Duong [10]. It does not depend explicitly in h, particularly on whether h is small or large. Since the support of S is related to the support of h, we have another manner to recover Isaacson and Isaacson's results [37] in statics. The difficulty of our source inverse problems relies on the non uniqueness of the solution. For example, in potential theory ($k = 0$), a unique point source or a concentric circular distributed source of the same global intensity corresponds to the same boundary data and thus the same R. Uniqueness of the solution S has been proved in Alves and Ha-Duong [10] for a finite number of point sources. Uniqueness also holds for the class of solutions of piecewise constant circular sources inscribed in regular square finite elements of size δ and centers a_i. However, the convergence of the solution when the size of elements tends to zero remains an open problem. The variational equation for S in a 2D problem is

$$\int_C v(\boldsymbol{x})S(\boldsymbol{x})d^2\boldsymbol{x} = \int_{\partial\Omega}(u^*\partial_n v - v\partial_n u^*)ds := R(v) \tag{75}$$

Eq. (75) shows that two sources S_1 and S_2 of distinct supports corresponding to the same R cannot exist, because otherwise $\int_\Omega (S_1 - S_2) v d^2\boldsymbol{x} = R(1) - R(2) = 0$, for any v which implies $S_1 - S_2 = 0$, which is a contradiction with our assumption. A finite linear system of algebraic equations is obtained by considering M adjoint fields $v^{(j)}$, j=1,., M such that the matrix of the discretized system is invertible, for $S = \sum_{i=1}^{M} \lambda_i \chi(a_i)$, where $\chi(a_i)$ is the characteristic function of the square element of centre a_i, λ_i is the intensity of the source.

$$\delta^2 \sum_{i=1}^{M} \lambda_i v^{(j)}(a_i) = R(v^{(j)}) \tag{76}$$

For $k = 0$, adjoint functions can be the real ou imaginary parts of polynomials of $z = x_1 + ix_2$. For $k \neq 0$ there are many possible adjoint fields. The first one is given by the real part of function, Ammari and Ramm [12]

$$v(\boldsymbol{x}, \boldsymbol{\xi}^{(j)}) = \exp[-i\boldsymbol{x}.(\boldsymbol{\xi}^{(j)} + i\gamma \boldsymbol{\xi}^{\perp(j)})] \quad j = 1, ..M \tag{77}$$

where $\boldsymbol{\xi}^{\perp(j)} = \boldsymbol{e}^3 \times \boldsymbol{\xi}^{(j)}$, $\gamma = (1/\|\boldsymbol{\xi}^{(j)}\|)\sqrt{\boldsymbol{\xi}^{(j)2} - 4k^2}$ if $\|\boldsymbol{\xi}^{(j)}\| > 2k$, and $\gamma = -i(1/\|\boldsymbol{\xi}^{(j)}\|)\sqrt{4k^2 - \boldsymbol{\xi}^{(j)2}}$ if $\|\boldsymbol{\xi}^{(j)}\| < 2k$.

The second one is the ξ-family of 2D fundamental solution of the Helmholtz equation with singular point $\boldsymbol{\xi}$ lying *outside* the domain

$$v(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{i}{4} H_0^1(k\|\boldsymbol{x} - \boldsymbol{\xi}\|), \quad \boldsymbol{\xi} \notin \Omega, \quad \boldsymbol{x} \in \Omega \tag{78}$$

with Hankel function of the 1rst kind and order 0. One chooses M different singular points $\boldsymbol{\xi}^{(j)}$ outside the domain and near its boundary.

It is of interest to solve numerically the source problem in a small region. Consider a small *window* which is discretized in regular meshes and solve numerically the source inverse problem for N point sources $S(\boldsymbol{x}) = \sum_{i=1}^{N} \lambda_i \delta(\boldsymbol{x} - \boldsymbol{a}_i)$, with source points at the centres of finite elements, and unknown amplitudes λ_i. Numerical solution is searched in the sense of the minimum norm of the errors. With a chosen window, we enforce the condition $S = 0$ outside it. For a large window enclosing the defect, it is shown in the paper [10] that the solution for a finite number N of sources approaching the source $S(\boldsymbol{x})$ exists and is unique. If the window does not contain entirely the source, we get a wrong solution and the corresponding image of the numerical solution is then blurred. Only in the case where the window contains the inclusion that a sharp image is obtained. This procedure resembles the medical echography imaging of a body. For example, by trials and errors, one moves the echography device on the body of an expectant mother in order to search its right location which reveals a sharp image of her foetus. In our example of the source problem, to study a tumor in life tissue or a damaged zone in materials, the moving window is a 4×5 mesh. For example, (Fig. 7a) corresponds to the wrong solution, while (Fig. 7b) is the correct one which is the input source for the reciprocity gap R.

9 Conclusions

In this paper, we make a review of some recent results in inverse problems in elasticity, in statics and dynamics, in acoustics, thermoelasticity and viscoelasticity. Crack inverse problems have been solved in closed form, by considering a nonlinear variational equation provided by the reciprocity gap functional. This equation involves the unknown geometry of the crack and the boundary data. It results from the symmetry lost between current fields and adjoint fields. The asymmetry already exists between spaces of current and adjoint fields, in

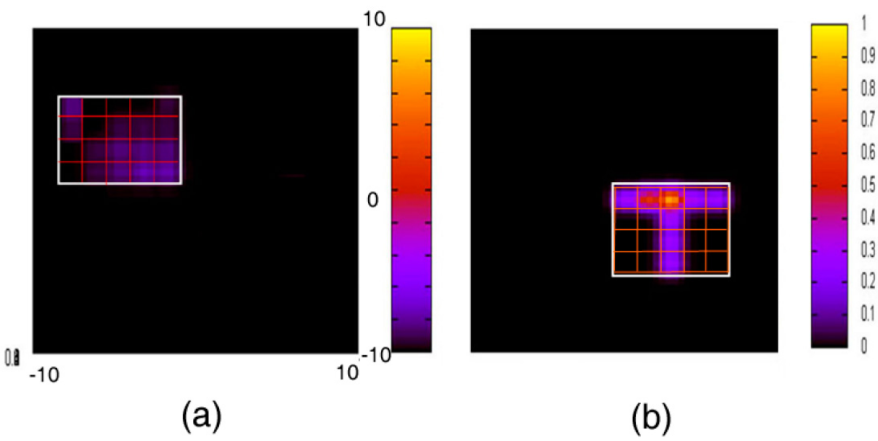

Figure 7: Imaging of a defect: (a) Bad window, wrong solution; (b) Correct window, good solution

the mathematical sense of duality and complementarity (Schwartz, Sobolev's sense). We are concerning solely with the asymmetry between the supports of current and adjoint fields. Cracks and defects can then be revealed. The nonlinear equation is solved step by step, by considering linear inverse problems. We also consider the problem of a volumetric defect viewed as the perturbation h of a material constant in elastic solids which satisfies the nonlinear Calderon equation. The nonlinear problem reduces to two successive ones: a source inverse problem and a Volterra integral equation of the first kind. The first problem provides information on the inclusion geometry supp(h). The second one provides the magnitude of h. We made a comparison between the geometry of an inclusion in the small perturbation case and the geometry in the nonlinear case and found that both inclusion geometries are identical for arbitrary loading and geometry of the solid. Our result elucidates the mystery of the linearized Calderon's solution for the geometry which works well for the nonlinear case, as revealed numerically by Isaacson and Isaacson [37] in the axisymmetric case.

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O nekim nelinearnim inverznim problemima u elastičnosti

Dat je pregled nekih inverznih problema u elastičnosti - u statici i dinamici, akustici, termoelastičnosti i viskoelastičnosti. Razmatrajući nelinearnu varijacionu jednačinu obezbedjenu funkcionalom recipročnog otvaranja inverzni problemi loma su rešeni u zatvorenom obliku. Ova jednačina uključuje nepoznatu geometriju prsline i granične podatke. Ona sledi iz simetrije izgubljene izmedju tekućih polja i susednih polja povezanih njihovim osloncem. Nelinearna jednačina je razmatranjem linearnih inverznih problema rešena korak po korak. Eksplicitno su odredjene: normala na ravan prsline, ravan prsline i geometrija prsline odredjene prekidom pomeranja prsline. Takodje se razmatra problem zapreminskog defekta vidjenog kao poremećaj materijalne konstante u elastičnim čvrstim telima koja zadovoljava nelinearnu Calderon-ovu jednačinu. Nelinearni problem se svodi na dva uzastopna: izvorni inverzni problem i Volterra-ovu integralnu jednačinu prve vrste. Prvi problem obezbedjuje informaciju o geometriji uključka. Drugi podaje veličinu poremećaja. Geometrija defekta u nelinearnom slučaju je dobijena u zatvorenom obliku i uporedjena sa linearizovanim Calderon-ovim rešenjem. Obe geometrije, u linearizovanom i nelinearnom slučaju, su nadjene iste.

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