Thermoinduced plastic flow and shape memory effects

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Abstract

We propose an enhanced form of thermocoupled J_2-flow models of finite deformation elastoplasticity with temperature-dependent yielding and hardening behaviour. The thermomechanical constitutive structure of these models is rendered free and explicit in the rigorous sense of thermodynamic consistency. Namely, with a free energy function explicitly introduced in terms of almost any given form of the thermomechanical constitutive functions, the requirements from the second law are identically fulfilled with positive internal dissipation. We study the case when a dependence of yielding and hardening on temperature is given and demonstrate that thermosensitive yielding with anisotropic hardening may give rise to appreciable plastic flow either in a process of heating or in a cyclic process of heating/cooling, thus leading to *the findings of one- and two-way thermoinduced plastic flow*. We then show that such theoretical findings turn out to be the effects found in shape memory materials, such as one- and two-way memory effects. Thus, shape memory effects may be explained to be thermoinduced plastic flow resulting from thermosensitive yielding and hardening behaviour. These and other relevant facts may suggest that, from a phenomenological standpoint, thermocoupled elastoplastic J_2-flow models with thermosensitive yielding and hardening may furnish natural, straightforward descriptions of thermomechanical behaviour of shape memory materials.

Keywords: thermomechanics, plasticity, shape memory, two-way memory.

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1 Introduction

It is well-known that, with moderate temperature changes, usual elastoplastic materials undergo merely small recoverable thermal deformations. It may be noticeable that appreciable plastic flows may be induced during a process of pure temperature change. This may be exemplified with a metal bar displaying appreciable lengthening or shortening effects at heating or cooling.

The main objective of this article is to show that thermocoupled finite strain elastoplastic J_2-flow models with thermosensitive yielding and hardening behaviour may indeed exhibit the noticeable effect mentioned above. Towards this objective, specific models with non-linear combined hardening should be established in a rigorous sense of thermodynamic consistency. This might be a formidable task in a broad sense without ad hoc assumptions or limitations.

In a latest development by Xiao et al. [1], a free, explicit formulation of thermocoupled finite elastoplasticity with non-linear combined hardening is proposed in a rigorous sense of thermodynamic consistency. It is demonstrated that such formulation identically fulfils the second law and leads to positive internal dissipation for any given form of the thermomechanical constitutive functions introduced, such as yield function and non-linear hardening functions.

In the subsequent development, we shall use thermocoupled finite strain J_2-flow models based on the aforementioned free, explicit formulation. With such models, we shall study thermosensitive yielding and hardening behaviour, and demonstrate that thermosensitive yielding with anisotropic hardening may give rise to appreciable plastic flow in a process of pure temperature change. This leads to the finding of *thermoinduced plastic flow resulting from thermosensitive yielding and hardening behaviour.* Then we shall show that such theoretical findings may become noticeable with the fact that they may be explained to be the shape memory effects found in these materials. Indeed, we shall indicate that, from a phenomenological standpoint, the aforementioned findings may furnish natural, straightforward descriptions and explanations of remarkable one- and two-way memory effects observed in shape memory materials (see, e.g., [2–4]).

The main content of this article will be organised as follows. In Section 2 we introduce an enhanced form of thermocoupled finite elastoplastic J_2-flow models with non-linear combined hardening. Then in Section 3 we study general 3-dimensional deformation response of these models in a process of temperature changes due to pure heating or cooling. We show that appreciable one-way plastic flow may be induced in a process of pure heating with no stress, thus

explaining the one-way effect from a straightforward phenomenological viewpoint. In Section 4 we further show that appreciable two-way plastic flow may be induced in a cyclic process of heating/cooling with constant stress. In Section 5, we demonstrate that these findings of thermoinduced plastic flows turn out to explain one- and two-way memory effects found in shape memory materials, from a phenomenological viewpoint. Simple forms of constitutive functions are presented as illustrative examples in Section 6, and a direct scheme for the determination of the constitutive parameters is proposed in Section 7. Finally, we discuss the main ideas and results in Section 8.

To conclude this introduction, we shortly explain notations that will frequently be used. A superposed dot means the material time derivative. The 2nd-order identity tensor is denoted $\boldsymbol{I}$. Moreover, let $\boldsymbol{S}$ and $\boldsymbol{T}$ be 2nd-order tensors. Then $\boldsymbol{S} : \boldsymbol{T}$ is the double dot product (scalar product) of $\boldsymbol{S}$ and $\boldsymbol{T}$, i.e. in a Cartesian coordinate system

$$\boldsymbol{S} : \boldsymbol{T} = S_{ij}\, T_{ij}.$$

In particular, the trace of tensor $\boldsymbol{S}$ is given by $\boldsymbol{S} : \boldsymbol{I}$ and denoted $\operatorname{tr}\boldsymbol{S}$. Moreover, $|\boldsymbol{S}|$ is used to designate the magnitude of tensor $\boldsymbol{S}$:

$$\operatorname{tr}\boldsymbol{S} = \boldsymbol{S} : \boldsymbol{I} = S_{ii}, \qquad |\boldsymbol{S}| = \sqrt{\boldsymbol{S} : \boldsymbol{S}} = \sqrt{\operatorname{tr}(\boldsymbol{S}\,\boldsymbol{S}^{\mathrm{T}})}.$$

The normalisation of non-vanishing tensor $\boldsymbol{S}$ is denoted $[\boldsymbol{S}]$ and given by

$$[\boldsymbol{S}] = \frac{\boldsymbol{S}}{|\boldsymbol{S}|}.$$

2 Enhanced form of thermocoupled finite strain J_2-flow models

Classical J_2-flow models for small deformation behaviour of elastoplastic materials may be found in many monographs, e.g., in [5–7], et al. It appears that a natural, straightforward extension of such models to finite deformations originated from [8–10], et al. Starting from the additive decomposition of the deformation rate (stretching), such an extension leads to Eulerian formulations of finite elastoplasticity. A basic issue in Eulerian formulations is the choice of objective tensor rates. In [11] it is demonstrated that a consistent Eulerian formulation may be established based on the newly discovered corotational logarithmic rate.

In a broad case with coupled thermal effects, general forms of thermomechanical constitutive relations of finite elastoplasticity should be placed on universal thermodynamic foundations centred on the internal dissipation and the irreversibility property of macroscopic material behaviour. Without ad hoc assumptions or simplifications, this might be a formidable task. Usually, implicit results are presented in conjunction with certain restrictions derived from the thermodynamic laws. Studies on thermodynamic foundations of elastoplastic formulations were made for small deformation case earlier by, e.g., [12–16] and recently by, e.g., [17–19], and for finite deformation case by [20–39].

A consistent, explicit Eulerian formulation of thermocoupled finite elastoplasticity is established in [1]. As mentioned earlier, this recent formulation automatically fulfils the second law with general free forms of thermomechanical constitutive functions introduced. In what follows we shall propose an enhanced form of thermocoupled finite strain J_2-flow models based on the new explicit formulation.

Consider a material body undergoing a process of finite deformations. Let $\boldsymbol{X}$ and $\boldsymbol{x}$ be the position vectors of a material point in an initial and a current configuration of this body, respectively. Then, the local deformation state of this body is described by the deformation gradient $\boldsymbol{F}$ and the changing rate of the deformation state is characterised by the velocity gradient $\boldsymbol{L}$

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}, \qquad \boldsymbol{L} = \frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{x}} = \dot{\boldsymbol{F}}\,\boldsymbol{F}^{-1}.$$

The symmetric and antisymmetric parts of the latter yield the stretching (resp. natural deformation rate, Eulerian strain rate, etc.) $\boldsymbol{D}$ and the vorticity tensor $\boldsymbol{W}$

$$\boldsymbol{D} = \frac{1}{2}\,(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}), \qquad \boldsymbol{W} = \frac{1}{2}\,(\boldsymbol{L} - \boldsymbol{L}^{\mathrm{T}}).$$

Let $\boldsymbol{\sigma}$ be the Cauchy stress. Then the Kirchhoff stress is

$$\boldsymbol{\tau} = J\,\boldsymbol{\sigma}$$

with $J = \det \boldsymbol{F}$ the volumetric ratio (Jacobian).

In [40] self-consistent Eulerian J_2-flow models of finite deformation elastoplasticity are established for the isothermal case, and a free, explicit Eulerian formulation of thermocoupled finite elastoplasticity is proposed in [1]. In what follows we shall propose an enhanced form of thermocoupled J_2-flow models.

2.1 Eulerian thermocoupled elastoplasticity with combined hardening

For a current elastoplastic deformation state, the stretching $\boldsymbol{D}$ is assumed to be decomposed into two parts

$$\boldsymbol{D} = \boldsymbol{D}^{\mathrm{e}} + \boldsymbol{D}^{\mathrm{p}} . \tag{1}$$

In the above, the thermoelastic part $\boldsymbol{D}^{\mathrm{e}}$ is related to thermoelastic behaviour, while the plastic part $\boldsymbol{D}^{\mathrm{p}}$ is responsible for plastic flow. Consistent constitutive characterisations for them should be presented, as will be shown below.

An objective Eulerian rate equation of Hookean type is used to relate the thermoelastic part $\boldsymbol{D}^{\mathrm{e}}$ and an objective stress rate $\overset{\mathrm{o}}{\boldsymbol{\tau}}$ as well as the temperature rate $\dot{T}$:

$$\boldsymbol{D}^{\mathrm{e}} = \frac{1}{2\,G} \overset{\mathrm{o}}{\boldsymbol{\tau}} - \frac{\nu}{E} \left(\mathrm{tr}\, \overset{\mathrm{o}}{\boldsymbol{\tau}} \right) \boldsymbol{I} + \beta\, \dot{T}\, \boldsymbol{I} . \tag{2}$$

Here, G, ν, E, β are the shear modulus, Poisson's ratio, Young's modulus and thermal expansion coefficient.

Prior to the yielding (hence $\boldsymbol{D}^{\mathrm{e}} = \boldsymbol{D}$), the elastic rate equation (2) should be exactly integrable to deliver a hyperelastic stress-strain relation based on an elastic stored-energy function. As demonstrated in [40–44], the just-mentioned requirement is fulfilled if and only if the stress rate $\overset{\mathrm{o}}{\boldsymbol{\tau}}$ in (2) is the corotational logarithmic rate as defined below (see, e.g., [45–47]):

$$\overset{\mathrm{o}}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\,\boldsymbol{\Omega} - \boldsymbol{\Omega}\,\boldsymbol{\tau} , \tag{3}$$

with the logarithmic spin:

$$\boldsymbol{\Omega} = \boldsymbol{W} + \sum_{r=1}^{m} \left(\sum_{s=1,\, s \neq r}^{m} \left(\frac{1 + (b_r/b_s)}{1 - (b_r/b_s)} + \frac{2}{\ln(b_r/b_s)} \right) \right) \boldsymbol{B}_r\, \boldsymbol{D}\, \boldsymbol{B}_s , \tag{4}$$

where b_t and $\boldsymbol{B}_t$ are the m distinct eigenvalues and the corresponding eigen-projections of the Cauchy-Green tensor $\boldsymbol{B} = \boldsymbol{F}\,\boldsymbol{F}^{\mathrm{T}}$, respectively.

As shown in [48] in isothermal case, a weakened form of Ilyushin postulate implies that the plastic part $\boldsymbol{D}^{\mathrm{p}}$ should be governed by the normality rule. For a general case with thermal effect, this leads to the following flow rule [1]:

$$\boldsymbol{D}^{\mathrm{p}} = \xi\, \frac{\hat{f}}{h}\, \frac{\partial f}{\partial \boldsymbol{\tau}} . \tag{5}$$

In the above, f is the yield function and $\hat{f}$ is given by

$$\hat{f} = \frac{\partial f}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}} + \frac{\partial f}{\partial T} \dot{T} \,. \tag{6}$$

h and ξ in eq. (5), known as plastic indicator and plastic modulus, will be given slightly later.

To characterise the hardening behaviour, namely the changing of the yield surface with the development of plastic flow, we introduce two hardening variables. One is the plastic work κ, as defined by the following rate equation:

$$\dot{\kappa} = \boldsymbol{\tau} : \boldsymbol{D}^{\mathrm{p}} \,. \tag{7}$$

This variable will enter the yield function and characterise isotropic hardening behaviour. The other hardening variable is known as back stress tensor. It was introduced originally by Prager [49] and intended for describing anisotropic or kinematic hardening, as exemplified by Bauschinger effects in uniaxial deformations of bars. In an idealised sense, such hardening effect is related to translation of the initial yield surface in stress space. The back stress $\boldsymbol{\alpha}$ is determined by the following evolution equation:

$$\overset{\circ}{\boldsymbol{\alpha}} = c \, \boldsymbol{D}^{\mathrm{p}} \,. \tag{8}$$

The variable c in the above may be referred to as Prager's hardening modulus, or simply Prager's modulus. Initially, it was taken to be constant with the dimension of stress. This case is known as linear anisotropic hardening. Later on, Prager's modulus c was taken to be dependent on the hardening variable κ; refer to, e.g., [7, 50–52]. In a general case with thermal effects, it may rely on both κ and temperature T. Then, we have

$$c = c(\kappa, T) \,. \tag{9}$$

It is demonstrated in [43] that the rate $\overset{\circ}{\boldsymbol{\alpha}}$ in eq. (8) should also be the corotational logarithmic rate, namely,

$$\overset{\circ}{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}} + \boldsymbol{\alpha}\,\boldsymbol{\Omega} - \boldsymbol{\Omega}\,\boldsymbol{\alpha}$$

with the logarithmic spin $\boldsymbol{\Omega}$ given by eq. (4).

2.2 Thermocoupled J_2-flow models with thermoinducer

With the two hardening variables κ and $\boldsymbol{\alpha}$, the yield function f in a general sense is of the form

$$f = f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa).$$

In the subsequent development, this function is taken to be *an enhanced form of von Mises yield function* below:

$$f = \frac{1}{2}\,|\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}|^2 - \frac{1}{3}\,r^2\,, \tag{10}$$

where $\tilde{\boldsymbol{\tau}}$ is the deviatoric stress, i.e.

$$\tilde{\boldsymbol{\tau}} = \boldsymbol{\tau} - \frac{1}{3}\,(\mathrm{tr}\,\boldsymbol{\tau})\,\boldsymbol{I}$$

and r is known as the current yield stress. Isotropic hardening behaviour means that the current yield stress r should rely on the plastic work κ (see, e.g. [10, 53, 54]). Generally, it depends on both κ and T

$$r = r(\kappa, T)\,. \tag{11}$$

Moreover, the scalar factor ϱ in eq. (10) is a non-dimensional temperature-dependent positive constitutive quantity, viz.

$$\varrho = \varrho(T) > 0\,. \tag{12}$$

The introduction of the factor ϱ, referred to as *thermoinducer*, is intended for enhancing and extending the classical yield function of von Mises type by allowing for coupling effects of temperature on yielding. For the isothermal case, ϱ is a constant, and then eq. (10) reduces to the classical case. For the non-isothermal case, taking a constant ϱ also leads to the classical case.

The question arises about the physical essence of the thermoinducer ϱ. It is known that for realistic materials the increase of temperature renders the occurrence of yielding easier. Then it is clear that the yield stress r should decrease with increasing temperature. However, it appears that this property alone could only represent one aspect of the coupling of temperature (heat) on yielding and could not catch all. An analogy of temperature change to load (stress) change may render this clear. For the sake of simplicity, consider an uniaxial deformation of a bar. On the one hand, in the case of pure load (stress) change, yielding is first induced in a process of tensile load and reverse

yielding is then induced in a following process of reverse, compressive load. On the other hand, in the case of pure temperature change, by analogy we may infer that plastic flow may first be induced at a process of heating and reverse plastic flow at a following process of cooling. Such two-way thermoinduced plastic flows would not be possible, according to classical form of von Mises yield function with a temperature-dependent yield stress r, since therein the effective stress $|\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}|$ keeps unchanged with constant $\boldsymbol{\tau}$ and $\boldsymbol{\alpha}$ at a process of pure temperature change. Thus, it follows that reverse plastic flow would not be induced in any process of cooling, since the yield condition could not be fulfilled.

An idea to bypass the above situation is to find out a certain means by which temperature change may be parallel to load change and then contribute to the effective stress. This idea leads to the introduction of the thermoinducer $\varrho = \varrho(T)$ and then to the enhanced form of von Mises yield function with the combined effective stress $|\varrho(T)\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}|$. Now the direct coupling of the current temperature T on the current stress level becomes clear. Also, it may be clear that temperature change indeed becomes parallel to load (stress) change. Here it may be essential that the geometrical meaning of both the yield stress as the radius of the current yield surface (work hardening) and the back stress $\boldsymbol{\alpha}$ as the center of the current yield surface (induced anisotropy) are kept intact as in the classical von Mises yield function, while the current temperature T and the current deviatoric stress $\tilde{\boldsymbol{\tau}}$ are combined in a parallel sense to yield a new, combined effective stress, namely, $|\varrho(T)\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}|$. On account of this, the latter may be referred to as *thermocoupled effective stress*. Then, according to the enhanced form of von Mises yield function, the yielding behaviour in a thermocoupled case is characterised by the thermocoupled effective stress with the thermoinducer ϱ, in conjunction with a temperature-dependent current yield stress r. Note that both ϱ and r are constitutive functions representing thermocoupled yielding behaviour.

It will be seen in Sections 4 and 5 that the factor ϱ will be essential for the findings of two-way thermoinduced plastic flows in a cyclic heating/cooling process.

With an enhanced form of the von Mises yield function given by eq. (10), we have the following derivatives

$$
\frac{\partial f}{\partial \boldsymbol{\tau}} = \varrho\,(\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}), \qquad\qquad \frac{\partial f}{\partial \boldsymbol{\alpha}} = -(\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}),
$$

$$
\frac{\partial f}{\partial \kappa} = -\frac{2}{3}\,r\,\frac{\partial r}{\partial T}, \qquad\qquad \frac{\partial f}{\partial T} = -\frac{2}{3}\,r\,\frac{\partial r}{\partial T} + \varrho'\,(\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}) : \boldsymbol{\tau}.
$$

From the first in the above and the normality rule (5), we obtain

$$\boldsymbol{D}^{\mathrm{P}} = \xi\,\varrho\,\frac{\hat{f}}{h}\,(\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha})\,. \tag{13}$$

Here and henceforth, the derivative of each function of temperature, $\phi = \phi(T)$, is designated by ϕ', namely,

$$\phi' = \frac{\mathrm{d}\,\phi}{\mathrm{d}\,T}.$$

Now we specify the plastic indicator ξ and the plastic modulus h. The latter is determined by the consistency condition for plastic flow

$$h = -\frac{\partial f}{\partial \kappa}\left(\boldsymbol{\tau} : \frac{\partial f}{\partial \boldsymbol{\tau}}\right) - c\,\frac{\partial f}{\partial \boldsymbol{\tau}} : \frac{\partial f}{\partial \boldsymbol{\alpha}}$$

for a yield function $f = f(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ in general form. For a von Mises yield function as given by eq. (10), using the above results for the derivatives, we deduce

$$h = \frac{2}{3}\,\varrho\,c\,r^2 + \frac{2}{3}\,\varrho\,r\,\frac{\partial r}{\partial \kappa}\,(\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}) : \boldsymbol{\tau}\,. \tag{14}$$

The plastic indicator ξ is associated with the loading-unloading conditions. It takes values 1 and 0 for loading and for unloading, respectively. Unified loading-unloading conditions for both strain-hardening ($h > 0$) and strain-softening ($h < 0$) materials are proposed in [1]. Following these conditions, we have

$$\xi = \begin{cases} 1 & \text{if} \quad f = 0, \quad \dfrac{\hat{f}}{h} \geq 0\,, \\[2mm] 0 & \text{if} \quad f < 0 \quad \text{or} \quad \left(f = 0,\ \hat{f} < 0\right). \end{cases} \tag{15}$$

In the above, the plastic modulus h is given by eq. (14), and the $\hat{f}$ is of the form:

$$\hat{f} = \varrho\,(\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}) : \overset{\mathrm{o}}{\boldsymbol{\tau}} + \left(\varrho'\,(\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}) : \boldsymbol{\tau} - \frac{2}{3}\,r\,\frac{\partial r}{\partial T}\right)\dot{T}\,. \tag{16}$$

Eqs. (1)–(16) establish thermocoupled J_2-flow models of finite elastoplasticity with nonlinear combined hardening. Their properties are determined by the two hardening functions $r(\kappa, T)$ and $c(\kappa, T)$ as well as the thermoinducer $\varrho(T)$.

A complete coupled thermomechanical system governing the fields of deformation, stress and temperature are presented by the foregoing thermomechanical constitutive equations, in conjunction with a heat conduction equation and an energy equilibrium equation in explicit form. Details are no longer given here and may be found in [1].

2.3   Thermodynamic consistency in free and explicit sense

Studies of elastoplastic behaviour and its models rarely make reference to thermodynamic requirements. It may not be clear whether the restrictions from thermodynamics would be fulfilled or not. As is usually known, a thermocoupled theory is not "free" and "explicit", in the sense that the constitutive functions involved as well as the thermomechanical process should be restricted by universal thermodynamic laws. Since the free energy and the entropy as central physical quantities are coupled with each other and may be elusive in concept, usually without additional assumptions and simplifications a constitutive formulation could not be freed from the second law formulated by the Clausius-Duhem inequality. That may be the case especially for thermocoupled elastoplasticity at finite deformations. In [1], for an Eulerian thermocoupled elastoplastic formulation of general form, a free energy function and then an entropy function are found in explicit form, which automatically fulfil the second law with positive intrinsic dissipation for any given forms of constitutive functions introduced.

Here, by applying the general result in the foregoing development, the enhanced Eulerian J_2-flow model of finite elastoplasticity based on the logarithmic rate is free and explicit from a standpoint of thermodynamic consistency. Namely, the second law and positive internal dissipation would automatically be guaranteed for any forms of hardening functions $r = r(\kappa, T) > 0$ and $c = c(\kappa, T) > 0$ as well as the thermoinducer $\varrho = \varrho(T)$. In fact, free energy ψ and specific entropy η per unit reference volume may be given in explicit forms, as shown below:

$$
\begin{aligned}
\psi = {} & \psi_0(T) + \frac{1}{4\,G}\,\mathrm{tr}\,\boldsymbol{\tau}^2 - \frac{\nu}{2\,E}\,(\mathrm{tr}\,\boldsymbol{\tau})^2 + \kappa \\
& - \frac{\lambda(T)}{c_0}\left(\varrho(T)\int_0^\kappa c(\kappa, T)\,\mathrm{d}\,\kappa - \frac{1}{2}\,\boldsymbol{\alpha}:\boldsymbol{\alpha}\right),
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\eta = {} & -\psi_0' + \beta\,(\mathrm{tr}\,\boldsymbol{\tau}) \\
& + \frac{\partial}{\partial T}\left(\frac{\lambda(T)}{c_0}\left(\varrho(T)\int_0^\kappa c(\kappa, T)\,\mathrm{d}\,\kappa - \frac{1}{2}\,\boldsymbol{\alpha}:\boldsymbol{\alpha}\right)\right),
\end{aligned}
\tag{18}
$$

where the $\psi_0(T)$ is the specific heat capacity, $\lambda(T) > 0$ is a free dimensionless quantity referred to as plastic dissipation factor and c_0 is Prager's modulus at a reference temperature. Then, the second law is fulfilled identically with the positive internal dissipation

$$\mathcal{D} \equiv \boldsymbol{\tau} : \boldsymbol{D} - (\dot{\psi} + \eta \dot{T}) = \xi \frac{\hat{f}}{h} \lambda \frac{c}{c_0} \, |\varrho \tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}|^2 = \frac{2}{3} \xi \lambda \frac{c}{c_0} \, r^2 \frac{\hat{f}}{h} \geq 0 \, .$$

In addition, the energy equilibrium equation (the first law) may be rendered explicit with the specific entropy and the internal dissipation given above. In fact, we have

$$T \dot{\eta} + J \boldsymbol{\Delta} \cdot \boldsymbol{q} - s = \mathcal{D} \, , \tag{19}$$

where $\boldsymbol{q}$ is the heat flux vector per unit current area and s is the heat supply per unit reference volume. Moreover, the heat flux $\boldsymbol{q}$ will be governed by a heat conduction equation (e.g., Fourier's equation). Details may be found in [1].

3 One-way thermoinduced plastic flow

In this section, we take into account a thermal process during which an elasto-plastic body with no boundary tractions and with no constraints for displacement is heated from an initial temperature T_0 to a temperature T_1. In such a process of pure heating, the body is stress-free at each instant but experiencing deformations resulting from temperature change. Usually, such deformation is not only small but recoverable. With the thermocoupled J_2-flow model proposed, in what follows we shall study the question whether or not appreciable plastic flow may be induced in a thermal process at issue.

An initial observation, however, would suggest that the answer for the above question should be negative. In fact, suppose that anisotropic hardening would play no role, i.e., $\boldsymbol{\alpha} = \boldsymbol{0}$. Then we have

$$f = -\frac{1}{3} \, r^2 < 0 \tag{20}$$

for a stress-free thermal process. Thus, it follows that yielding would never occur in the absence of anisotropic hardening, no matter how the yield stress r changes with temperature. On the other hand, in the presence of anisotropic hardening, suppose that the initial value of the back stress $\boldsymbol{\alpha}$ at the initial temperature T_0 is zero, namely, $\boldsymbol{\alpha}|_{T=T_0} = \boldsymbol{0}$. Then, we again derive (20) for

a thermal process starting from T_0. In this case, no plastic flow would be induced, either.

The above discussion implies that *anisotropic hardening with a non-vanishing initial value of the back stress is necessary for thermoinduced plastic flows in a process of pure heating with no stress.*

Now let

$$\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}|_{T=T_0} \neq \boldsymbol{0} \,. \tag{21}$$

To generate such a non-vanishing initial value, we introduce an elastoplastic loading-unloading process just preceding the thermal process indicated at the outset of this section. Starting from a natural state, the body is loaded well beyond the initial yielding and then unloaded at temperature T_0. At the end of this process, the value of the back stress is given by eq. (21) and the plastic deformation by

$$\boldsymbol{F}_0 = \boldsymbol{F}|_{T=T_0} \,. \tag{22}$$

Since plastic deformation is isochoric, we have

$$J_0 = \det \boldsymbol{F}_0 = 1.$$

In what follows, the natural state at the beginning of the loading-unloading process is taken as the initial configuration. All the deformation quantities, such as the initial plastic deformation $\boldsymbol{F}_0$, will be referred to this configuration.

We examine the changing of the yield function during a heating process. It follows from eq. (7) that the plastic work κ keeps constant. Then, either of the two hardening functions r and c reduces to a function of temperature T. At temperature T_0, i.e., at the end of the foregoing loading-unloading process, we have

$$f = \frac{1}{2}\, |\boldsymbol{\alpha}_0|^2 - \frac{1}{3}\, (r(T_0))^2 < 0 \tag{23}$$

i.e.,

$$r(T_0) > \sqrt{\frac{3}{2}}\, |\boldsymbol{\alpha}_0| \,. \tag{24}$$

As the temperature is changing from T_0, the back stress $\boldsymbol{\alpha}$ keeps constant up to the thermoinduced plastic flow begins to emerge at a certain temperature T_{h}, whereas the yield stress $r = r(T)$ is always changing with changing temperature

T. The just-mentioned temperature T_h at the start of yielding should be derived from the yield condition

$$\frac{1}{2}\,|\boldsymbol{\alpha}_0|^2 - \frac{1}{3}\,(r(T_\mathrm{h}))^2 = 0\,, \tag{25}$$

viz.,

$$r(T_\mathrm{h}) = \sqrt{\frac{3}{2}}\,|\boldsymbol{\alpha}_0|\,. \tag{26}$$

Whenever a temperature T_h meeting eq. (25) is found in the interval $[T_0, T_1)$, the yield condition is met, but this does not guarantee the emergence of thermoinduced plastic flow, as can be seen from eq. (15). A further condition is

$$\frac{\hat{f}}{h} > 0.$$

Utilizing eqs. (14) and (16) and noting $\boldsymbol{\tau} = \mathbf{0}$, we have

$$\hat{f} = -\frac{2}{3}\,r\,r'\,\dot{T}, \qquad h = \frac{2}{3}\,\varrho\,c\,r^2\,. \tag{27}$$

Since $r > 0$ and $c > 0$, the foregoing condition reduces to

$$r'\,\dot{T} < 0 \tag{28}$$

for $\varrho > 0$.

We need to examine whether the conditions (24), (25) and (28) can be satisfied with monotonically increasing temperature T, i.e.

$$\dot{T} > 0.$$

Then, the condition (28) yields

$$r' < 0\,. \tag{29}$$

This requires that the yield stress $r = r(T)$ should decrease with increasing temperature T. In this case, it is indeed possible that the yield stress $r(T)$ is decreasing from an initial value at T_0 meeting (24) up to a value at $T_\mathrm{h} > T_0$ meeting (25). As a result, the plastic flow will be induced as from $T = T_\mathrm{h}$.

Condition (29) is derived for heating. It may be reasonable for most metals and alloys, as observed in relevant experiments. The corresponding condition for cooling with $\dot{T} < 0$ is just the opposite, viz.

$$r' > 0 \,, \tag{30}$$

which requires that the yield stress $r = r(T)$ would increase with increasing temperature. It appears that this would not be consistent with realistic material behaviour. Consequently, here only heating is taken into consideration.

We proceed to find out the deformation response from T_0 to T_1. From T_0 to T_{h}, only recoverable thermal deformations emerge. We have

$$\boldsymbol{D} = \beta \dot{T} \boldsymbol{I} \,. \tag{31}$$

Let $\boldsymbol{Q}$ be the log-rotation specified by

$$\dot{\boldsymbol{Q}} = -\boldsymbol{Q} \boldsymbol{\Omega} \,, \qquad \boldsymbol{Q}|_{T=T_0} = \boldsymbol{Q}_0 \,, \tag{32}$$

where $\boldsymbol{Q}_0$ is a given rotation tensor. Then, utilizing the relation between the stretching $\boldsymbol{D}$ and the Hencky strain

$$\boldsymbol{h} = \frac{1}{2} \ln(\boldsymbol{F} \boldsymbol{F}^{\mathrm{T}}),$$

namely (see, e.g., [47]),

$$\boldsymbol{Q} \boldsymbol{D} \boldsymbol{Q}^{\mathrm{T}} = \overline{\boldsymbol{Q} \dot{\boldsymbol{h}} \boldsymbol{Q}^{\mathrm{T}}} \,, \tag{33}$$

we may convert eq. (31) into

$$\overline{\dot{\boldsymbol{h}^*}} = \beta \dot{T} \boldsymbol{I} \,. \tag{34}$$

Here and henceforth, we denote

$$\boldsymbol{A}^* = \boldsymbol{Q} \boldsymbol{A} \boldsymbol{Q}^{\mathrm{T}} \tag{35}$$

for each 2nd-order tensor $\boldsymbol{A}$ of interest. From eq. (34) we obtain

$$\boldsymbol{h}^* = \boldsymbol{h}_0^* + \beta \left(T - T_0\right) \boldsymbol{I} \,, \qquad T \leq T_{\mathrm{h}} \,, \tag{36}$$

where $\boldsymbol{h}_0^*$ is the value of $\boldsymbol{h}^*$ at T_0

$$\boldsymbol{h}_0^* = \frac{1}{2} \boldsymbol{Q}_0 \ln \left(\boldsymbol{F}_0 \boldsymbol{F}_0^{\mathrm{T}}\right) \boldsymbol{Q}_0^{\mathrm{T}}.$$

Now we study the thermoinduced plastic flow starting from T_h. With eq. (27) and $\xi = 1$ as well as $\boldsymbol{\tau} = \boldsymbol{0}$, from eq. (13) we derive

$$\boldsymbol{D}^\text{p} = \frac{r'\dot{T}}{c\,r}\,\boldsymbol{\alpha}\,. \tag{37}$$

Substituting this into eq. (8), we obtain

$$\overset{\text{o}}{\boldsymbol{\alpha}} = \frac{r'\dot{T}}{r}\,\boldsymbol{\alpha}\,. \tag{38}$$

Using $\dot{r} = r'\dot{T}$ and the identity

$$\boldsymbol{Q}\,\overset{\text{o}}{\boldsymbol{\alpha}}\,\boldsymbol{Q}^\text{T} = \overline{\boldsymbol{Q}\,\dot{\boldsymbol{\alpha}}\,\boldsymbol{Q}^\text{T}},$$

we recast eq. (38) in the form

$$\overline{\dot{\boldsymbol{\alpha}^*}} - \frac{\dot{r}}{r}\,\boldsymbol{\alpha}^* = \boldsymbol{0}\,, \tag{39}$$

namely,

$$\overline{r^{-1}\,\dot{\boldsymbol{\alpha}^*}} = \boldsymbol{0}\,. \tag{40}$$

From this we arrive at

$$\boldsymbol{\alpha}^* = \frac{r}{r_\text{h}}\,\boldsymbol{\alpha}_0^*\,. \tag{41}$$

In the above,

$$r_\text{h} = r(T_\text{h}), \qquad \boldsymbol{\alpha}^* = \boldsymbol{Q}\,\boldsymbol{\alpha}\,\boldsymbol{Q}^\text{T}, \qquad \boldsymbol{\alpha}_0^* = \boldsymbol{Q}_0\,\boldsymbol{\alpha}_0\,\boldsymbol{Q}_0{}^\text{T}.$$

Now we are in a position to find out the deformation response from eqs. (1), (2) and (13). With eqs. (27), (34), (35) and (41) as well as $\xi = 1$ and $\boldsymbol{\tau} = \boldsymbol{0}$, we derive the following equation

$$\overline{\dot{\boldsymbol{h}^*}} = \beta\,\dot{T}\,\boldsymbol{I} + \frac{r'\,\dot{T}}{c\,r_\text{h}}\,\boldsymbol{\alpha}_0^*\,. \tag{42}$$

Integrating this equation from T_h to $T \geq T_\text{h}$ and using eq. (36), we arrive at

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I} + \boldsymbol{h}_0^* + \left(\frac{1}{r_\text{h}}\int_{T_\text{h}}^T \frac{r'}{c}\,\text{d}\,T\right)\boldsymbol{\alpha}_0^*\,. \tag{43}$$

Here, the first term is the usual recoverable thermal deformation, while the last term is the contribution from the thermoinduced plastic flow.

4 Two-way thermoinduced plastic flows

In the last section, we showed that thermoinduced plastic flow may be induced in a process of pure heating with no stress, when the yield stress $r = r(T)$ is a monotonically decreasing function of temperature T, as indicated by eq. (29). In this section, we shall study a cyclic process of heating/cooling with a given initial stress. It will be shown that a perhaps interesting phenomenon may emerge. Namely, under suitable conditions, two-way plastic flow may be induced. As will be seen, here the thermoinducer ϱ, viz. eq. (10), will play an essential role. Indeed, with the temperature-dependent thermoinducer ϱ therein, the term $|\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}|$ is changing in a cyclic process of pure temperature change, even if the stress is held fixed.

In what follows the main effort will be devoted to deriving exact closed-form solutions at multiaxial finite deformations and arbitrary temperature changes. This will be done for any given forms of constitutive functions introduced. Such solutions are derivable by directly integrating the rate constitutive equations with the yield conditions. This may suggest the simplicity of the proposed model. Nevertheless, the procedures will not be so concise as in treating one-way plastic flow at heating. However, the main idea may become clear from an analogy to the two-way plastic flow induced in a process of tensile/compressive loading at uniaxial deformations. In a process of pure (isothermal) load change, the thermocoupled effective stress $|\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}|$ is changing with constant ϱ. In parallel to this, in a process of pure temperature change, the thermocoupled effective stress is changing with constant $\tilde{\boldsymbol{\tau}}$. Since in a parallel sense $|\varrho\,\tilde{\boldsymbol{\tau}} - \boldsymbol{\alpha}|$ changes either in a process of pure load change or in a process of pure temperature change, the analyses of the two-way plastic flows in the two processes at issue may be made in a parallel sense. As a result, with analogy to the well-known treatment for two-way plastic flows in a process of tensile/compressive load, the subsequent treatment for two-way plastic flows at a process of heating/cooling may readily be understood.

We consider *a cyclic thermal process in the presence of constant stress*, namely, heating from T_0 to T_1 and then cooling down from T_1 to T_0 with a constant stress[1], namely,

$$\boldsymbol{\tau}^* = \boldsymbol{\tau}_0^*, \qquad \text{i.e.} \qquad \overset{\text{o}}{\boldsymbol{\tau}} = \mathbf{0}\,, \tag{44}$$

in the whole process at issue. In the presence of a constant stress, the plastic work κ will be changing with the development of plastic flow, as can be seen

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[1]Note that an objective Eulerian tensor quantity, such as the Kirchhoff stress $\boldsymbol{\tau}$, could not be set to be constant in a general sense of change of frame.

from eq. (7). In the subsequent analysis we shall assume that the yield stress r and Prager's modulus c are independent of κ and hence either of them is a function of temperature T alone. Then, from eqs. (14), (16) and (44) we deduce

$$h = \frac{2}{3}\,\varrho\,c\,r^2, \qquad \hat{f} = \left(\varrho'\,(\varrho\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^*) : \boldsymbol{\tau}_0^* - \frac{2}{3}\,r\,r'\right)\dot{T}. \tag{45}$$

4.1 Plastic flow induced at heating

Now we study the heating process, i.e. $\dot{T} > 0$, from T_0 to T_1. At the initial state at $T = T_0$, the initial values are given by

$$\boldsymbol{F}|_{T=T_0} = \boldsymbol{F}_0, \qquad \boldsymbol{\alpha}|_{T=T_0} = \boldsymbol{\alpha}_0, \tag{46}$$

At $T = T_0$, we assume either unloading or yielding

$$\frac{1}{2}\,|\varrho_0\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}_0^*|^2 - \frac{1}{3}\,r_0^2 \le 0. \tag{47}$$

From $T = T_0$, a usual process of recoverable thermal deformations occurs up to $T = \hat{T}_{\rm h}$ and, then, thermoinduced plastic flow emerges from $T = \hat{T}_{\rm h}$. This will be studied below.

The temperature $\hat{T}_{\rm h}$ at the beginning of the foregoing plastic flow should satisfy the yield condition:

$$\frac{1}{2}\left|\varrho(\hat{T}_{\rm h})\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}_0^*\right|^2 - \frac{1}{3}\left(r(\hat{T}_{\rm h})\right)^2 = 0, \quad \hat{T}_{\rm h} > T_0. \tag{48}$$

Starting from $T = \hat{T}_{\rm h}$, plastic flow will be induced with the loading conditions $f = 0$ and $\hat{f}/h > 0$. From eq. (44) and $\dot{T} > 0$ we infer that the latter two are given by

$$\begin{cases} \frac{1}{2}\,|\varrho(T)\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^*|^2 - \frac{1}{3}\,(r(T))^2 = 0, & T \ge \hat{T}_{\rm h}, \\[2mm] \varrho'\,(\varrho\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^*) : \boldsymbol{\tau}_0^* - \frac{2}{3}\,r\,r' > 0, & T \ge \hat{T}_{\rm h}. \end{cases} \tag{49}$$

Whenever the conditions as given by eqs. (47)–(49) are met, plastic flow is induced from $\hat{T}_{\rm h}$ to T_1 in the heating process. The recoverable thermoelastic deformation prior to $\hat{T}_{\rm h}$ is given by eq. (36). Thus, the deformation for each

$T \in [\hat{T}_{\mathrm{h}}, T_{\mathrm{l}}]$ is determined by

$$\overline{\dot{\boldsymbol{h}}^*} = \boldsymbol{D}^{\mathrm{e}*} + \boldsymbol{D}^{\mathrm{p}*}, \tag{50}$$

$$\boldsymbol{D}^{\mathrm{e}*} = \beta \dot{T} \boldsymbol{I}, \tag{51}$$

$$\boldsymbol{D}^{\mathrm{p}*} = \dot{T} \frac{1}{c} \left(\frac{3}{2} \frac{\varrho'}{r^2} \left(\varrho \tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^* \right) : \boldsymbol{\tau}_0^* - \frac{r'}{r} \right) \left(\varrho \tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^* \right), \tag{52}$$

$$\overline{\dot{\boldsymbol{\alpha}}^*} = c \boldsymbol{D}^{\mathrm{p}*}. \tag{53}$$

The above equations may be derived from eqs. (1), (2), (13) and (8) by left- and right-multiplying $\boldsymbol{Q}$ and $\boldsymbol{Q}^{\mathrm{T}}$ and then using eqs. (33), (35), (44) and (45).

Now we are in a position to derive the deformation response from eqs. (50)–(53). First, from eqs. (52) and (53) we deduce

$$\frac{\mathrm{d}\boldsymbol{\alpha}^*}{\mathrm{d}T} = \left(\frac{3}{2} \frac{\varrho'}{r^2} \left(\varrho \tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^* \right) : \boldsymbol{\tau}_0^* - \frac{r'}{r} \right) \left(\varrho \tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^* \right), \tag{54}$$

Let

$$\boldsymbol{\Phi}^* = \varrho \tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^*. \tag{55}$$

Then we may recast eq. (54) in the form

$$\frac{\mathrm{d}\boldsymbol{\Phi}^*}{\mathrm{d}T} = \left(\frac{r'}{r} - \frac{3}{2} \frac{\varrho'}{r^2} \boldsymbol{\tau}_0^* : \boldsymbol{\Phi}^* \right) \boldsymbol{\Phi}^* + \varrho' \tilde{\boldsymbol{\tau}}_0^*. \tag{56}$$

This is a non-linear tensorial differential equation of Riccati type. Its analytic solution in explicit form may be crucial to achieving definite results in the succeeding study. However, usually even it would be difficult to tackle a single Riccati equation in a single scalar unknown, let alone here a system of coupled Riccati equations in the six components of the tensor unknown $\boldsymbol{\Phi}^*$.

Fortunately, here with the help of a key transformation we may circumvent the aforementioned difficulty. In fact, performing the scalar product of the two sides with constant stress $\boldsymbol{\tau}_0^*$ and denoting

$$y = \boldsymbol{\tau}_0^* : \boldsymbol{\Phi}^*, \tag{57}$$

we derive a single Riccati equation in y as follows:

$$y' = \left(\frac{r'}{r} - \frac{3}{2} \frac{\varrho'}{r^2} y \right) y + \varrho' |\tilde{\boldsymbol{\tau}}_0^*|^2. \tag{58}$$

This equation, together with the initial value

$$y_{\mathrm{h}} = y|_{T=\hat{T}_{\mathrm{h}}} = \varrho(\hat{T}_{\mathrm{h}}) \, |\tilde{\boldsymbol{\tau}}_0^*|^2 - \boldsymbol{\tau}_0^* : \boldsymbol{\alpha}_0^* \tag{59}$$

yields a solution (see Appendix 8)

$$y = y(T), \quad T \geq \hat{T}_{\mathrm{h}} \,. \tag{60}$$

Then, with

$$g = -\frac{r'}{r} + \frac{3}{2} \frac{\varrho'}{r^2} \, y \tag{61}$$

we may convert eq. (56) to the following form:

$$\frac{\mathrm{d}}{\mathrm{d}T} \left(\mathrm{e}^{\omega} \, \boldsymbol{\Phi}^* \right) = \varrho' \, \mathrm{e}^{\omega} \, \boldsymbol{\tau}_0^*, \qquad \omega = \int_{\hat{T}_{\mathrm{h}}}^{T} g \, \mathrm{d}T \,. \tag{62}$$

Integrating eq. (62) from $\hat{T}_{\mathrm{h}}$ to T and using the initial value

$$\boldsymbol{\Phi}^*|_{T=\hat{T}_{\mathrm{h}}} = \varrho(\hat{T}_{\mathrm{h}}) \, \tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}_0^*$$

at initial yielding with $T = \hat{T}_{\mathrm{h}}$, we arrive at

$$\boldsymbol{\Phi}^* = \left(-\boldsymbol{\alpha}_0^* + \left(\varrho(\hat{T}_{\mathrm{h}}) + \int_{\hat{T}_{\mathrm{h}}}^{T} \varrho' \, \mathrm{e}^{\omega} \, \mathrm{d}T \right) \tilde{\boldsymbol{\tau}}_0^* \right) \mathrm{e}^{-\omega} \,. \tag{63}$$

On the other hand, with eqs. (55) and (61), from eqs. (50)-(52) we deduce

$$\frac{\mathrm{d}\boldsymbol{h}^*}{\mathrm{d}T} = \beta \, \boldsymbol{I} + \frac{g}{c} \, \boldsymbol{\Phi}^* \,. \tag{64}$$

Integrating this equation from $\hat{T}_{\mathrm{h}}$ to T and using eqs. (63) and (36), we obtain the deformation

$$\boldsymbol{h}^* = \beta \, (T - T_0) \, \boldsymbol{I} + \boldsymbol{h}_0^* + p \, \boldsymbol{\alpha}_0^* + q \, \tilde{\boldsymbol{\tau}}_0^*, \quad T \geq \hat{T}_{\mathrm{h}} \,, \tag{65}$$

In the above,

$$p = -\int_{\hat{T}_{\mathrm{h}}}^{T} \frac{g}{c} \, \mathrm{e}^{-\omega} \, \mathrm{d}T, \quad q = \int_{\hat{T}_{\mathrm{h}}}^{T} \frac{g}{c} \left(\varrho(\hat{T}_{\mathrm{h}}) + \int_{\hat{T}_{\mathrm{h}}}^{T} \varrho' \, \mathrm{e}^{\omega} \, \mathrm{d}T \right) \mathrm{e}^{-\omega} \, \mathrm{d}T, \tag{66}$$

and ω is given with eq. (62)$_2$.

A reduced form of the above solution as given by eqs. (65) and (66) may be derived by means of eq. (58). To this end, we rewrite eq. (58) into

$$(y\,\mathrm{e}^\omega)' = \varrho'\,\mathrm{e}^\omega\,|\tilde{\boldsymbol\tau}_0^*|^2\,.$$

From this and the initial value given by eq. (59), we obtain

$$\left(\varrho(\hat{T}_\mathrm{h}) + \int_{\hat{T}_\mathrm{h}}^T \varrho'\,\mathrm{e}^\omega\,\mathrm{d}\,T\right)|\tilde{\boldsymbol\tau}_0^*|^2 = y\,\mathrm{e}^\omega + \boldsymbol\alpha_0^*:\tilde{\boldsymbol\tau}_0^*\,. \tag{67}$$

Substituting this in eq. $(66)_2$ gives

$$|\tilde{\boldsymbol\tau}_0^*|^2\,q = \int_{\hat{T}_\mathrm{h}}^T \frac{g}{c}\left(y + (\boldsymbol\alpha_0^*:\tilde{\boldsymbol\tau}_0^*)\,\mathrm{e}^{-\omega}\right)\mathrm{d}\,T\,. \tag{68}$$

Thus, expression (65) may be recast in the form

$$\boldsymbol h^* = \beta\,(T-T_0) + \boldsymbol h_0^* + p\,(\boldsymbol\alpha_0^* - (\boldsymbol\alpha_0^*:[\tilde{\boldsymbol\tau}_0^*])\,[\tilde{\boldsymbol\tau}_0^*])$$
$$+ \left(\int_{\hat{T}_\mathrm{h}}^T \frac{g}{c}\frac{y}{|\tilde{\boldsymbol\tau}_0^*|}\,\mathrm{d}\,T\right)[\tilde{\boldsymbol\tau}_0^*], \qquad T \geq \hat{T}_\mathrm{h}\,. \tag{69}$$

4.2 Plastic flow induced at cooling

Now, we study the cooling process ($\dot{T} < 0$) from $T = T_1$ back to $T = T_0$, immediately following the heating from T_0 to T_1. In the following we will study the possibility of reverse plastic flow in the process of cooling, namely, unloading will take place from T_1 up to a certain temperature $T_\mathrm{c} < T_1$ and then reverse plastic flow will be induced starting from T_c.

The reverse plastic flow may be induced only in the case when the loading condition as given in eq. $(15)_1$ is fulfilled. Moreover, it is noted that the yield condition $f = 0$ is met at $T = T_1$, i.e., at the end of heating and the outset of cooling. This means that the yield condition $f = 0$ should be met at both $T = T_1$ and $T = T_\mathrm{c}$ and, besides, that the loading condition, i.e., $f = 0$ and $\hat{f}/h > 0$ for $T \leq T_\mathrm{c}$, should further be satisfied.

At the outset of cooling, i.e. at the end of heating, we have

$$\frac{1}{2}\,|\varrho(T_1)\,\tilde{\boldsymbol\tau}_0^* - \boldsymbol\alpha_1^*|^2 - \frac{1}{3}\,(r(T_1))^2 = 0\,. \tag{70}$$

and at the start of the reverse plastic flow

$$\frac{1}{2}\,|\varrho(T_\mathrm{c})\,\tilde{\boldsymbol\tau}_0^* - \boldsymbol\alpha_1^*|^2 - \frac{1}{3}\,(r(T_\mathrm{c}))^2 = 0, \qquad T_\mathrm{c} < T_1\,. \tag{71}$$

Herein, $\boldsymbol{\alpha}_1^*$ is the value of $\boldsymbol{\alpha}^*$ at $T = T_1$, i.e.,

$$\boldsymbol{\alpha}_1^* = \boldsymbol{\alpha}^*|_{T=T_1} = \varrho(T_1)\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\Phi}^*|_{T=T_1}\,.$$

Hence, by using eqs. (63) and (67) we obtain

$$\boldsymbol{\alpha}_1^* = \left(\varrho_1 - \frac{y_1}{|\tilde{\boldsymbol{\tau}}_0^*|^2}\right)\tilde{\boldsymbol{\tau}}_0^* + \mathrm{e}^{-\omega_1}\left(\boldsymbol{\alpha}_0^* - (\boldsymbol{\alpha}_0^* : [\tilde{\boldsymbol{\tau}}_0^*])\,[\tilde{\boldsymbol{\tau}}_0^*]\right), \qquad \omega_1 = \int_{\hat{T}_\mathrm{h}}^{T_1} g(T)\,\mathrm{d}\,T. \tag{72}$$

From T_1 to T_c, unloading occurs and the back stress $\boldsymbol{\alpha}$ keeps constant, i.e., $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1$. Therefore, we have

$$\frac{1}{2}\,|\varrho(T)\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}_1^*|^2 - \frac{1}{3}r^2(T) < 0, \qquad T_\mathrm{c} < T < T_1\,. \tag{73}$$

Furthermore, from eq. (45) and $\dot{T} < 0$ we derive the loading condition for the reverse plastic flow starting from $T = T_\mathrm{c}$ as follows:

$$\begin{cases} \dfrac{1}{2}\,|\varrho(T)\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^*|^2 - \dfrac{1}{3}\,(r(T))^2 = 0, & T \leq T_\mathrm{c}\,, \\[3mm] \varrho'\,(\varrho\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^*) : \boldsymbol{\tau}_0^* - \dfrac{2}{3}\,r\,r' < 0, & T \leq T_\mathrm{c}\,. \end{cases} \tag{74}$$

The occurrence of reverse plastic flow in the cooling process is guaranteed by the three conditions[2] (70), (71) and (74). On the other hand, three conditions have been presented for the emergence of plastic flow in a process of heating, as given by eqs. (47)–(49). Then the question arises, whether all these can be satisfied? Of them, conditions $(49)_2$ and $(74)_2$ may be essential, while the others simply present conditions for yielding and are already incorporated into the governing equations for plastic flow. Here, the thermoinducer $\varrho = \varrho(T)$ would play an essential role in fulfilling the two essential conditions just indicated. In fact, even if the stress is kept constant, the term $\varrho(T)\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^*$ in the foregoing conditions is changing in a process of either heating or cooling. Accordingly, all the foregoing conditions may be met by suitably choosing the factor $\varrho = \varrho(T)$, as will be seen next section. This may readily be understood

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[2]Condition (73) will be automatically satisfied and hence will no longer be mentioned. In fact, T_1 marks the end of the heating process and also the start of the cooling process. For $T = T_1$ as the end of heating, we have $\hat{f} > 0$ with $\dot{T} > 0$. Since here $\hat{f}$ is given by $(45)_2$, we have $\hat{f} < 0$ with $\dot{T} < 0$ for $T = T_1$ as the start of cooling. Hence, unloading occurs and we have $\dot{f} = \hat{f} < 0$ at $T = T_1$ as the start of cooling. Moreover, we have either $f < 0$ or $f > 0$ for $T_\mathrm{c} < T < T_1$. From this and $\dot{f} < 0$ at $T = T_1$, we infer that $f < 0$ for $T_\mathrm{c} < T < T_1$.

by comparing the reverse plastic flow induced in a bar simply by changing the axial stress in isothermal case.

Suppose that reverse plastic flow may be induced in the cooling process from T_1 to T_0. Now we find out the deformation in this process. For the unloading process from T_1 to T_c, the deformation is given by

$$\boldsymbol{h}^* = \beta\,(T - T_1) + \boldsymbol{h}_1^*, \qquad T \in [T_c, T_1]. \tag{75}$$

Here $\boldsymbol{h}_1^*$ is obtained by setting $T = T_1$ in eq. (69), viz.

$$\boldsymbol{h}_1^* = \boldsymbol{h}_0^* - \left(\int_{\hat{T}_h}^{T_1} \frac{g}{c}\,\mathrm{e}^{-\omega}\,\mathrm{d}\,T\right)(\boldsymbol{\alpha}_0^* - (\boldsymbol{\alpha}_0^* : [\tilde{\boldsymbol{\tau}}_0^*])\,[\tilde{\boldsymbol{\tau}}_0^*])$$

$$+ \left(\int_{\hat{T}_h}^{T_1} \frac{g}{c}\,\frac{y_h(T)}{|\tilde{\boldsymbol{\tau}}_0^*|}\,\mathrm{d}\,T\right)[\tilde{\boldsymbol{\tau}}_0^*]. \tag{76}$$

The reverse plastic flow starting from $T = T_c$ is governed also by eqs. (50)–(53), but with the initial conditions given by

$$\boldsymbol{\alpha}^*|_{T=T_c} = \boldsymbol{\alpha}_1^*, \qquad \boldsymbol{h}^*|_{T=T_c} = \beta\,(T_c - T_0)\,\boldsymbol{I} + \boldsymbol{h}_1^*. \tag{77}$$

Following the same procedures as in the last subsection, we derive the following initial value problem of Riccati equation:

$$y' = \left(\frac{r'}{r} - \frac{3}{2}\frac{\varrho'}{r^2}\,y\right)y + \varrho'\,|\tilde{\boldsymbol{\tau}}_0^*|^2, \quad y_c = y|_{T=T_c} = \varrho(T_c)\,|\tilde{\boldsymbol{\tau}}_0^*|^2 - \boldsymbol{\tau}_0^* : \boldsymbol{\alpha}_1^*, \tag{78}$$

with y defined by eqs. (55) and (57). From this we obtain a solution (see Appendix 8)

$$\bar{y} = \bar{y}(T), \qquad T \leq T_c. \tag{79}$$

After that, the same procedure as used in deriving eq. (62) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\,T}\left(\mathrm{e}^{\bar{\omega}}\,\boldsymbol{\Phi}^*\right) = \varrho'\,\mathrm{e}^{\bar{\omega}}\,\boldsymbol{\tau}_0^*, \tag{80}$$

but here with the $\bar{\omega}$ and the initial values given by

$$\bar{\omega} = \int_{T_c}^{T} \bar{g}\,\mathrm{d}\,T, \quad \bar{g} = -\frac{r'}{r} + \frac{3}{2}\frac{\varrho'}{r^2}\,\bar{y}, \quad \boldsymbol{\Phi}^*|_{T=T_c} = \varrho(T_c)\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}_1^*. \tag{81}$$

Hence we arrive at

$$\boldsymbol{\Phi}^* = \left(-\boldsymbol{\alpha}_1^* + \left(\varrho(T_\mathrm{c}) + \int_{T_\mathrm{c}}^{T} \varrho' \, \mathrm{e}^{\bar{\omega}} \, \mathrm{d}\,T\right) \tilde{\boldsymbol{\tau}}_0^*\right) \mathrm{e}^{-\bar{\omega}}, \qquad T \leq T_\mathrm{c} \,. \qquad (82)$$

Finally, we come to the differential equation as given by eq. (64). From this and the initial values given by eq. (77) as well as eq. (82), we obtain

$$\boldsymbol{h}^* = \beta \, (T - T_0) \, \boldsymbol{I} + \boldsymbol{h}_1^* + (\bar{p} \, \boldsymbol{\alpha}_1^* + \bar{q} \, \tilde{\boldsymbol{\tau}}_0^*) \,, \qquad T \leq T_\mathrm{c} \,, \qquad (83)$$

In the above,

$$\bar{p} = -\int_{T_\mathrm{c}}^{T} \frac{\bar{g}}{c} \, \mathrm{e}^{-\bar{\omega}} \, \mathrm{d}\,T \,, \quad \bar{q} = \int_{T_\mathrm{c}}^{T} \frac{\bar{g}}{c} \left(\varrho(T_\mathrm{c}) + \int_{T_\mathrm{c}}^{T} \varrho' \, \mathrm{e}^{\bar{\omega}} \, \mathrm{d}\,T\right) \mathrm{e}^{-\bar{\omega}} \, \mathrm{d}\,T \,. \qquad (84)$$

Following the same procedure as used in deriving (69), we derive a reduced form of the above solution as given by eqs. (83) and (84) as follows:

$$\begin{aligned} \boldsymbol{h}^* = & \beta \, (T - T_0) + \boldsymbol{h}_1^* + \bar{p} \, (\boldsymbol{\alpha}_1^* - (\boldsymbol{\alpha}_1^* : [\tilde{\boldsymbol{\tau}}_0^*]) \, [\tilde{\boldsymbol{\tau}}_0^*]) \\ & + \left(\int_{T_\mathrm{c}}^{T} \frac{\bar{g}}{c} \, \frac{\bar{y}}{|\tilde{\boldsymbol{\tau}}_0^*|} \, \mathrm{d}\,T\right) [\tilde{\boldsymbol{\tau}}_0^*], \quad T \leq T_\mathrm{c} \,, \end{aligned} \qquad (85)$$

where $\boldsymbol{\alpha}_1^*$ and $\boldsymbol{h}_1^*$ are given by eqs. (73) and (76), respectively.

5 One- and two-way shape memory effects

Results in the last two sections suggest that plastic flow may possibly be induced either in a process of heating or in a cyclic process of heating/cooling. However, both the emergence and the magnitude of such thermoinduced plastic flow relies essentially on how the yield stress r changes with temperature. Usually, this change is not appreciable for usual metals and alloys. This implies that a considerable change in temperature would be needed for ensuring a transition from condition (23) at initial temperature T_0 to condition (25) at temperature T_h at the start of yielding. And even after such a considerable temperature change, the induced plastic flow would not be appreciable and even negligible for vanishingly small r'. Then, no noticeable thermoinduced plastic flow could be observed.

Whenever the yield stress r may markedly change with temperature in a certain range, however, appreciable thermoinduced plastic flow may emerge in this temperature range as natural consequence of elastoplastic models. It may

probably be intriguing to notice that such thermoinduced plastic flow turns out to exhibit one- and two-way memory effects, as observed in certain shape memory materials (see, e.g., [2–4]). The main objective of this section is to uncover natural relationships between thermoinduced plastic flows and shape memory effects.

5.1 One-way memory

As in section 3, plastic deformation $\boldsymbol{F}_0$ and initial back stress $\boldsymbol{\alpha}_0$ are generated by a loading-unloading process in an elastoplastic body and, then, this body is heated from temperature T_0 to T_1, in the absence of stress. Starting from temperature T_h meeting eq. (26), plastic flow is induced. The plastic deformation at each $T \geq T_\mathrm{h}$ is given by the last two terms in eq. (43). Of them, the last term results from the thermoinduced plastic flow. Since $c > 0$, it may be evident that the coefficient in the last term is always positive for any yield stress r decreasing with increasing temperature. From this we come to the following conclusion:

> The strain recovery or shape memory effect in a heating process turns out to be a natural consequence implied by a decreasing yield stress $r = r(T)$.

To know to what extent the strain will be recovered, we shall evaluate the magnitude of the plastic strain given in eq. (43). Towards this goal, let

$$\boldsymbol{A} = \boldsymbol{h}_0^* + x\,[\boldsymbol{\alpha}_0^*]\,, \tag{86}$$

with

$$x = \sqrt{\frac{2}{3}} \int_{T_\mathrm{h}}^{T} \frac{r'}{c}\,\mathrm{d}\,T\,. \tag{87}$$

In deriving the above, eq. (26) has been used. Since $r' < 0$ and $c > 0$, the scalar factor x in eq. (86) is negative and, starting from 0, the magnitude $|x|$ is growing as heating is in progress. Thus, the thermoinduced plastic flow in a process of heating, i.e. the second term in eq. (86), turns out to display strain recovery or shape memory effect.

Indeed, the above fact may be rendered clear by studying the magnitude of the plastic deformation given by eq. (86). We have

$$|\boldsymbol{A}|^2 = (x + \boldsymbol{h}_0^* : [\boldsymbol{\alpha}_0^*])^2 + |\boldsymbol{h}_0^*|^2 - (\boldsymbol{h}_0^* : [\boldsymbol{\alpha}_0^*])^2\,. \tag{88}$$

It may be clear that the minimum of the magnitude $|\boldsymbol{A}|$ is given by

$$\min |\boldsymbol{A}|^2 = |\boldsymbol{h}_0^*|^2 - (\boldsymbol{h}_0^* : [\boldsymbol{\alpha}_0^*])^2 = |\boldsymbol{h}_0^*|^2 \left(1 - ([\boldsymbol{h}_0^*] : [\boldsymbol{\alpha}_0^*])^2\right), \tag{89}$$

at

$$x = -\boldsymbol{h}_0^* : [\boldsymbol{\alpha}_0^*],$$

namely,

$$\int_{T_{\rm h}}^{T} \frac{r'}{c} \, {\rm d}\,T = -\sqrt{\frac{3}{2}} \, \boldsymbol{h}_0^* : [\boldsymbol{\alpha}_0^*] \,. \tag{90}$$

The minimum given by eq. (89) is always smaller than the magnitude $|\boldsymbol{h}_0^*|$, since

$$([\boldsymbol{h}_0^*] : [\boldsymbol{\alpha}_0^*])^2 \leq 1.$$

From the above it follows that, as heating is in progress, the magnitude of the total plastic strain $|\boldsymbol{A}|$ is monotonically decreasing from its initial value $|\boldsymbol{h}_0^*|$ and eventually attains its minimum as given by eq. (89). This may suggest that the thermoinduced plastic flow is going back towards recovering the original shape as heating is going on, thus explaining the shape memory effect. The smaller the minimum $\min|\boldsymbol{A}|$, the more appreciable the strain recovery or shape memory effect. The perfect recovery or memory will happen when

$$\min |\boldsymbol{A}| = 0, \qquad \text{i.e.} \qquad ([\boldsymbol{h}_0^*] : [\boldsymbol{\alpha}_0^*])^2 = 1.$$

The latter leads to

$$\boldsymbol{\alpha}_0 = c_0 \, \boldsymbol{h}_0 \,. \tag{91}$$

This relation holds always true in uniaxial strain case. Generally, it is noted that this relationship comes just from the simplest anisotropic hardening rule, viz. Prager's hardening rule (8) with constant modulus $c = c_0$ in isothermal case. In fact, with negligible elastic deformation and with $c = c_0$, the hardening rule (8) becomes

$$\overset{\circ}{\boldsymbol{\alpha}} = c_0 \, \boldsymbol{D} \,, \tag{92}$$

in an isothermal loading process preceding the heating. Then, utilizing eqs. (33) and (35), we deduce

$$\overline{\dot{\boldsymbol{\alpha}^*}} = c_0 \, \overline{\dot{\boldsymbol{h}^*}} \,. \tag{93}$$

Thus, integrating the latter with an original natural state we derive eq. (91). Moreover, for perfect strain recovery, i.e., for perfect shape memory, eq. (90) reduces to

$$-\sqrt{\frac{2}{3}} \int_{T_{\mathrm{h}}}^{T} \frac{r'}{c} \, \mathrm{d}\,T = |\boldsymbol{h}_0^*| \; . \tag{94}$$

Both conditions (26) for yielding and (94) for perfect recovery introduce certain requirements for the forms of the yield stress $r = r(T)$ and Prager's modulus $c = c(T)$. In fact, the simplest case for anisotropic hardening is when modulus c is constant. This would imply that the anisotropic hardening behaviour would never be thermosensitive but the opposite. In this case, we have $c = c_0$. Then, the condition (94) for perfect strain recovery yields

$$r(T_{\mathrm{h}}) - r(T) = \sqrt{1.5} \, c_0 \, |\boldsymbol{h}_0^*|.$$

This and eqs. (26) and (91) together result in a vanishing yield stress $r(T) = 0$, which would not be possible except for a temperature close to the melting point.

From the above we may conclude that, according to eq. (94), *a thermosensitive anisotropic hardening modulus c would be essential for ensuring perfect strain recovery (shape memory).* In fact, modulus c should be monotonically increasing with increasing temperature. This means that, within a certain temperature range, the induced anisotropy should become more and more appreciable with increasing temperature.

In summary, one-way shape memory (strain recovery) effects may be natural consequences of thermocoupled elastoplastic J_2-flow models with thermosensitive yielding and hardening behaviour. Of course, such thermosensitive behaviour usually manifests itself within a certain temperature range and, accordingly, shape memory effects may be observed only in such range. Beyond this range emerge simply usual effects of elastoplastic deformations.

5.2 Two-way memory

Now we explain the two-way thermoinduced plastic flow found in section 4. We therefore explore whether *a two-way thermoinduced plastic flow in a cyclic thermal process will give rise to a strain cycle (hysteresis)*, such that in a process of heating from T_0 to T_1 the thermoinduced plastic flow leads to a strain recovery from the initial plastic state at T_0 to an original state and, then, in a succeeding process of cooling from T_1 to T_0 the reverse thermoinduced plastic flow results in a return to the initial strain at T_0.

First, we study the plastic flow during heating from T_0 to T_1. As has been shown in the last subsection, the relation as given by eq. (91) leads to the maximum strain recovery. With this relation and the solution given by eqs. (65) and (66), the plastic strain at the end of heating, i.e. at $T = T_1$, is given by

$$\boldsymbol{h}_1^* = (1 + c_0\,p)\,\boldsymbol{h}_0^* + q\,\tilde{\boldsymbol{\tau}}_0^*\,, \tag{95}$$

where p and q are given by eq. (66). The minimum of the magnitude $|\boldsymbol{h}_1^*|$ may be attained whenever $\boldsymbol{\tau}_0^*$ and $\boldsymbol{h}_0^*$ are linearly dependent, namely,

$$\tilde{\boldsymbol{\tau}}_0^* = \tau_0\,[\boldsymbol{h}_0^*]\,. \tag{96}$$

This and eq. (91) require that any two of the three tensors $\boldsymbol{h}_0^*$, $\boldsymbol{\alpha}_0^*$, $\tilde{\boldsymbol{\tau}}_0^*$ should be linearly dependent. Now expression (76) for the plastic strain $\boldsymbol{h}_1^*$ at the end of heating reduces to

$$\boldsymbol{h}_1^* = \left(|\boldsymbol{h}_0| + \int_{\hat{T}_{\mathrm{h}}}^{T_1} \frac{g}{c}\,\frac{y}{|\tau_0|}\;\mathrm{d}\,T\right)[\boldsymbol{h}_0^*]\,. \tag{97}$$

Using the equality

$$g = \frac{3}{2}\,\frac{1}{r^2}\left(\varrho'\,(\varrho\,\tilde{\boldsymbol{\tau}}_0^* - \boldsymbol{\alpha}^*) : \boldsymbol{\tau}_0^* - \frac{2}{3}\,r\,r'\right)\,,$$

we deduce the following equivalence relation:

$$(49)_2 \iff g > 0\,. \tag{98}$$

Then, from this and eq. (97) it may be clear that strain recovery at heating is possible whenever

$$y < 0\,, \qquad \hat{T}_{\mathrm{h}} \le T \le T_1\,. \tag{99}$$

Here, g and y may further be simplified by using the two relations shown by eqs. (91) and (96). In fact, the solution of the Riccati equation (58) with (59) is of the form (see Appendix 8)

$$y = \sqrt{\frac{2}{3}}\,r\,|\boldsymbol{\tau}_0|\,\mathrm{sgn}(y_{\mathrm{h}})\,, \tag{100}$$

with

$$y_{\mathrm{h}} = \tau_0\,(\varrho(\hat{T}_{\mathrm{h}})\,\tau_0 - c_0\,|\boldsymbol{h}_0|)\,. \tag{101}$$

This and eq. (61) produce

$$g = -\frac{r'}{r} + \sqrt{\frac{3}{2}} \frac{\varrho'}{r} |\boldsymbol{\tau}_0| \, \mathrm{sgn}(y_{\mathrm{h}}) \,. \tag{102}$$

Thus, condition (99) with (100) leads to $\mathrm{sgn}(y_{\mathrm{h}}) = -1$, i.e.

$$\tau_0 \left(\varrho(\hat{T}_{\mathrm{h}}) \, \tau_0 - c_0 \, |\boldsymbol{h}_0| \right) \leq 0 \,. \tag{103}$$

Then, condition (98) results in

$$-r' - \sqrt{1.5} \, \varrho' \, |\tau_0| > 0, \qquad \hat{T}_{\mathrm{h}} \leq T \leq T_1 \,. \tag{104}$$

Moreover, by using (103) we infer that condition (48) for the start of plastic flow at heating reduces to

$$\varrho(\hat{T}_{\mathrm{h}}) \, \tau_0 - c_0 \, |\boldsymbol{h}_0| = -\sqrt{\frac{2}{3}} \, r(\hat{T}_{\mathrm{h}}) \, \mathrm{sgn}(\tau_0) \,, \tag{105}$$

which specifies the temperature $\hat{T}_{\mathrm{h}}$.

With conditions (104) and (105), both thermoinduced plastic flow and strain recovery are possible at heating. Now, expression (97) for the plastic flow at heating is of the form:

$$\boldsymbol{h}^* = \beta \, (T - T_0) \, \boldsymbol{I}$$
$$+ \left(|\boldsymbol{h}_0^*| - \sqrt{\frac{2}{3}} \int_{\hat{T}_{\mathrm{h}}}^{T} \frac{-r' - \sqrt{1.5} \, |\tau_0| \, \varrho'}{c} \, \mathrm{d}T \right) [\boldsymbol{h}_0^*], \quad T \in [\hat{T}_{\mathrm{h}}, T_1] \,. \tag{106}$$

At $T = T_1$ we have

$$h_{2\mathrm{W}} = \sqrt{\frac{2}{3}} \int_{\hat{T}_{\mathrm{h}}}^{T_1} \frac{-r' - \sqrt{1.5} \, |\tau_0| \, \varrho'}{c} \, \mathrm{d}T \,. \tag{107}$$

In expression (106), the contribution to the strain from the last term is always negative (cf. eq. (104)) and hence indicating a strain recovery effect. As will be seen, the second presents the maximum magnitude of the strain change in a cyclic process displaying two-way memory effect.

Next, we study the plastic flow during cooling from T_1 to T_0. With relations (91) and (96) as well as eq. (100) with $\mathrm{sgn}(y_{\mathrm{h}}) = -1$, we infer that eq. $(72)_1$ reduces to

$$\boldsymbol{\alpha}_1^* = c_1 \, [\boldsymbol{h}_0^*],$$

with

$$c_1 = \varrho_1 \tau_0 + \sqrt{\frac{2}{3}} \, r_1 \, \text{sgn}(\tau_0).$$

From this and eq. (96) we deduce

$$\boldsymbol{\alpha}_1^* - (\boldsymbol{\alpha}_1^* : [\tilde{\boldsymbol{\tau}}_0^*]) \, [\tilde{\boldsymbol{\tau}}_0^*] = \mathbf{0},$$

together with eq. (85) leading to

$$\boldsymbol{h}^* = \beta \, (T - T_0) \, \boldsymbol{I} + \left(|\boldsymbol{h}_0^*| - h_{2\mathrm{W}} + \int_{T_\mathrm{c}}^{T} \frac{\bar{g}}{c} \, \frac{\bar{y}}{|\tilde{\boldsymbol{\tau}}_0^*|} \, \mathrm{d}T \right) [\boldsymbol{h}_0^*], \quad T \le T_\mathrm{c} \quad (108)$$

for the plastic flow at cooling. The following equivalence relation holds true:

$$(74)_2 \iff g < 0. \tag{109}$$

From this and eq. (108) we infer that the reverse plastic flow at cooling goes back towards the initial strain whenever

$$\bar{y} > 0, \quad T_0 \le T \le T_\mathrm{c}. \tag{110}$$

With eqs. (91) and (96), the solution $\bar{y}$ of Riccati equation (78) is given by

$$\bar{y} = \sqrt{\frac{2}{3}} \, r \, |\boldsymbol{\tau}_0| \, \text{sgn}(y_\mathrm{c}), \qquad y_\mathrm{c} = \tau_0 \, (\varrho(T_\mathrm{c}) \, \tau_0 - c_1). \tag{111}$$

From this and eq. $(81)_2$ we have

$$\bar{g} = -\frac{r'}{r} + \sqrt{\frac{3}{2}} \, \frac{\varrho'}{r} \, |\boldsymbol{\tau}_0| \, \text{sgn}(y_\mathrm{c}). \tag{112}$$

Hence, condition (110) gives $\text{sgn}(y_\mathrm{c}) = 1$, i.e.

$$\tau_0 \, (\varrho(T_\mathrm{c}) \, \tau_0 - c_1) > 0,$$

i.e.

$$(\varrho(T_\mathrm{c}) - \varrho(T_1)) \, |\tau_0| > \sqrt{\frac{2}{3}} \, r(T_1), \tag{113}$$

and then condition (109) is of the form:

$$r' - \sqrt{1.5} \, \varrho' \, |\tau_0| > 0, \qquad T_0 \le T \le T_\mathrm{c}. \tag{114}$$

In addition, the start temperature T_c of reverse plastic flow at cooling is determined by (cf. (71))

$$\varrho(T_c)\,\tau_0 - c_1 = \sqrt{\frac{2}{3}}\,r(T_c)\,\mathrm{sgn}(\tau_0),$$

namely,

$$(\varrho(T_c) - \varrho(T_1))\,|\tau_0| = \sqrt{\frac{2}{3}}\,(r(T_1) + r(T_c)).\tag{115}$$

With conditions (113) and (114), reverse thermoinduced plastic flow with strain recovery is possible at cooling. Now, we have

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I}\tag{116}$$
$$+ \left(|\boldsymbol{h}_0^*| - h_{2\mathrm{W}} + \sqrt{\frac{2}{3}}\int_T^{T_c}\frac{r' - \sqrt{1.5}\,|\tau_0|\,\varrho'}{c}\,\mathrm{d}\,T\right)[\boldsymbol{h}_0^*],\quad T \in [T_0,\,T_c]\,,$$

for the plastic flow at cooling. Let the plastic strain at the return of temperature to T_0 be $\hat{\boldsymbol{h}}_0^*$. Then, setting $T = T_0$ in the above we obtain

$$\hat{\boldsymbol{h}}_0^* = \boldsymbol{h}_0^* + \left(\sqrt{\frac{2}{3}}\int_{T_0}^{T_c}\frac{r' - \sqrt{1.5}\,|\tau_0|\,\varrho'}{c}\,\mathrm{d}\,T - h_{2\mathrm{W}}\right)[\boldsymbol{h}_0^*].\tag{117}$$

Thus, from this and eq. (107) it follows that the condition for generating a strain cycle in a cyclic thermal process is given by

$$\int_{\hat{T}_h}^{T_1}\frac{-r' - \sqrt{1.5}|\,\tau_0|\,\varrho'}{c}\,\mathrm{d}\,T = \int_{T_0}^{T_c}\frac{r' - \sqrt{1.5}\,|\tau_0|\,\varrho'}{c}\,\mathrm{d}\,T\,.\tag{118}$$

With this condition, eqs. (106) and (116) may be recast in the following forms:

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I}\tag{119}$$
$$+ \left(|\boldsymbol{h}_0^*| - h_{2\mathrm{W}}\frac{\int_{\hat{T}_h}^{T}\frac{-r' + \sqrt{1.5}\,|\tau_0|\,\varrho'}{c}\,\mathrm{d}\,T}{\int_{\hat{T}_h}^{T_1}\frac{-r' + \sqrt{1.5}\,|\tau_0|\,\varrho'}{c}\,\mathrm{d}\,T}\right)[\boldsymbol{h}_0^*],\quad T \in [\hat{T}_h,\,T_1]$$

for the plastic flow at heating, and

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I}\tag{120}$$
$$+ \left(|\boldsymbol{h}_0^*| - h_{2\mathrm{W}}\frac{\int_{T_0}^{T}\frac{r' - \sqrt{1.5}\,|\tau_0|\,\varrho'}{c}\,\mathrm{d}\,T}{\int_{T_0}^{T_c}\frac{r' - \sqrt{1.5}\,|\tau_0|\,\varrho'}{c}\,\mathrm{d}\,T}\right)[\boldsymbol{h}_0^*],\quad T \in [T_0,\,T_c]$$

for the plastic flow at cooling.

Eqs. (119) and (120) supply explicit closed-form solutions for two-way memory effects in cyclic thermal processes. These solutions have been derived in a broad sense for multiaxial finite deformations and for any forms of the three constitutive functions r, c and ϱ, with the conditions (104) and (105) for plastic flow at heating and (114) and (115) for plastic flow at cooling as well as eq. (118) for cyclic strains. Choices of r, c and ϱ will provide sufficient possibilities in matching any shape of strain recovery loop for two-way memory. In the next section it will be shown that even a single function may be introduced to serve for this purpose.

6 Yield stress, Prager's modulus and the thermoinducer

The key role of the thermoinducer $\varrho = \varrho(T)$, introduced in an enhanced form of von Mises yield function in eq. (10), may be clear in the preceding study. Moreover, thermosensitive yield stress $r = r(T)$ and Prager's modulus $c = c(T)$ are also essential. Here, we proceed to find out further forms of the foregoing three quantities for describing two-way memory effects. The three quantities r, c and ϱ may be general functions of the temperature. We shall treat a simplified case when these three are given in terms of a single monotonically decreasing function of T. As will be seen, even this case is broad enough to characterise two-way memory effects.

6.1 Simplified conditions and general results

First, for the initial values $\boldsymbol{h}_0$, $\boldsymbol{\tau}_0$ and $\boldsymbol{\alpha}_0$, we have

$$\boldsymbol{\tau}_0 = \tau_0 \left[\boldsymbol{h}_0\right], \qquad \boldsymbol{\alpha}_0 = \alpha_0 \left[\boldsymbol{h}_0\right], \qquad \alpha_0 = c_0 \left|\boldsymbol{h}_0\right| > 0 \,.$$

Next, from eq. (103) with $c_0 > 0$ we infer that $\tau_0 > 0$, where the trivial case $\tau_0 = 0$ has been neglected. Then, it may be shown that, with $\tau_0 > 0$, condition (103) may be derived from condition (105) and will no longer be needed. That is also the case for condition (113), which may be derived from condition (115). As a consequence, the six conditions given by eqs. (103)–(105) and (113)–(115) may be reduced to eqs. (104) and (105) as well as (114) and

(115) with $\tau_0 > 0$, namely,

$$\begin{cases} \varrho(\hat{T}_{\mathrm{h}})\,\tau_0 - \alpha_0 = -\sqrt{\dfrac{2}{3}}\,r(\hat{T}_{\mathrm{h}})\,, \\ -r' - \sqrt{1.5}\,\varrho'\,\tau_0 > 0\,, \qquad \hat{T}_{\mathrm{h}} \le T \le T_1\,; \end{cases} \qquad (121)$$

$$\begin{cases} (\varrho(T_{\mathrm{c}}) - \varrho(T_1))\,\tau_0 = \sqrt{\dfrac{2}{3}}\,(r(T_{\mathrm{c}}) + r(T_1))\,, \\ r' - \sqrt{1.5}\,\varrho'\,\tau_0 > 0\,, \qquad T_0 \le T \le T_{\mathrm{c}}\,. \end{cases} \qquad (122)$$

Now it is possible to describe the property of the thermoinducer $\varrho = \varrho(T)$. Since $r' < 0$, condition (122) implies that

$$\varrho' < 0\,, \qquad T_0 \le T \le T_{\mathrm{c}},$$

while ϱ', according to condition (121), may be either negative or positive outside the temperature range (T_0, T_{c}).

Generally, it may be demonstrated that the thermoinducer $\varrho = \varrho(T)$ fulfilling conditions (121) and (122) may be found for any given monotonically decreasing yield stress $r = r(T) > 0$. In fact, the latter may be given by

$$\frac{r}{r_0} = 1 - \gamma\,F\,, \qquad (123)$$

where $F = F(T)$ is any chosen function of the following properties:

$$F' \ge 0\,, \qquad 0 \le F \le 2\,, \qquad (124)$$

and γ is a dimensionless parameter meeting the condition

$$0 < \gamma < \frac{1}{2}\,. \qquad (125)$$

Let modulus $c = c(T)$ and thermoinducer $\varrho = \varrho(T)$ be of the forms:

$$\frac{c}{c_0} = 1 + \eta\,F\,, \qquad \eta > -\frac{1}{2}\,; \qquad (126)$$

$$\frac{\varrho}{\varrho_0} = 1 - \rho\,F\,, \qquad 0 < \rho < \frac{1}{2}\,. \qquad (127)$$

Here and henceforth, r_0, c_0 and ϱ_0 are positive constants. It may readily be shown that the c of the above form is monotonically increasing and positive, while the ϱ of the above form is monotonically decreasing and also positive.

Let

$$\tau_{0_0} = \varrho_0 \, \tau_0.$$

Then, conditions (121) and (122) yield

$$F(\hat{T}_{\mathrm{h}}) = \frac{1 + \sqrt{1.5} \, \frac{\tau_{0_0} - \alpha_0}{r_0}}{\gamma + \sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0}} \,, \tag{128}$$

$$F(T_{\mathrm{c}}) = \frac{\sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0} + \gamma}{\sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0} - \gamma} \, F(T_1) - \frac{2}{\sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0} - \gamma} \,, \tag{129}$$

$$\sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0} - \gamma > 0 \,. \tag{130}$$

Note that the second condition in eq. (121) is already satisfied. The first two above specify the two start temperatures $\hat{T}_{\mathrm{h}}$ and T_{c}.

Let $P(T_a, T_b)$ be defined by

$$P(T_a, T_b) = \ln \frac{1 + \eta \, F(T_a)}{1 + \eta \, F(T_b)}. \tag{131}$$

The deformation for the plastic flow at heating is given by (cf. eq. (106))

$$\boldsymbol{h}^* = \beta \, (T - T_0) \, \boldsymbol{I} \tag{132}$$

$$+ \left(|\boldsymbol{h}_0| - \sqrt{\frac{2}{3}} \, \frac{1}{\eta} \, \frac{r_0}{c_0} \left(\sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0} + \gamma \right) P(T, \hat{T}_{\mathrm{h}}) \right) [\boldsymbol{h}_0^*], \quad T \in [\hat{T}_{\mathrm{h}}, T_1] \,,$$

and the deformation for the plastic flow at cooling by (cf. eq. (116))

$$\boldsymbol{h}^* = \beta \, (T - T_0) \, \boldsymbol{I} \tag{133}$$

$$+ \left(|\boldsymbol{h}_0| - h_{2\mathrm{W}} - \sqrt{\frac{2}{3}} \, \frac{1}{\eta} \, \frac{r_0}{c_0} \left(\sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0} - \gamma \right) P(T, T_{\mathrm{c}}) \right) [\boldsymbol{h}_0^*], \quad T \in [T_0, T_{\mathrm{c}}] \,,$$

where the two-way strain change $h_{2\mathrm{W}}$ is given by (cf. eq. (107))

$$h_{2\mathrm{W}} = \sqrt{\frac{2}{3}} \, \frac{1}{\eta} \, \frac{r_0}{c_0} \left(\sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0} + \gamma \right) P(T_1, \hat{T}_{\mathrm{h}}) \,. \tag{134}$$

The condition for a strain cycle is as follows:

$$\left(\sqrt{1.5} \, \rho \, \frac{\tau_0}{r_0} + \gamma \right) P(T_1, \hat{T}_{\mathrm{h}}) = \left(\sqrt{1.5} \, \rho \, \frac{\tau_{0_0}}{r_0} - \gamma \right) P(T_{\mathrm{c}}, T_0) \,. \tag{135}$$

With this condition, eqs. (132) and (133) may be recast in the following forms (cf. eqs. (119) and (120)):

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I} + \left(|h_0| - h_{2\mathrm{W}}\,\frac{P(T,\hat{T}_{\mathrm{h}})}{P(T_1,\hat{T}_{\mathrm{h}})}\right)[\boldsymbol{h}_0^*], \quad T \in [\hat{T}_{\mathrm{h}},\,T_1]\,, \qquad (136)$$

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I} + \left(|h_0| - h_{2\mathrm{W}}\,\frac{P(T,T_0)}{P(T_{\mathrm{c}},T_0)}\right)[\boldsymbol{h}_0^*], \quad T \in [T_0,\,T_{\mathrm{c}}]\,. \qquad (137)$$

The above results may be derived by utilizing the following integrations:

$$\int_{\hat{T}_{\mathrm{h}}}^{T} \frac{-r' + \sqrt{1.5}\,\tau_{0_0}\,\varrho'}{c}\,\mathrm{d}T = \frac{r_0}{c_0}\left(\gamma + \sqrt{1.5}\,\frac{\tau_{0_0}}{r_0}\,\rho\right)\int_{\hat{T}_{\mathrm{h}}}^{T}\frac{F'\,\mathrm{d}T}{1 + \eta\,F}$$

$$= \frac{1}{\eta}\frac{r_0}{c_0}\left(\gamma + \sqrt{1.5}\,\frac{\tau_{0_0}}{r_0}\,\rho\right)\ln\frac{1 + \eta\,F(T)}{1 + \eta\,F(\hat{T}_{\mathrm{h}})},$$

$$\int_{T_{\mathrm{c}}}^{T} \frac{r' - \sqrt{1.5}\,\tau_{0_0}\,\varrho'}{c}\,\mathrm{d}T = \frac{r_0}{c_0}\left(-\gamma + \sqrt{1.5}\,\frac{\tau_{0_0}}{r_0}\,\rho\right)\int_{T_{\mathrm{c}}}^{T}\frac{F'\,\mathrm{d}T}{1 + \eta\,F}$$

$$= \frac{1}{\eta}\frac{r_0}{c_0}\left(\gamma - \sqrt{1.5}\,\frac{\tau_{0_0}}{r_0}\,\rho\right)\ln\frac{1 + \eta\,F(T)}{1 + \eta\,F(T_{\mathrm{c}})}.$$

Thus, it may be concluded from the above that if a function $F = F(T)$ of the properties as given by eq. (124) is chosen to ensure that eqs. (128) and (129) yield solutions for $\hat{T}_{\mathrm{h}}$ and T_{c} for suitable parameters $0 < \gamma < \frac{1}{2}$, $\eta > 0$ and $0 < \rho < \frac{1}{2}$ as well as suitable initial values τ_0 and α_0, then a strain cycle with two-way memory, as given by eqs. (136) and (137), is indeed possible in a process of heating/cooling. Examples will be presented next subsection.

It should be pointed out that these results are derived for any form of function $F = F(T)$ meeting eq. (124), but they are merely particular cases of the general results given last section. The above results may be extended to a case with a more general form of Prager's modulus c:

$$c = \Phi(F) > 0,$$

where $\Phi(F)$ may be any form of positive function. Clearly, eq. (126) provides only an example. In this extended case, eqs. (128)–(130) still hold true, and, moreover, simplified results may also be derived from eqs. (118)–(120). In fact, we have

$$\int \frac{\pm r' - \sqrt{1.5}\,\tau_{0_0}\,\varrho'}{c}\,\mathrm{d}T = (\pm\gamma r_0 + \sqrt{1.5}\,\rho\,\tau_{0_0})\int \frac{\mathrm{d}F}{\Phi(F)}.$$

Further development in this respect will not be pursued here.

6.2 Examples

Let $[T_0, T_1]$ be a temperature range of interest. In the following analysis, we introduce the dimensionless temperature:

$$\theta = \frac{2\,T - T_1 - T_0}{T_1 - T_0}\,.$$

It runs over the range $[-1, 1]$ when temperature T runs over the range $[T_0, T_1]$.

A general property for the three temperature-dependent constitutive quantities r, c and ϱ is as follows: each of them is changing appreciably with changing temperature within the range $[T_0, T_1]$, whereas outside this range their dependence on temperature is negligible in an idealised sense. With this in mind, as an illustrative example we set

$$F = F(T) = 1 + \tanh(\lambda\,\theta) \tag{138}$$

in the general results derived last subsection. Namely, we take into account the following forms of yield stress $r = r(T)$ and Prager's modulus $c = c(T)$ as well as thermoinducer $\varrho = \varrho(T)$:

$$\frac{r}{r_0} = 1 - \gamma\,(1 + \tanh(\lambda\,\theta))\,, \tag{139}$$

$$\frac{c}{c_0} = 1 + \eta\,(1 + \tanh(\lambda\,\theta))\,, \tag{140}$$

$$\frac{\varrho}{\varrho_0} = 1 - \rho\,(1 + \tanh(\lambda\,\theta))\,. \tag{141}$$

Herein, λ, γ, η and ρ are positive dimensionless constitutive constants. Of them, λ is greater than 1, so that each quantity displays a thermosensitivity property merely within temperature range $[T_0, T_1]$. Specifically, we have

$$\lambda > 1, \qquad 0 < \gamma < \frac{1}{2}, \qquad \eta > -\frac{1}{2}, \qquad 0 < \rho < \frac{1}{2}.$$

The property for the above three functions is shown in Figs. 1 and 2. It may be seen that each function displays the general property mentioned before. Note that both the yield stress r and the thermoinducer ϱ monotonically decrease with increasing temperature, whereas Prager's modulus c increases with increasing temperature. Their rates of changing with temperature are characterised by the parameters γ, η and ρ.

The derivatives of r and ϱ are given by

$$\frac{r'}{r_0} = -\xi\,\frac{\gamma}{(\cosh(\lambda\,\theta)^2}\,, \qquad \frac{\varrho'}{\varrho_0} = -\xi\,\frac{\rho}{(\cosh(\lambda\theta))^2}\,.$$
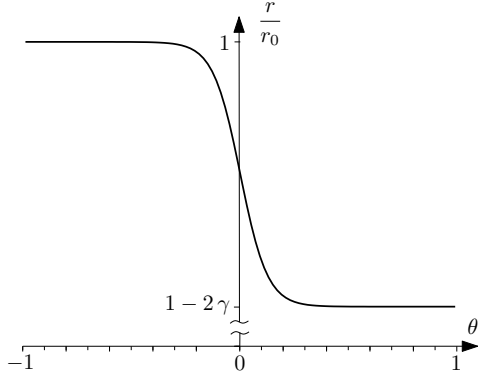
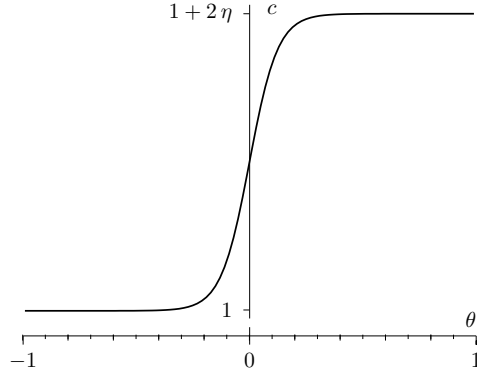Figure 1: Monotonically decreasing yield stress $r(T)$ (and, similarly, thermoinducer $\varrho(T)$)

Figure 2: Monotonically increasing Prager's modulus $c(T)$

Here $\xi = 2\,\lambda/(T_1 - T_0) > 0$. Hence, we have

$$-r' - \sqrt{1.5}\,\tau_{0_0}\,\varrho' = \xi\,\frac{\gamma\,r_0 + \sqrt{1.5}\,\rho\,\tau_{0_0}}{(\cosh(\lambda\theta))^2},$$

$$r' - \sqrt{1.5}\,\tau_0\,\varrho' = \xi\,\frac{-\gamma\,r_0 + \sqrt{1.5}\,\rho\,\tau_0}{(\cosh(\lambda\theta))^2}.$$

Evidently, the former is always positive, and the latter is also positive whenever

$$\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} > \gamma\,. \tag{142}$$

Now we derive the results for the two-way plastic flow for the constitutive quantities introduced. First, from condition $(121)_1$ we infer

$$\sqrt{1.5}\,\frac{\tau_{0_0}}{r_0}\left(1 - \rho - \rho\,\tanh(\lambda\hat\theta_{\mathrm h})\right) - \sqrt{1.5}\,\frac{\alpha_0}{r_0} = \gamma\left(1 + \tanh(\lambda\hat\theta_{\mathrm h})\right) - 1\,. \tag{143}$$

Then, the start temperature $\hat\theta_{\mathrm h}$ of the plastic flow at heating is given by (cf. eq. (128))

$$\tanh(\lambda\hat\theta_{\mathrm h}) = \frac{1 + \sqrt{1.5}\,\frac{\tau_{0_0} - \alpha_0}{r_0}}{\gamma + \sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0}} - 1\,. \tag{144}$$

Since $|\tanh x| \leq 1$ for any real number x, such a solution is possible whenever

$$-1 \leq \sqrt{1.5}\,\frac{\tau_{0_0} - \alpha_0}{r_0} \leq 2\,\gamma - 1 + 2\,\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0}\,. \tag{145}$$

Besides, τ_{0_0} and α_0 should also be restricted by condition (47), i.e.

$$\sqrt{1.5}\,\frac{|\tau_{0_0} - \alpha_0|}{r_0} \leq 1\,. \tag{146}$$

Next, from condition $(122)_1$ and eqs. (139)–(141) we deduce

$$\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0}\,(\tanh\lambda - \tanh(\lambda\,\theta_c)) = 2 - 2\,\gamma - \gamma\,(\tanh\lambda + \tanh(\lambda\theta_c))\,. \tag{147}$$

From this we infer that the start temperature θ_c of plastic flow at cooling is given by (cf. eq. (129))

$$\tanh(\lambda\,\theta_c) = \frac{\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} + \gamma}{\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} - \gamma}\,\tanh\lambda + \frac{2\,(\gamma - 1)}{\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} - \gamma}\,. \tag{148}$$

The condition for such a solution is as follows:

$$\left|\left(\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} + \gamma\right)\tanh\lambda - 2\,(1 - \gamma)\right| < \left|\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} - \gamma\right|\,.$$

For $\lambda \gg 1$, we have $\tanh\lambda = 1$. Then, the latter produces

$$\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} > 1 - \gamma\,. \tag{149}$$

Since $\gamma < \frac{1}{2}$, condition (142) is implied by the above condition. Moreover, by using condition (149) we infer that condition (145) is implied by condition (146). From these it follows that conditions (142), (145), (146) and (149) may be reduced to

$$\sqrt{1.5}\,\frac{\tau_{0_0}}{r_0} > \frac{1 - \gamma}{\rho}\,, \qquad \sqrt{1.5}\,\frac{|\tau_{0_0} - \alpha_0|}{r_0} \leq 1\,. \tag{150}$$

With the two start temperatures $\hat{\theta}_h$ (i.e. $\hat{T}_h$) and θ_c (i.e. T_c), strain responses can be determined for plastic flow induced at heating and cooling. Results are as follows (cf. eqs. (132) and (133)):

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I} + (|\boldsymbol{h}_0| - \phi(\theta))\,[\boldsymbol{h}_0^*]\,, \tag{151}$$

$$\phi(\theta) = \sqrt{\frac{2}{3}}\,\frac{r_0}{c_0}\,\frac{1}{\eta}\left(\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} + \gamma\right)\ln\frac{1 + \eta\,(1 + \tanh(\lambda\theta))}{1 + \eta\left(1 + \tanh\lambda\,\hat{\theta}_h\right)}\,, \quad \theta \in [\hat{\theta}_h, 1]\,,$$

for the plastic flow at heating, and

$$\boldsymbol{h}^* = \beta\left(T - T_0\right)\boldsymbol{I} + \left(|\boldsymbol{h}_0| - h_{2\mathrm{W}} - \varphi(\theta)\right)\left[\boldsymbol{h}_0^*\right], \tag{152}$$

$$\varphi(\theta) = \sqrt{\frac{2}{3}}\,\frac{r_0}{c_0}\,\frac{1}{\eta}\left(\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} - \gamma\right)\ln\frac{1 + \eta\left(1 + \tanh(\lambda\,\theta)\right)}{1 + \eta\left((1 + \tanh(\lambda\,\theta_{\mathrm{c}})\right)},\quad \theta \in [-1, \theta_{\mathrm{c}}],$$

for the plastic flow at cooling. In the above, the maximal two-way strain change $h_{2\mathrm{W}}$ is given by (cf. eq. (134))

$$h_{2\mathrm{W}} = \phi(1) = \sqrt{\frac{2}{3}}\,\frac{r_0}{c_0}\,\frac{1}{\eta}\left(\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} + \gamma\right)\ln\frac{1 + \eta\left(1 + \tanh\lambda\right)}{1 + \eta\left(1 + \tanh(\lambda\,\hat{\theta}_{\mathrm{h}})\right)}. \tag{153}$$

Moreover, condition (118) for a strain cycle is of the form:

$$\phi(1) + \varphi(-1) = 0,$$

namely (cf. eq. (135)),

$$\left(\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} + \gamma\right)\ln\frac{1 + \eta\left(1 + \tanh\lambda\right)}{1 + \eta\left(1 + \tanh(\lambda\,\hat{\theta}_{\mathrm{h}})\right)} =$$

$$\left(\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} - \gamma\right)\ln\frac{1 + \eta\left(1 + \tanh\theta_{\mathrm{c}}\right)}{1 + \eta\left(1 - \tanh\lambda\right)}. \tag{154}$$

From the conditions given by eq. (150), it may be seen that both the initial stress τ_0 and the initial back stress α_0 should be suitably generated to ensure two-way memory effects. It is noted that sometimes processes of " training" may be introduced for this purpose.

With four dimensionless parameters γ, η, ρ and c_0/r_0, the three constitutive quantities given by eqs. (139)–(141) are broad enough to describe two-way memory effects. Of them, c_0 may be used to control the amount of the maximal two-way strain change $h_{2\mathrm{W}}$, while the other three may be used to match the two start temperatures $\hat{T}_{\mathrm{h}}$ and T_{c} as well as the cyclic condition (140). In addition, the initial values τ_0 and α_0 should be suitably chosen to meet the conditions given by eq. (150). Further results and numerical examples will be presented next section.

7 Determination of parameters and numerical examples

The two-way memory effect is characterised by the two start temperatures $\hat{T}_{\mathrm{h}}$ and T_{c} as well as the initial plastic strain $|\boldsymbol{h}_0|$ and the maximal two-way

strain change h_{2W}. These characteristic quantities may be determined from the constitutive parameters introduced last section. The reverse problem is as follows: with given values of these characteristic quantities as well as η and λ, in this section we shall show how to determine the constitutive parameters.

7.1 Determination of parameters

First, from eqs. (147) and (154) we may derive γ and $\sqrt{1.5}\,\rho\,\tau_{0_0}/r_0$:

$$\gamma = \frac{p(\theta_c, -1) + p(\hat{\theta}_h, 1)}{(1 + \tanh\lambda)\,p(\theta_c, -1) + (1 + \tanh(\lambda\,\theta_c))\,p(\hat{\theta}_h, 1)}\,, \tag{155}$$

$$\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} = \frac{p(\theta_c, -1) - p(\hat{\theta}_h, 1)}{(1 + \tanh\lambda)\,p(\theta_c, -1) + (1 + \tanh(\lambda\,\theta_c))\,p(\hat{\theta}_h, 1)}\,. \tag{156}$$

In the above,

$$\hat{\theta}_h = \frac{2\,\hat{T}_h - T_1 - T_0}{T_1 - T_0}, \qquad \theta_c = \frac{2\,T_c - T_1 - T_0}{T_1 - T_0},$$

$$p(\theta_a, \theta_b) = \ln\frac{1 + \eta\,(1 + \tanh(\lambda\,\theta_a))}{1 + \eta\,(1 + \tanh(\lambda\,\theta_b))}.$$

Then, from eqs. (143), (155) and (156) we deduce

$$\sqrt{1.5}\,\frac{\tau_{0_0} - \alpha_0}{r_0} = \tag{157}$$

$$\frac{\left(1 - \tanh\lambda + 2\,\tanh(\lambda\,\hat{\theta}_h)\right) p(\theta_c, -1) - (1 + \tanh(\lambda\,\theta_c))\,p(\hat{\theta}_h, 1)}{(1 + \tanh\lambda)\,p(\theta_c, -1) + (1 + \tanh(\lambda\,\theta_c))\,p(\hat{\theta}_h, 1)}.$$

Next, with the maximal two-way strain change h_{2W}, from eqs. (153), (155) and (156) we may find out α_0/r_0 directly:

$$\sqrt{1.5}\frac{\alpha_0}{r_0} = -\frac{2}{\eta}\,\frac{h_0}{h_{2W}}\,\frac{p(\theta_c, -1)\,p(\hat{\theta}_h, 1)}{(1 + \tanh\lambda)\,p(\theta_c, -1) + (1 + \tanh(\lambda\,\theta_c))\,p(\hat{\theta}_h, 1)}\,. \tag{158}$$

Hence, we have

$$\sqrt{1.5}\,\frac{\tau_{0_0}}{r_0} = \sqrt{1.5}\,\frac{\tau_{0_0} - \alpha_0}{r_0} + \sqrt{1.5}\,\frac{\alpha_0}{r_0}\,. \tag{159}$$

In the above, the two terms of the right-hand side have been given by eqs. (157) and (158) and, besides, $h_0 = |\boldsymbol{h}_0|$ is the magnitude of the initial plastic strain.

Finally, ρ is given by

$$\rho = \frac{\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0}}{\sqrt{1.5}\,\frac{\tau_{0_0}}{r_0}}\,. \tag{160}$$

Note in the above that the nominator and denominator have been given by eqs. (156) and (159), respectively.

Since $0 < \gamma < \frac{1}{2}$, from eq. (155) we derive the following condition for the two start temperatures $\hat{T}_\mathrm{h}$ and T_c:

$$p(\theta_\mathrm{c}, -1) > p(1, \hat{\theta}_\mathrm{h})\,. \tag{161}$$

It may readily be shown that γ and $\sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0}$ given by eqs. (155) and (156) fulfil inequality $(150)_1$. In addition, the left-hand side of eq. (157) should meet the second inequality $(150)_2$. This places further restrictions on the two start temperatures:

$$\frac{p(\theta_\mathrm{c}, -1)}{1 + \tanh(\lambda\,\theta_\mathrm{c})} \geq \frac{p(\hat{\theta}_\mathrm{h}, 1)}{\tanh(\lambda\,\hat{\theta}_\mathrm{h}) - \tanh\lambda}\,. \tag{162}$$

Finally, ρ given by eqs. (157)–(160) should not exceed 0.5.

With these results and upon using eqs. (155), (156) and (158), expressions (151) and (152) for two-way plastic flows may be simplified to give:

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I} + \left(h_0 - h_{2\mathrm{W}}\,\frac{p(\theta, \hat{\theta}_\mathrm{h})}{p(1, \hat{\theta}_\mathrm{h})}\right)[\boldsymbol{h}_0^*], \qquad \theta \in [\hat{\theta}_\mathrm{h}, 1] \tag{163}$$

for the plastic flow at heating, and

$$\boldsymbol{h}^* = \beta\,(T - T_0)\,\boldsymbol{I} + \left(h_0 - h_{2\mathrm{W}}\,\frac{p(\theta, -1)}{p(\theta_\mathrm{c}, -1)}\right)[\boldsymbol{h}_0^*], \qquad \theta \in [-1, \theta_\mathrm{c}] \tag{164}$$

for the plastic flow at cooling. It may be seen from above two expressions that a strain cycle is indeed generated in a thermal cycle of heating/cooling. This may also be directly seen in Figure 3 presented in next subsection.

7.2 Solution and example

A solution is found for any given two start temperatures T_h and T_c as well as for any given ratio $h_0/h_{2\mathrm{W}}$ meeting the condition $(166)_7$ below. In fact, let

$$\lambda = 3, \quad \eta = 0.1, \quad \lambda\,\theta_\mathrm{h} = 0, \quad \lambda\,\theta_\mathrm{c} = 0.6\,. \tag{165}$$

Then, following the procedures described last subsection, we obtain

$$\gamma = 0.367682, \quad \sqrt{1.5}\,\rho\,\frac{\tau_{0_0}}{r_0} = 1.510943, \quad \sqrt{1.5}\,\frac{\tau_{0_0} - \alpha_0}{r_0} = 0.878624\,,$$

$$\sqrt{1.5}\,\frac{\alpha_0}{r_0} = 1.634617\,\frac{h_0}{h_{2\mathrm{W}}}, \quad \sqrt{1.5}\,\frac{\tau_{0_0}}{r_0} = 0.878624 + 1.634617\,\frac{h_0}{h_{2\mathrm{W}}}, \qquad (166)$$

$$\rho = \frac{1.510943}{0.878624 + 1.634616\,\frac{h_0}{h_{2\mathrm{W}}}}, \quad \frac{h_0}{h_{2\mathrm{W}}} > 1.311716\,.$$

These results fulfil all the conditions. Note that the last condition ensures $\rho < \frac{1}{2}$. Moreover, the temperatures T_0 and T_1 are given by

$$\begin{cases} T_0 = 2\,\frac{(\theta_c + 1)\,T_{\mathrm{h}} - (\theta_h + 1)\,T_{\mathrm{c}}}{\theta_c - \theta_h} = 6\,T_{\mathrm{h}} - 5\,T_{\mathrm{c}}\,, \\[2mm] T_1 = 2\,\frac{(\theta_c - 1)\,T_{\mathrm{h}} - (\theta_h - 1)\,T_{\mathrm{c}}}{\theta_c - \theta_h} = 5\,T_{\mathrm{c}} - 4\,T_{\mathrm{h}}\,. \end{cases} \qquad (167)$$

The curves for the two-way plastic flows are given by eqs. (163) and (164). Results are shown in Figure 3 for

$$h_0 = 0.06, \qquad h_{2\mathrm{W}} = 0.04.$$

For these values we have

$$\sqrt{1.5}\,\frac{\alpha_0}{r_0} = 2.451925, \qquad \sqrt{1.5}\,\frac{\tau_{0_0}}{r_0} = 3.330549, \qquad \rho = 0.453662.$$

It may be interesting to note that in producing Figure 3 no direct reference is made to the values of the two pairs of temperatures, i.e. $(T_{\mathrm{h}}, T_{\mathrm{c}})$ and (T_0, T_1). They are integrated into a non-dimensional procedure introduced by the expression just preceding eq. (138). That is also the case in general. The pair $(T_{\mathrm{h}}, T_{\mathrm{c}})$ may take any values from experiment, and then follows the pair (T_0, T_1) given by eq. (167). Also, h_0 and $h_{2\mathrm{W}}$ may take any values obeying eq. (166).

8 Discussion

In the past decades there have been numerous contributions in modelling the thermomechanical behaviour of shape memory materials. Details may be found in the latest review articles by Lagoudas, Patoor et al. [55, 56]. For representatives of recent contributions in the case of 3-dimensional deformations with thermal effects, reference may be made to, e.g., [57–72] and many others.
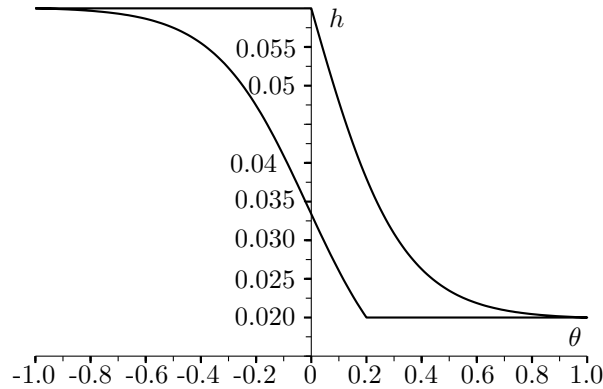
Figure 3: Two-way memory with two-way thermoinduced plastic flows

Prior to possible theories and models for explaining and simulating noticeable thermomechanical deformation behaviour of shape memory materials, such as strain recovery (shape memory) effects etc., let us first take a closer look at the two main stages of these phenomena itself from a phenomenological standpoint. At the first stage, an initial irrecoverable plastic strain will be generated for a material specimen in an isothermal loading/unloading process, and then, at the second stage, this initial plastic strain will be totally or partially recovered in a thermal process with or without initial stress. Here, the first stage in isothermal case may clearly indicate that the material should reveal isothermal elastoplastic behaviour. Then, a perhaps natural idea would be that the second stage with strain recovery (shape memory) in a thermal process should be non-isothermal elastoplastic behaviour. With this idea in mind[3], both isothermal and non-isothermal behaviour of the material specimen may be described by a straightforward thermocoupled elastoplasticity model. Then arises a perhaps interesting question: would it be possible to derive the strain recovery effects from the latter?

From purely phenomenological standpoint *exclusively* based on thermo-coupled elastoplasticity in a classical sense, we have presented a report on the theoretical findings of *recoverable plastic flows* induced in processes of pure temperature change and then on the direct, unexpected correlations of these

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[3]Ingo Müller [73] pioneered the idea of applying elastoplastic models to simulating shape memory behaviour. With temperature-dependent statistical distributions and expectations of displacements, he showed how the snap-spring model for an elastoplastic body, proposed in [74], could be used to model shape memory bodies; refer to [75] for further developments based on statistical thermodynamics.

findings to one- and two-way memory effects that have been found experimentally in shape memory alloys. The key point is a new, enhanced form of von Mises yield function with a temperature-dependent constitutive quantity named thermoinducer. This quantity is introduced for the first time to display a direct coupling effect of the temperature on the current deviatoric stress level and leads to the thermocoupled effective stress. It has been shown that the latter plays a crucial role in the finding of thermoinduced two-way plastic flows exhibiting two-way memory effects in a cyclic process of heating/cooling.

The study in preceding sections suggests that both one- and two-way memory effects may be explained to be thermoinduced plastic flows in elastoplastic materials with thermosensitive yielding and hardening behaviour, with no direct reference to phase changes related to martensite and austenite as commonly assumed as starting point in microstructure-based models. This may suggest that complicated phenomenological consequences from processes of phase changes occurring at a microstructural level in SMAs may be characterised and modelled straightforwardly by applying classical thermocoupled elastoplastic J_2-flow models[4] simply with three temperature-dependent constitutive quantities, namely, the yield stress $r = r(T)$, Prager's modulus $c = c(T)$ and the thermoinducer $\varrho = \varrho(T)$. It may be perhaps noticeable that strain recovery (shape memory) effects would be natural consequences derived just from well-known facts concerning temperature-dependent yielding behaviour of elastoplastic materials, namely, the yield stress is monotonically decreasing with increasing temperature. Whenever such dependence properties of yield stress and hardening modulus on temperature are appreciable and even sensitive within a certain temperature range, then appreciable plastic flows may be induced in a process of pure temperature change within this range, and, accordingly, appreciable strain recovery (shape memory) effects may be observed.

It is noted that strain recovery features of shape memory materials are observed in cases of either pure temperature changes or pure isothermal load changes, known as shape memory effects by temperature change and pseudoelastic hysteresis by isothermal load change. The study here suggests that both one- and two-way strain recovery features due to temperature change may be explained to be thermoinduced plastic flow. On the other hand, latest studies

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[4]To the authors' knowledge, the first systematic treatment in applying straightforward phenomenological elastoplasticity models in an extended sense to simulating phenomenological behaviour of SMAs at small strains is presented by Bertram [76], where two temperature-dependent yield limits as well as a limit concerning plastic strain are introduced for describing phase transformation strains during forward and reverse phase changes.

[77, 78] suggest that the strain recovery feature due to isothermal load changes may also be explained to be two-way plastic flows. In coupled cases of both load change and temperature change, both finite deformation effects and thermal effects may be essential and should be taken into account in a consistent thermocoupled elastoplastic model at finite deformations. The thermocoupled elastoplastic J_2-flow model proposed here may further be used to study complicated thermomechanical behaviour of shape memory materials in general cases of thermocoupled deformations.

Appendix: Solutions of Riccati equation (58)

In this appendix, we present the solutions of Riccati equation (58) with initial values (59) and (78). Generally, it may be impossible to get an exact solution for a Riccati equation. However, it is indeed possible to obtain the exact solution following a standard procedure, whenever a particular solution is available. Fortunately, that is the case for Riccati equation (58).

In fact, it may readily be shown that the expression

$$\phi = \pm\sqrt{\frac{2}{3}}\, r\, |\tilde{\boldsymbol{\tau}}_0^*|$$

provides two particular solutions of Riccati equation (58). Then, with the transformation

$$y = \phi + u,$$

we recast eq. (58) in the form

$$u' + \left(3\frac{\varrho'}{r^2}\phi - \frac{r'}{r}\right)u = -\frac{3}{2}\frac{\varrho'}{r^2}u^2. \tag{A1}$$

This is a Bernoulli equation. To find out solutions with the initial values (59) and $(78)_2$, we need to distinguish two different cases.

The first case is that a particular solution as shown before can satisfy the initial conditions at issue. In this case, we have

$$\frac{2}{3}\,r^2\,|\tilde{\boldsymbol{\tau}}_0^*|^2 = \left(\varrho\,|\tilde{\boldsymbol{\tau}}_0^*|^2 - \tilde{\boldsymbol{\tau}}^*\right)^2$$

for $T = \hat{T}_{\mathrm{h}},\, T_{\mathrm{c}}$ and $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}_0^*,\, \boldsymbol{\alpha}_1^*$, respectively. From this and the yield conditions (48) and (71) we deduce

$$|\tilde{\boldsymbol{\tau}}_0^*|^2 \cdot |\boldsymbol{\alpha}^*|^2 = (\tilde{\boldsymbol{\tau}}_0^* : \boldsymbol{\alpha}^*)^2.$$

This means that the two tensors $\tilde{\boldsymbol{\tau}}_0$ and $\boldsymbol{\alpha}^*$ should be linearly dependent, namely,

$$\boldsymbol{\tau}_0^* = \gamma\,\boldsymbol{\alpha}^*.$$

With this relation, it may be proved that the two particular solutions

$$y = \phi_{\mathrm{h}} = \sqrt{\frac{2}{3}}\,r\,|\tilde{\boldsymbol{\tau}}_0^*|\,\mathrm{sgn}(y_{\mathrm{h}}),$$

$$\bar{y} = \phi_{\mathrm{c}} = \sqrt{\frac{2}{3}}\,r\,|\tilde{\boldsymbol{\tau}}_0^*|\,\mathrm{sgn}(y_{\mathrm{c}}),$$

can fulfil the initial conditions (59) and (78)$_2$, respectively. Therefore, the above two provide the solutions needed.

The other case is when tensors $\tilde{\boldsymbol{\tau}}_0$ and $\boldsymbol{\alpha}^*$ are linearly independent, namely,

$$|\tilde{\boldsymbol{\tau}}_0^*|^2 \cdot |\boldsymbol{\alpha}^*|^2 - (\tilde{\boldsymbol{\tau}}_0^* : \boldsymbol{\alpha}^*)^2 \neq 0.$$

In this case, we infer that the foregoing Bernoulli equation will supply a non-vanishing solution. Hence, with the transformation

$$u = \frac{1}{z}$$

we convert the Bernoulli equation (A1) to the following linear equation:

$$z' + \left(\frac{r'}{r} - 3\frac{\varrho'}{r^2}\,\phi\right) z = \frac{3}{2}\frac{\varrho'}{r^2}\,. \tag{A2}$$

Here, T_i may be taken as either T_0 or $\hat{T}_{\mathrm{h}}$ and, accordingly, ϕ_i is taken as either ϕ_{h} or ϕ_{c} given before. Integrating the latter from T_i to T, we derive

$$z = \left(z_i + \frac{3}{2}\int_{T_i}^T \frac{\varrho'}{r^2}\,\mathrm{e}^w\,\mathrm{d}T\right)\mathrm{e}^{-w},$$

where

$$w = \int_{T_i}^T \left(\frac{r'}{r} - 3\frac{\varrho'}{r^2}\,\phi_i\right)\mathrm{d}T\,,\qquad z_i = \frac{1}{y_i - \phi_i}\,,$$

for $T_i = \hat{T}_h$, T_c, separately. Then, the solutions of Riccati equation (58) with initial values y_h and y_c (refer to eqs. (59) and (78)$_2$) are given by

$$
\begin{cases}
y = \sqrt{\dfrac{2}{3}}\, r\, |\tilde{\boldsymbol{\tau}}_0^*|\, \mathrm{sgn}(y_h) + \mathrm{e}^w \left(z_h + \dfrac{3}{2} \int_{\hat{T}_h}^{T} \dfrac{\varrho'}{r^2}\, \mathrm{e}^w\, \mathrm{d}\,T \right)^{-1}, \\[4mm]
w = \displaystyle\int_{\hat{T}_h}^{T} \left(\dfrac{r'}{r} - \sqrt{6}\, |\tilde{\boldsymbol{\tau}}_0^*|\, \dfrac{\varrho'}{r}\, \mathrm{sgn}(y_h) \right) \mathrm{d}\,T, \\[4mm]
z_h = \left(y_h - \sqrt{\dfrac{2}{3}}\, |\tilde{\boldsymbol{\tau}}_0^*|\, r(\hat{T}_h)\, \mathrm{sgn}(y_h) \right)^{-1},
\end{cases}
$$

for heating and

$$
\begin{cases}
\bar{y} = \sqrt{\dfrac{2}{3}}\, r\, |\tilde{\boldsymbol{\tau}}_0^*|\, \mathrm{sgn}(y_c) + \mathrm{e}^w \left(z_c + \dfrac{3}{2} \int_{T_c}^{T} \dfrac{\varrho'}{r^2}\, \mathrm{e}^w\, \mathrm{d}\,T \right)^{-1}, \\[4mm]
w = \displaystyle\int_{T_c}^{T} \left(\dfrac{r'}{r} - \sqrt{6}\, |\tilde{\boldsymbol{\tau}}_0^*|\, \dfrac{\varrho'}{r}\, \mathrm{sgn}(y_c) \right) \mathrm{d}\,T, \\[4mm]
z_c = \left(y_c - \sqrt{\dfrac{2}{3}}\, |\tilde{\boldsymbol{\tau}}_0^*|\, r(T_c)\, \mathrm{sgn}(y_c) \right)^{-1},
\end{cases}
$$

at cooling.

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Termoizazvano plastično tečenje i efekti pamćenja oblika

Predložen je poboljšani oblik termospregnutih J_2 modela tečenja elastoplastičnosti konačnih deformacija sa početkom tečenja i deformacionim ocvrscavanjem zavisnim od temperature. Termomehanička konstitutivna struktura ovih modela je formulisana u rigoroznom i konsistentnom termodinamičkom obliku. Naime, sa opstim oblikom funkcije slobodne energije, zavisne od uopstenih oblika konstitutivnih funkcija, zahtevi drugog zakona termodinamike su identički zadovoljeni sa pozitivnom unutrašnjom disipacijom. Pokazuje se da termoosetljivi početak tečenja sa anizotropnim ojačanjem može dovesti do značajnog plastičnog tečenja kako pri zagrevanju tako i pri cikličnom zagrevanju/hladjenju. Kao posledica, ovo vodi termoizazvanom plastičnom tečenju u jednom ili oba pravca. Ispostavlja se da ovakva teorijska razmatranja su analogna analizi efekata prisutnih u materijalima koji pamte oblik, takve kao efekti pamcenja jednog ili dva načina. Konsekventno, efekti pamćenja oblika mogu se objasniti termoizazvanim plastičnim tečenjem koje sledi iz termoosetljivog početka tečenja i deformacionog ocvrscavanja. Ove i druge povezane činjenice mogu sugerisati da termospregnuti J_2 modeli tečenja sa termoosetljivim početkom tečenja i očvršćavanja mogu da obezbede prirodan i i jednostavan opis termomehaničkog ponašanja materijala koji pamte oblik.