Finite elastic-plastic deformations: beyond the plastic spin

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Abstract

One important aspect of finite elastic-plastic deformation constitutive theories is addressed in this work, namely the appropriate embedding of tensor-valued internal variables into the plastic deformation continuum description, which has been called physico-geometrical coupling reflecting the relation between geometry of deformation and the physical nature of an internal variable. In the past it was assumed hat such embedding was co-rotational with a material substructure, rotating independently from the continuum, which required the introduction of the concepts of constitutive and plastic spins for each internal variable. This assumption is now extended to cases where the embedding is convected with the plastic deformation, and it is possible to obtain a common formulation for both rotational and convected embeddings. Explicit expressions are obtained for the plastic multiplier (or loading index) from the consistency condition and the free energy function, making use of certain analytical properties of isotropic scalar and tensor valued functions of scalar and tensor-valued variables, such isotropy arising from the need to satisfy objectivity.

Keywords: finite deformations; plasticity; plastic spin; objective rates; anisotropy.

1 Introduction

The formulation of finite elastic-plastic deformation theories traces a long orbit since decades ago, and is characterized by numerous arguments and

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counterarguments over various issues, but mainly over the kinematics associated with it. In regards to kinematics one can distinguish two debatable issues. First, is the issue of geometrical separation (decomposition) of the total deformation and/or its rate into elastic and plastic parts, the two being interconnected. Second, and most recent, is the issue of physico-geometrical coupling on the way the embedding of internal state variables, scalar or tensor-valued, is defined in regards to the plastic and elastic deformation, and the ensuing appropriate rate of evolution of such variables which arises as a result of such embedding. Both issues, and in particular the first, have been addressed by a plethora of publications which is vast enough to defy a comprehensive review within the limits of this work. Nevertheless, some very fundamental works must be mentioned as a precursor to what follows.

Two basic schools of thought can be distinguished in regards to the decomposition of finite elastic-plastic deformations. The first proposes the additive decomposition of an appropriately defined finite strain tensor into elastic and plastic parts, representative work for which is the classical work by Green and Naghdi (1965); the rate decomposition follows directly from the strain decomposition. While rigorous mathematically, this approach lacks the capability to incorporate typical continuum crystal plasticity, a benchmark problem for any plasticity theory. The second school of thought proposes the multiplicative decomposition of the deformation gradient into elastic and plastic parts, representative work for which is the other classical work by Lee (1969) (see also following works by Lubarda and Lee(1981) and Lubarda (1991)). This multiplicative decomposition was in fact motivated by the deformation process in crystal plasticity, where the continuum deforms firstly plastically by shearing on lattice slip-planes, and subsequently the plastically deformed continuum is mapped by an elastic deformation gradient onto the current configuration. This approach incorporates almost by definition the benchmark case of crystal plasticity, and has been adopted by the majority of researchers in the field of finite elastoplastic deformation. Furthermore the concept of the multiplicative decomposition of the deformation gradient has been appropriately extended to thermo-elasticity and mass growth in biomechanics, Rodriguez et al (1994), Lubarda (2004). One aspect which is debatable though in the case of elastoplasticity, is that when one forms the velocity gradient of the total deformation and proceeds to decompose additively the total rate of deformation tensor, he obtains a plastic rate of deformation which is not purely plastic because the elastic deformation gradient intervenes in its definition. This aspect has been ad-

dressed by Dafalias (1998) and it will be elaborated again in subsequent sections.

The second kinematical issue is associated with the embedding of the tensor-valued internal variables into the kinematics of the continuum, and the appropriate objective rate which must be used in their equations of evolution. It is important to understand that a proposition of an objective rate is tantamount to a specific embedding, and vice-versa an embedding during plastic deformation, proposed based on some physical argument, implies a specific rate of evolution. Often the connection between embedding and resulting rate has not been recognized, and the usual approach was to simply choose an objective rate to be used in evolution equation of an internal variable, such as a Jaumann corotational rate in regards to the material spin or an Oldroyd or Trusdell convected rate, and then proceed without examining the consequences of the implied embedding. This was the reason for obtaining the surprising and unexpected stress oscillations obtained during simple shear of a linear kinematically hardening material, for which a Jaumann rate was postulated for the evolution of the back-stress, Nagtegaal and de Jong (1982).

This strange response spurred a plethora of relevant publications, first of which were the ones by Lee et al (1983) and Dafalias (1983), appearing in sequence in the same journal. The theme of the first publication was to suggest the spin tensor of the principal back-stress directions as the one in regards to which the corotational rate of the back stress is defined, while the second publication used the spin associated with the rate of the orthogonal part of the polar decomposition of the deformation gradient. Most importantly, in this second publication a constitutive equation for the plastic spin was proposed for the first time, a concept derived from the fundamental works of Mandel (1971) and Kratochvil (1973), to be defined in the sequel. In fact a more general approach for the plastic spin was presented earlier in two conferences in June 1983 by Dafalias (1985a, 1993a), before publication of the aforementioned article, Dafalias (1983), while the same theme was addressed independently by Loret (1983). A comprehensive and complete presentation of the concept appeared finally in Dafalias (1985b) where the name "Plastic Spin" was coined while its extension to Viscoplasticity was made in Dafalias (1990). In simple terms the idea, motivated by crystal plasticity again, was that an internal variable is embedded in a substructure represented by a triad of director vectors and being conceptually associated with the crystal lattice, and that this triad spins at a rate which is different

that the material spin of the continuum; the difference of these two spins is the plastic spin for which constitutive relations must be provided as done for the plastic rate of deformation. This type of "rotational" embedding of the internal variable requires, based on physical grounds, to use for its evolution a corotational rate associated with the triad spin. This scheme can eventually eliminate the aforementioned unwanted oscillations because now the constitutively defined spinning of the background manifold under simple shear can be controlled unlike the material spin which continues for ever.

In subsequent works by Dafalias (1984, 1987, 1988, 1993a,b, 1998) and Cho and Dafalias (1996), the notion of the director triad spin was extended, applied and generalized to that of different constitutive spins, each one associated with a different internal variable, thus, introducing correspondingly multiple plastic spins (to each constitutive spin there corresponds a plastic spin and vice-versa). As mentioned before, any such constitutive spin implies that the corresponding internal variable is embedded into its own rotating constitutive frame (its own director triad), hence, it can be said it is rotationally embedded. But this approach cannot address the possibility of having an internal variable embedded in a convected way (contravariant, covariant, mixed). This is not just a technical detail but reflects a particular physico-geometrical coupling that cannot be addressed by the theory of constitutive and plastic spins. Thus, going beyond plastic spin, the present work will focus on extending the embedding of internal variables from rotational to convected ways, and derive the full set of the corresponding constitutive equations. The latter requires the specification of the plastic multiplier from the consistency condition, which will be seen to be not a trivial matter.

It should be mentioned that after the original works on plastic spin in the early 1980's, there was a large number of significant publications on the subject by various authors who used the concept of plastic spin in relation to several important and practical issues in mechanics such as shear banding (Tvergaard and van der Giessen, 1991, Kuroda, 1997), crystal plasticity and variations of it (Zbib and Aifantis, 1988, Rashid, 1992, Prantil et al, 1993, Dafalias, 1993c), fibrous composites (Fares and Dvorak, 1991), polymers (Boyce et al, 1988, Aravas, 1994), sheet metals (Dafalias, 2000) and several other areas which simply cannot be covered here due to the restricted scope and length of this work. Nevertheless, the interested reader can find an extensive, if not exhaustive, review in Dafalias (1998).

In terms of notation the simplest possible avenue will be followed. Scalars

and tensors of any order will be denoted by same direct italics notation, and the distinction of the tensorial degree will be deduced based on their original definition and structure of corresponding equations. Since direct notation rather than indicial will be used, juxtaposition of tensors implies summation of neighboring indices, while a : between two tensors implies the trace operation executed over two neighboring pairs of indices. For example one has $AB \Leftrightarrow A_{ij}B_{jk}$ for the ik component of the product, and $A : B = tr(AB) = A_{ij}B_{ji}$. Similar equations hold when one has higher order tensors (third and fourth). In more complex expressions detailed explanation of index summation will be provided. Last, single subscripts will be used not as tensorial indices, but as indicators of the plurality associated with a tensor variable symbol. For example A_i is one of many tensors the plurality of which is defined by the values i can take. In such cases the standard rule is that no summation is implied over repeated indices, i.e. $A_i\omega_i$ does not imply summation over i. If summation is necessary, in particular in differentiations or when one index appears three times, it will be explicitly declared. For example for $(\partial f/\partial A_i)\bar{A}_i$ or $A_i(\partial f/\partial A_i)\bar{A}_i$ it will be said that summation over i is implied, even when i appears three times.

2   Kinematics

The multiplicative decomposition $F = F^e F^p$ of the deformation gradient F between initial reference and current configurations is adopted which defines an intermediate reference unstressed (or relaxed) configuration obtained from the initial reference configuration by plastic deformation and rotation defined by F^p, and a current configuration obtained by the subsequent elastic deformation and rotation of the intermediate configuration defined by F^e. The intermediate configuration has an entirely arbitrary orientation, such arbitrariness accommodated by the choice of the rotational parts of the polar decomposition of F^e and F^p. This scheme introduces the natural concept of an elastic Green strain tensor $E^e = (1/2)(C^e - I)$ with $C^e = F^{eT}F^e$ in regards to the intermediate configuration, used as the kinematic variable in the elastic constitutive laws. Similarly a plastic Green strain tensor can be defined by $E^p = (1/2)(C^p - I)$ with $C^p = F^{pT}F^p$ in regards to the initial configuration, but such tensor is not used in the plastic constitutive relations which require a rate form of the deformation measure.

With this in mind and based on the foregoing multiplicative decomposition, one can define first the velocity gradient at the current configuration

by

$$\dot{F}F^{-1} = (\dot{F}F^{-1})_s + (\dot{F}F^{-1})_a = D + W = \dot{F}^e F^{e-1} + F^e \dot{F}^p F^{p-1} F^{e-1} \quad (1)$$

where a superposed dot implies the material time derivative or rate, subscripts s and α denote symmetric and antisymmetric parts, respectively, D is the symmetric rate of deformation tensor and W is the antisymmetric material spin tensor. One can take the symmetric part of the last two members of Eq.(1) in order to define elastic and plastic parts of the rate of deformation tensor according to

$$D = D^e + D^p = (\dot{F}^e F^{e-1})_s + (F^e \dot{F}^p F^{p-1} F^{e-1})_s \quad (2)$$

which is the usual way most finite strain elastoplastic theories formulate their kinematics. However, the physical meaning of $D^p = (F^e \dot{F}^p F^{p-1} F^{e-1})_s$ has been criticized because of two reasons. First, it includes in its definition the elastic deformation gradient F^e which acts as a transport agent of the plastic velocity gradient $\dot{F}^p F^{p-1}$ from intermediate to current configuration, which is counterintuitive when a constitutive law is postulated for D^p because plastic deformation is assumed to be kinematically independent of the elastic deformation. Second, both symmetric and antisymmetric parts of $\dot{F}^p F^{p-1}$ contribute to the definition of the symmetric D^p, while it is only the symmetric part of $\dot{F}^p F^{p-1}$ which defines the purely plastic rate of deformation at the intermediate configuration. One may also observe that elasticity is a potential and not a rate theory in its constitutive foundation and does not need an elastic rate of deformation tensor $D^e = (\dot{F}^e F^{e-1})_s$ as per Eq.(2). On the other hand plasticity theory is a rate theory and in order to couple these two separate theories one needs to find a common platform to connect their kinematics, and as such Eq.(2) has been almost universally adopted despite all foregoing criticisms.

In order to avoid these physically based objections Dafalias (1998) followed a different approach, as follows. First the plastic velocity gradient was defined at the intermediate configuration by the usual expression

$$L_0^p = \dot{F}^p F^{p-1} = (\dot{F}^p F^{p-1})_s + (\dot{F}^p F^{p-1})_a = D_0^p + W_0^p \quad (3)$$

where D_0^p is the symmetric plastic rate of deformation tensor and W_0^p is the antisymmetric plastic material spin tensor (not to be confused with the plastic spin tensor to be defined in the sequel) at the intermediate configuration. Notice that contrary to the definition of D^p, for the definition of

both D_0^p and W_0^p only the plastic deformation gradient F^p and its rate are used, rendering such definition kinematically uncoupled from the elastic deformation. One can now postulate purely plastic constitutive relations for D_0^p while the W_0^p can acquire any value given the arbitrary orientation of the intermediate configuration, as discussed in Lubarda and Shih (1994) and Dafalias(1998).

However, the question still remains as to how one can relate the plastic rate of deformation D_0^p at the intermediate configuration with the elastic kinematics expressed by E^e. Such necessary relation should certainly be in terms of rates, but such rates must involve uncoupled elastic or plastic quantities. Following Dafalias (1998) one can premultiply by F^{eT} and postmultiply by F^e the last two members of Eq.(1) and subsequently take the symmetric part of the ensuing relation and based on the important observation that $\dot{E}^e = (F^{eT}\dot{F}^e)_s$, obtain the key kinematic relation $\overset{\vee}{E}^e = \dot{E}^e + L_0^{pT}E^e + E^e L_0^p = F^{eT}DF^e$ for the covariant rate of E^e in regards to F^p, which based on Eq.(3) can be re-written in the more pertinent form

$$\overset{\nabla}{E}^e = \dot{E}^e - W_0^p E^e + E^e W_0^p = F^{eT}DF^e - (C^e D_0^p)_s \qquad (4)$$

where a superposed ∇ implies the Jaumann corotational rate in regards to a spin tensor, in this case the W_0^p. Eq.(4) involves quantities and rates of quantities which are purely elastic or purely plastic or total in their kinematical definitions. For example D is the total rate of deformation tensor at the current configuration, E^e and its rate as well as F^e and $C^e = F^{eT}F^e$, are entirely elastic quantities, and most importantly the D_0^p is a purely plastic rate of deformation tensor at the intermediate configuration without any elastic kinematical coupling which might jeopardize the physical meaning of its constitutive expressions as in the previous case with D^p. As already mentioned, the $W_0^p = (\dot{F}^p F^{p-1})_a$ is a kinematically free to choose quantity and does not require any constitutive expression, as D_0^p does; an example is the so-called spinless configuration for which one sets $W_0^p = 0$. Eq. (4) re-written as $F^{eT}DF^e = \overset{\nabla}{E}^e + (C^e D_0^p)_s$ substitutes for the usual rate of deformation additive decomposition $D = D^e + D^p$ of Eq.(2), and although the latter has a simpler form, the former is addressing correctly the issue of pure kinematical decoupling of elastic and plastic deformations, an issue of paramount importance for the correct physical meaning attributed to elastic and plastic constitutive relations.

3   Embedding of internal variables, their rates and the concept of plastic spin

The state of the material will be characterized by the stress and a set of internal variables defined at the intermediate reference configuration. These variables will be either tensor-valued (restricted to second order symmetric tensors) and denoted by A_i, or scalar-valued and denoted by K_i. The physical meaning of these variables is multiple; it ranges from internal stresses to anisotropic directions (a vector direction enters the state functions as a tensor product with itself, thus it becomes a tensor) for tensor-valued, and from damage measure to dislocation density for scalar-valued. No matter what the physical interpretation may be, the values of these variables at the intermediate configuration are related to their "image" at the initial reference configuration by an appropriate plastic embedding reflecting their physical interpretation. This issue has been examined from the perspective of elastic embedding from the intermediate to the current configuration in what has been called "physico-geometrical coupling" by Dafalias (2001). Here the same issue is pursued from the perspective of plastic embedding described above. One very important corollary of such embedding is the ensuing convected or corotational rate at the intermediate configuration corresponding to the rate at the initial configuration. In other words the definition of an embedding implies the definition of a specific rate at the intermediate configuration and vice-versa.

The analytical expression of the above can be succinctly stated as follows. Let A_i^r denote the value of A_i at the initial reference configuration. The possible plastic convected embeddings relating A_i^r to A_i are as follows.

Convected embeddings:

$$(a) \text{ Contravariant:} \qquad A_i^r = |F^p|^{w_i} \, F^{p-1} A_i F^{p-T} \qquad (5a)$$

$$(b) \text{ Covariant:} \qquad A_i^r = |F^p|^{w_i} \, F^{pT} A_i F^p \qquad (5b)$$

$$(c) \text{ Mixed } \ 1: \qquad A_i^r = |F^p|^{w_i} \, F^{pT} A_i F^{p-T} \qquad (5c)$$

$$(d) \text{ Mixed } \ 2: \qquad A_i^r = |F^p|^{w_i} \, F^{p-1} A_i F^p \qquad (5d)$$

with w_i the weight of a relative tensor A_i and $|F^p|$ denoting the determinant of F^p.

A totally different "embedding" is related to a rotation from initial to intermediate configuration measured by an orthogonal tensor R_i ($R_i R_i^T = I$) associated with each internal variable and being in general different from the

orthogonal part R^p of the polar decomposition $F^p = R^p U^p = V^p R^p$. Hence one can express the rotational embedding as follows.

Rotational embedding:

$$A_i^r = |F^p|^{w_i} R_i^T A_i R_i \qquad (6)$$

Notice that in past publications the term $|F^p|^{w_i}$ of the relative tensor was missing; this is a particular case of Eq.(6) with $w_i = 0$.

Finally the embedding of a scalar-valued internal variable K_i is expressed by

Scalar embedding:

$$K_i^r = |F^p|^{w_i} K_i \qquad (7)$$

It takes now only some straightforward algebra to take the rate of each one of Eqs.(5) and obtain the expressions $\dot{A}_i^r = |F^p|^{w_i} F^{p^x} \overset{\vee}{A}_i F^{p^y}$ where the exponents x and y correspond to the first and second exponents, respectively, of F^p in each one of the Eqs. (5), and a superposed $\vee$ denotes the corresponding convected rate defined according to

$$\begin{aligned}
\overset{\vee}{A}_i &= \dot{A}_i + s_1^{(i)}(A_i D_0^p + s_2^{(i)} D_0^p A_i) + (A_i W_0^p - W_0^p A_i) + w_i A_i tr D_0^p \\
&= \overset{\triangledown}{A}_i + s_1(A_i D_0^p + s_2 D_0^p A_i) + w_i A_i tr D_0^p
\end{aligned} \qquad (8)$$

where in reference to Eqs.(5) the $s_1^{(i)}$ and $s_2^{(i)}$ acquire the values $s_1^{(i)} = +1$ for cases (b) and (d); $s_1^{(i)} = -1$ for cases (a) and (c); $s_2^{(i)} = +1$ for cases (a) and (b); $s_2^{(i)} = -1$ for cases (c) and (d).

The Jaumann rate

$$\overset{\triangledown}{A}_i = \dot{A}_i - W_0^p A_i + A_i W_0^p \qquad (9)$$

is defined in regards to the plastic material spin $(\dot{F}^p F^{p-1})_a = W_0^p$ as it was done for $\overset{\triangledown}{E}^e$ in Eq.(4). In the foregoing and several equations to follow, use of the relation $|\dot{F}^p| = |F^p| tr D_0^p$ is made.

For the rotational embedding of Eq.(6) the same process is followed, but now two new concepts arise, those of the constitutive and plastic spins as follows. The rate of Eq.(6) yields $\dot{A}_i^r = |F^p|^{w_i} R_i^T \overset{\circ}{A}_i R_i$ with a superposed $\circ$ denoting the corotational rate $\overset{\circ}{A}_i = \dot{A}_i - \omega_i A_i + A_i \omega_i + w_i A_i tr D_0^p$ in regards

to the so-called constitutive spin $\omega_i = \dot{R}_i R_i^T$ (Dafalias, 1998). It is clear that the definition of R_i implies the definition of ω_i by differentiation with respect to time, and vice-versa for a given ω_i one obtains a corresponding R_i by integration in time. The definition of either ω_i or R_i is a constitutive ingredient associated with the rotational embedding of the corresponding internal variable A_i. There are in general as many ω_i as the rotationally embedded A_i. The first introduction of the concept of constitutive spin (although not named likewise) was made by Mandel (1971) who associated it with the spin of a triad of director vectors, usually associated with the lattice spin of crystal plasticity. In doing so Mandel has introduced a common constitutive spin for all internal variables, a concept which was later generalized to multiple constitutive spin by Dafalias (1993b).

Now it is important to focus on the definition of the plastic spin which has been so often misinterpreted in several publications. The ω_i is defined at the intermediate configuration as something totally different than the plastic material spin $(\dot{F}^p F^{p-1})_a = W_0^p$, hence, one can always write (Dafalias, 2000):

$$W_0^p = \omega_i + W_i^p \tag{10}$$

with W_i^p being the plastic spin associated with a specific rotationally embedded internal variable A_i, different from the plastic material spin W_0^p. A common error is to consider the plastic material spin W_0^p as the plastic spin, instead of W_i^p, possibly because of the similarity of name and because the former is created by the rate of F^p. The W_0^p is a kinematical variable associated with the orientation rate (but not deformation rate) of the intermediate configuration, while the plastic spin W_i^p is in general different for each rotationally embedded internal variable (multiple constitutive and plastic spins) and represents a constitutive ingredient of the rate equation of evolution of the corresponding internal variable as shown in the sequel. Notice that when there is zero plastic rate of deformation, i.e. $(\dot{F}^p F^{p-1})_s = D_0^p = 0$, the plastic spin $W_i^p = 0$ as part of its constitutive nature, while the plastic material spin W_0^p may be non zero since the intermediate configuration is allowed to take arbitrary orientations by arbitrary spinning measured by W_0^p. To this extend zero plastic rate of deformation does not necessarily mean $\dot{F}^p = 0$, as erroneously is often assumed bypassing the possibility of spinning without deformation. One specific orientation of the intermediate configuration may be associated with the so-called spinless configuration for which $W_0^p = 0$ always; in this case it follows from Eq.(10) that $\omega_i = -W_i^p$.

With the above explanations and the key Eq.(10) exhibiting the funda-

mental difference between the plastic material spin and the plastic spin for each rotationally embedded internal variable, one can rewrite the corotational rate $\overset{\circ}{A_i}$ by substituting $\omega_i = W_0^p - W_i^p$ and obtain

$$\overset{\circ}{A_i} = \dot{A_i} - \omega_i A_i + A_i \omega_i + w_i A_i tr D_0^p = \overset{\triangledown}{A_i} - (A_i W_i^p - W_i^p A_i) + w_i A_i tr D_0^p \quad (11)$$

in terms of $\overset{\triangledown}{A_i}$ as defined in Eq.(9). Eq.(11) with the corotational rate for the rotational embedding of A_i, is the counterpart of Eq. (8) with the convected rate for the convected embedding of A_i.

Finally the scalar embedding of K_i as per Eq.(7) yields the rate equation $\dot{K_i^r} = |F^p|^{w_i} \overset{\vee}{K_i}$ with

$$\overset{\vee}{K_i} = \dot{K_i} + w_i K_i tr D_0^p \quad (12)$$

The importance of the rate expressions (8), (11) and (12), lies in their use for the rate equations of evolution of the corresponding internal variables during plastic loading; in other words these are the proper constitutive rates for those variables emerging directly from the physico-geometrical coupling of their embedding from the initial to the relaxed intermediate configuration. Notice that while the corotational rate of Eq.(11) was considered in past theories associated with the plastic and constitutive spins, here one adds a new family of constitutive rates, the convected ones expressed by Eqs. (8) and being independent of plastic spin, hence the words "beyond the plastic spin" of the title of this work. In simple terms one may state that the constitutive and plastic spins express the need for differentiating the deformation and rotation of the continuum from an underlying substructure (Dafalias, 1987, 1998) with which the rotationally embedded internal variables are associated, while the plastic convected embedding is related to internal variables which evolve in association with the plastic deformation of the continuum with no reference to any substructure. Clearly these two types of embedding reflect very different physics characterizing the definition and rate evolution of the corresponding variables.

4   Rate equations and loading index

4.1   Rate equations

The plasticity equations are in essence rate equations for the kinematical and internal variables. The kinematical variables were defined by Eq. (3) as

the plastic rate of deformation D_0^p and the plastic material spin W_0^p at the intermediate configuration. It is very important to understand that only D_0^p is defined by constitutive relations while the W_0^p acquires any value one desires reflecting the arbitrariness of the orientation of the intermediate configuration; one such value is $W_0^p = 0$ for the so-called spinless configuration, Dafalias (1998). Thus, one can write for D_0^p

$$D_0^p = <\lambda> N_0^p \tag{13}$$

where N_0^p is a tensor-valued function of the state variables (stress and internal variables), part of the constitutive formulation, and λ is the scalar-valued plastic multiplier or loading index whose sign defines loading/unloading in conjunction with the Macauley brackets $<>$, where $<\lambda> = \lambda$ if $\lambda > 0$, and $<\lambda> = 0$ if $\lambda \leq 0$. The value of λ will be defined in the sequel in association with the consistency condition.

For the convected embedding of internal variables according to Eqs.(5), the rate equation of evolution is expressed in terms of the corresponding convected rate of Eq.(8) and reads

$$\overset{\vee}{A}_i = <\lambda> \bar{A}_i \tag{14}$$

where $\bar{A}_i$ is a tensor-valued function of the state variables (stress and internal variables). By stating Eq.(14) it is tacitly assumed that the same $<\lambda>$ controls simultaneously the appearance of plastic rate of deformation as per Eq.(13), and the evolution of internal variables. In other words it is only when plastic deformation takes place that internal variables evolve in the way expressed by Eq. (14). In order to appreciate also the constitutive meaning of Eq.(14) notice that according to the relation in the paragraph preceding Eq.(8), the $\overset{\vee}{A}_i = 0$ at the intermediate configuration implies that $\dot{A}_i^r = 0$ at the initial configuration. This interrelation expresses the corresponding physico-geometrical coupling for the internal variable A_i. Since the $\overset{\vee}{A}_i$ definition in Eq.(8) depends on D_0^p, it follows that the completeness of Eq.(14) relies on Eq. (13).

For the rotationally embedded internal variables according to Eq.(6), the corresponding co-rotational rate of Eq.(11) is the one used in the rate equation of evolution according to

$$\overset{\circ}{A}_i = <\lambda> \bar{A}_i \tag{15}$$

where again $\bar{A}_i$ is a tensor-valued function of the state variables and the appearance of $< \lambda >$ implies the simultaneous occurrence of plastic deformation. Notice that trD_0^p enters the definition of $\overset{\circ}{A}_i$, hence, Eq.(13) must be employed in Eq.(15). Most importantly, the definition of $\overset{\circ}{A}_i$ in Eq.(11) includes also the plastic spin W_i^p, thus, Eq.(15) is not complete until one specifies the plastic spin associated with the corresponding internal variable. Here is where the theory of Plastic Spin (Dafalias, 1985) enters the picture. The plastic spin determination by a constitutive equation is not but a constitutive ingredient of the constitutive relation expressed by Eq.(15), and has been again erroneously considered in many past publications as an issue of kinematics rather than a constitutive issue. Following the development and notation of Dafalias (1985, 1998) one can then write

$$W_i^p =< \lambda > \Omega_i^p \qquad (16)$$

where Ω_i^p is a tensor-valued antisymmetric function of the state variables (stress and internal variables), and the appearance of $< \lambda >$ implies the simultaneous occurrence of plastic deformations according to Eq.(13) for plastic spin to be non-zero. Use of Eq.(16) renders Eq.(15) complete.

Finally for the scalar-valued internal variable K_i one can write the following rate equation based on Eq.(12)

$$\overset{\vee}{K}_i =< \lambda > \bar{K}_i \qquad (17)$$

with simultaneous use of Eq. (13) for D_0^p entering the expression of $\overset{\vee}{K}_i$, and $\bar{K}_i$ a scalar-valued function of its argument.

The convected and corotational rates of Eqs. (14) and (15) are objective, thus, it follows that the corresponding $\bar{A}_i$ are isotropic symmetric tensor-valued functions of their arguments in order to satisfy objectivity. Similarly, in Eq.(10) the spins W_0^p and ω_i are not objective but their difference, the plastic spin W_i^p, can be shown to be objective (Dafalias, 1983), thus, the Ω_i^p is also objective, which means it must be an isotropic antisymmetric tensor-valued function of the state variables. The isotropy of $\bar{A}_i$ and Ω_i^p allows their analytical expressions to be obtained rigorously from the representation theorems of isotropic functions (Wang, 1970, Smith, 1971), and such expressions were first proposed in Dafalias (1983, 1984, 1985a,b) and Loret (1983). Objectivity imposes also isotropy for $\bar{K}_i$ of Eq.(17).

It will be expedient now to put all above rate equations for the tensor-valued internal variables A_i into a common expression, irrespective of the

kind of embedding. This will be achieved by solving for the Jaumann coro-tational rate $\overset{\triangledown}{A}_i$ (Eq. (9)) which appears in both Eqs. (8) and (11), and simultaneously use Eqs. (13), (14), (15) and (16), to finally obtain after some algebra

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\begin{aligned}
\overset{\triangledown}{A}_i &= \dot{A}_i - W_0^p A_i + A_i W_0^p \\
&= <\lambda> [\bar{A}_i - s_1^{(i)} \varepsilon (A_i N_0^p + s_2^{(i)} N_0^p A_i) \\
&\quad + (1 - \varepsilon)(A_i \Omega_i^p - \Omega_i^p A_i) - w_i A_i tr N_0^p]
\end{aligned}
\tag{18}
$$

where $s_1^{(i)}$ and $s_2^{(i)}$ acquire the values +1 or -1 according to the kind of convected embedding of A_i as described after Eq.(8), $\varepsilon = 1$ for convected embedding and corresponding convected rates of Eqs. (5) and (8), respectively, and $\varepsilon = 0$ for rotational embedding and corresponding corotational rate of Eqs. (6) and (11), respectively.

4.2   Loading index

With the constitutive functions N_0^p, Ω_i^p and $\bar{A}_i$ given, the plastic rate of deformation tensor D_0^p in Eq.(13) and the evolution Eq.(18) for each internal variable A_i, requires only the specification of the loading index λ in order to become complete since the plastic material spin W_0^p can be given any value, Dafalias (1998). It will be necessary to first introduce the yield surface in stress space whose analytical expression is given by

$$
f(\Pi, A_i, K_i) = 0
\tag{19}
$$

where Π is the 2nd Piola-Kirchhoff stress tensor, related to the Cauchy stress tensor σ according to $\Pi = |F^e| F^{e-1} \sigma F^{e-T}$ and being conjugate of the Green strain tensor E^e in regards to a Helmholtz free energy function $\Psi(E^e, A_i, K_i)$ according to

$$
\Pi = \rho_0 \frac{\partial \Psi}{\partial E^e}
\tag{20}
$$

where ρ_0 is the mass density at the intermediate configuration changing during plastic deformation. For future reference notice that $\dot{\rho}_0 = -\rho_0 tr D_0^p = - <\lambda> \rho_0 N_0^p$.

In order to satisfy objectivity the f and Ψ must be isotropic functions of their arguments Π, A_i, K_i, E^e (recall that the intermediate configuration where the Π, A_i, K_i, E^e are defined, can be arbitrarily oriented). For a scalar-valued or a second order tensor-valued function, Dafalias (1985) has shown

that the rate of such function can be expressed in terms of any corotational rate for the function and its arguments (of course the rate of a scalar-valued entity is always the usual material time derivative). This property has been extended to an arbitrary order tensor-valued function by Hashiguchi (2003). Applying this result to the rate of f, the consistency condition of plasticity is expressed as

$$\dot{f} = \frac{\partial f}{\partial \Pi} : \dot{\Pi} + \frac{\partial f}{\partial A_i} : \dot{A}_i + \frac{\partial f}{\partial K}\dot{K} = \frac{\partial f}{\partial \Pi} : \overset{\nabla}{\Pi} + \frac{\partial f}{\partial A_i} : \overset{\nabla}{A}_i + \frac{\partial f}{\partial K_i}\dot{K}_i = 0 \quad (21)$$

where the corotational rates $\overset{\nabla}{\Pi}$ and $\overset{\nabla}{A}_i$ are defined in regards to the plastic material spin W_0^p (see also Eq.(9)). Summation over i is now implied. It suffices now to introduce in Eq.(21) the expression (18) for $\overset{\nabla}{A}_i$ and the expression $\dot{K}_i = < \lambda > (\bar{K}_i - w_i K_i tr N_0^p)$ for $\dot{K}_i$ obtained from Eqs.(12), (13) and (17), and making repeated use of the trace property of the product of tensors $AB : C = A : BC = BC : A = B : CA$, solve for λ to obtain

$$\lambda = \frac{N_0^n : \overset{\nabla}{\Pi}}{\Delta} \qquad (22)$$

$$\Delta = H + s_1^{(i)}\varepsilon(s_2^{(i)} A_i \frac{\partial f}{\partial A_i} + \frac{\partial f}{\partial A_i}A_i) : N_0^p$$

$$+ (1 - \varepsilon)(A_i \frac{\partial f}{\partial A_i} - \frac{\partial f}{\partial A_i}A_i) : \Omega_i^p \qquad (23a)$$

$$+ (w_i \frac{\partial f}{\partial A_i} : A_i + w_i \frac{\partial f}{\partial K_i}K_i)tr N_0^p$$

$$H = -(\frac{\partial f}{\partial A_i} : \bar{A}_i + \frac{\partial f}{\partial K_i}\bar{K}_i) \qquad (23b)$$

where $N_0^n = \partial f/\partial \Pi$ represents the gradient to the yield surface at Π, H is the plastic modulus, often symbolized also by K_p, but here H is preferred in agreement with the notation in Dafalias (1998) for ease of comparison. In the above equations summation over i is implied even if i is repeated more than two times in a term, and this includes the terms with $s_1^{(i)}$ and $s_2^{(i)}$.

With λ given by Eqs. (22) and (23) in terms of the stress rate $\overset{\nabla}{\Pi}$, Eq.(13) allows the calculation of the plastic rate of deformation tensor D_0^p. What is desirable however is to have an expression of $\overset{\nabla}{\Pi}$ in terms of the total rate

of deformation tensor D, since this relation is fundamental in numerical calculations. To this extend the first step is to express λ in terms of D. To achieve this goal one takes the rate of Eq. (20) accounting for the fact that Ψ is an isotropic function of E^e, A_i, K_i and so is its derivative with respect to E^e, hence, its rate can be expressed in terms of any corotational rate as done before for the rate of f. Considering such corotational rate in relation to the plastic material spin W_0^p and accounting for the relation $\partial\Pi/\partial\rho_0 = \Pi/\rho_0$, write

$$\overset{\nabla}{\Pi} = \frac{\dot{\rho}_0}{\rho_0}\Pi + \frac{\partial\Pi}{\partial E^e} : \overset{\nabla}{E^e} + \frac{\partial\Pi}{\partial A_i} : \overset{\nabla}{A_i} + \frac{\partial\Pi}{\partial K_i}\dot{K}_i \qquad (24)$$

If one multiplies both sides of Eq.(24) by $N_0^n = \partial f/\partial\Pi$ and then take the trace of the product, the left hand side of the resulting equation will provide the numerator of Eq.(22) which equals the loading index λ multiplied by the denominator of Eq.(22). The right hand side of the resulting equation will involve the rates $\dot{\rho}_0$ expressed by $\dot{\rho}_0 = -\rho_0 tr D_0^p$ reflecting the mass conservation in the intermediate configuration, $\overset{\nabla}{E^e}$ expressed by Eq. (4), $\overset{\nabla}{A_i}$ expressed by Eq.(18) and $\dot{K}_i$ expressed by Eq. (12) in conjunction with Eq.(17). The D_0^p in these equations is expressed by Eq.(13) in terms of λ. Consequently, the right hand side will also involve λ, hence, together with the left hand side they provide an equation which can be solved for λ, but now instead of the stress rate the term $F^{eT}DF^e$ of Eq.(4) will appear expressing λ in terms of D, as intended. This new expression for λ reads as follows:

$$\lambda = \frac{N_0^n : L_0 : F^{eT}DF^e}{\Delta + N_0^n : Z} \qquad (25a)$$

$$\begin{aligned}
Z = \ &L_0 : (C^e N_0^p)_s - (C_{A_i} : \bar{A}_i + C_{K_i}\bar{K}_i) \\
&+ s_1^{(i)}\varepsilon(s_2^{(i)}A_i C_{A_i} + C_{A_i}A_i) : N_0^p \\
&+ (1-\varepsilon)(A_i C_{A_i} - C_{A_i}A_i) : \Omega_i^p \\
&+ (\Pi + w_i C_{A_i} : A_i + w_i C_{K_i}K_i)tr N_0^p
\end{aligned} \qquad (25b)$$

where Δ is given by Eq. (23a), the summation over i is implied in Eq.(25b) even if i is repeated more than two times in a term, and

$$L_0 = \frac{\partial\Pi}{\partial E^e} = \rho_0\frac{\partial\Psi}{\partial E^e \otimes \partial E^e} \qquad (26)$$

represents the fourth order tensor of elastic tangent moduli while

$$C_{A_i} = \frac{\partial \Pi}{\partial A_i} = \rho_0 \frac{\partial^2 \Psi}{\partial E^e \otimes \partial A_i}; \; C_{K_i} = \frac{\partial \Pi}{\partial K_i} = \rho_0 \frac{\partial^2 \Psi}{\partial E^e \otimes \partial K_i} \qquad (27)$$

are the fourth and third order elastoplastic coupling tensors, respectively, due to the effect of the internal tensor-valued variables A_i and scalar-valued variables K_i on the Helmholtz free energy Ψ, and by extension on the elastic properties. Attention should be given as to how the various multiplications and trace operations among tensor-valued quantities are executed in Eq.(25b) when they involve the fourth order tensors $L_0 = \partial \Pi / \partial E^e$, $C_{A_i} = \partial \Pi / \partial A_i$ and the third order tensor $C_{K_i} = \partial \Pi / \partial K_i$. Observe that all these tensors are produced by partial differentiation of Π, thus, based on the derivation of Eq.(25b) it follows that the trace operation of the term $N_0^n : Z$ of Eq.(25a) takes place in regards to the first two indices of the fourth and third order tensors, i.e to the indices of Π, while all other operations of multiplication and trace appearing in Eq.(25b) are executed in regards to the last two indices of the fourth and last index of the third order tensors. For example, omitting for simplicity the factor $s_2^{(i)}$ and the indices of A_i, N_0^p, C_{A_i}, and considering one terms of Z as an example, one can express as follows in indicial notation the operation $(AC + CA) : N = (A_{ij}C_{abjk} + C_{abim}A_{mk})N_{ki}$. Observe that the indices a, b of the fourth order tensor C_{abjk} do not participate in the operations and are kept to be used in the trace operation $N_0^n : Z$.

Substitution of Eq.(25a) in the terms of the right hand side of Eq.(24) which involve λ in conjunction with Eq.(4) yields after some easy manipulation the sought result

$$\overset{\triangledown}{\Pi} = \Lambda : F^{eT}DF^e = (L_0 - \frac{Z \otimes N_0^n : L_0}{\Delta + N_0^n : Z}) : F^{eT}DF^e \qquad (28)$$

where Λ, as expressed clearly in Eq.(28), is the fourth order tensor of tangent elastoplastic moduli. As already pointed out in Dafalias (1998), the quantity $F^{eT}DF^e$ is the work conjugate to Π because it can be shown that $(\Pi : F^{eT}DF^e)/\rho_0 = (\sigma : D)/\rho$ with ρ_0 and ρ the mass densities at the intermediate and current configurations, respectively.

The new aspects of the foregoing formulation in comparison with the formulation in Dafalias(1998) are as follows: (i) the internal variables A_i may be embedded in a convected way with the plastically deforming continuum, in addition to the rotational embedding associated with constitutive and

plastic spins, hence, going beyond the plastic spin. This is reflected by the value $\varepsilon = 1$ in Eqs. (23a) and (25b); (ii) scalar-valued internal variables K_i were explicitly considered in addition to the tensor-valued A_i; (iii) all internal variables, including convected and rotationally embedded as well as scalar-valued, can be considered as relative scalars or tensors of order w_i; this is reflected by the non-zero value of w_i in the relevant equations. Thus, if one sets $\varepsilon = 0$ and $w_i = 0$ as well as $K_i = \bar{K}_i = 0$ in Eqs.(23a) and (25b), the denominators of Eqs. (21) and (25) in Dafalias (1998) are retrieved as expected, since in this reference only rotationally embedded tensor-valued internal variables were considered of weight $w_i = 0$.

4.3   Alternative derivation of plastic multiplier

The derivation of the expressions (22) and (25) for the plastic multiplier or loading index λ was largely based on applying the property of scalar or tensor valued isotropic functions to have their rate expressed in terms of any corotational rate, i.e a Jaumann rate in regards to any spin (Dafalias, 1985, 1988, Hashiguchi, 2003). This property was in fact applied in Eqs. (21) and (24). An alternative way to derive the expression for λ will be briefly presented in the sequel.

It was shown in Dafalias (1998) that for a scalar-valued isotropic function of tensor valued variables such as the expression (19) for the yield surface $f(\Pi, A_i, K_i)$, used here for convenience, the following identity holds

$$\Pi\frac{\partial f}{\partial \Pi} - \frac{\partial f}{\partial \Pi}\Pi + A_i\frac{\partial f}{\partial A_i} - \frac{\partial f}{\partial A_i}A_i \equiv 0 \tag{29}$$

where summation over i applies. The presence of the scalar valued K_i in f, not considered in Dafalias (1998), has no effect on the validity of identity (29).

Similarly, for a tensor-valued function, chosen here for convenience to be the stress tensor Π which is function of E^e, A_i, K_i since so is Ψ, Eq. (20), Dafalias (1998) has proved the identity

$$\left[E^e\frac{\partial \Pi}{\partial E^e} - \frac{\partial \Pi}{\partial E^e}E^e + A_i\frac{\partial \Pi}{\partial A_i} - \frac{\partial \Pi}{\partial A_i}A_i\right] : \omega + \Pi\omega - \omega\Pi \equiv 0 \tag{30}$$

for *any* spin (antisymmetric tensor) ω, where summation over i applies. The presence of the scalar-valued K_i in the arguments of Π in this presentation, not considered explicitly in Dafalias (1998), does not invalidate identity

(30). Similarly to the comments after Eq.(27) on the implied indices operations of Eq.(25), in Eq.(30) the multiplication of the partial derivatives of Π with other tensors and the subsequent trace operation with ω take place without involving the indices of Π which remain as were before the partial differentiation.

Consider now the consistency condition $\dot{f} = 0$ as in Eq.(21), but now use the usual rates of the arguments Π, A_i, K_i of $f = 0$ instead of corotational rates, and substitute for $\dot{A}_i$ the expression obtained from Eq.(18), and for $\dot{K}_i$ the expression obtained from Eq. (12) in conjunction with Eq.(17) to obtain

$$\dot{f} = \frac{\partial f}{\partial \Pi} : \dot{\Pi} + \frac{\partial f}{\partial A_i} : \dot{A}_i + \frac{\partial f}{\partial K_i} \dot{K}_i = \frac{\partial f}{\partial \Pi} : \dot{\Pi} + \frac{\partial f}{\partial A_i} : (W_0^p A_i - A_i W_0^p)$$

$$+ <\lambda> \frac{\partial f}{\partial A_i} : [\bar{A}_i - s_1^{(i)} \varepsilon (A_i N_0^p + s_2^{(i)} N_0^p A_i) + (1 - \varepsilon)(A_i \Omega_i^p - \Omega_i^p A_i)$$

$$- w_i A_i tr N_0^p] + <\lambda> \frac{\partial f}{\partial K_i}(\bar{K}_i - w_i K_i tr N_0^p)$$

$$= \frac{\partial f}{\partial \Pi} : \dot{\Pi} + (A_i \frac{\partial f}{\partial A_i} - \frac{\partial f}{\partial A_i} A_i) : W_0^p + <\lambda> [\frac{\partial f}{\partial A_i} : \bar{A}_i + \frac{\partial f}{\partial K_i} \bar{K}_i$$

$$- s_1^{(i)} \varepsilon (s_2^{(i)} A_i \frac{\partial f}{\partial A_i} + \frac{\partial f}{\partial A_i} A_i) : N_0^p - (1 - \varepsilon)(A_i \frac{\partial f}{\partial A_i} - \frac{\partial f}{\partial A_i} A_i) : \Omega_i^p$$

$$- (w_i \frac{\partial f}{\partial A_i} : A_i + w_i \frac{\partial f}{\partial K_i} K_i) tr N_0^p] = 0$$

(31)

Based on identity (29), the first two terms of the last member of Eq.(31) can be written as:

$$\frac{\partial f}{\partial \Pi} : \dot{\Pi} + (A_i \frac{\partial f}{\partial A_i} - \frac{\partial f}{\partial A_i} A_i) : W_0^p = \frac{\partial f}{\partial \Pi} : \dot{\Pi} - (\Pi \frac{\partial f}{\partial \Pi} - \frac{\partial f}{\partial \Pi} \Pi) : W_0^p$$

$$= \frac{\partial f}{\partial \Pi} : (\dot{\Pi} - W_0^p \Pi + \Pi W_0^p) = \frac{\partial f}{\partial \Pi} : \overset{\nabla}{\Pi}$$

(32)

Inserting the expression (32) in Eq.(31) and solving for λ yields Eqs.(22) and (23a,b) exactly, QED.

In what follows let us re-write for simplicity of notation Eq. (18) as $\dot{A}_i = <\lambda> \bar{A}_i^c + W_0^p A_i - A_i W_0^p$, where the $\bar{A}_i^c$ includes all terms in [ ] of Eq.(18), and also rewrite $\dot{K}_i = <\lambda> (\bar{K}_i - w_i K_i tr N_0^p) = <\lambda> \bar{K}_i^c$ with obvious definition of $\bar{K}_i^c$. Take now the rate of Π, Eq.(20), as done in Eq.(24), but now use regular rates for the variables E^e, A_i instead of corotational

rates, express the rates of $\dot{A}_i$ and $\dot{K}_i$ by the aforementioned equations , use the mass conservation equation $\dot{\rho}_0 = -\rho_0 tr D_0^p = - <\lambda> \rho_0 tr N_0^p$ and finally use the identity (30) with $\omega = W_0^p$ to obtain

$$
\begin{aligned}
\dot{\Pi} &= \frac{\dot{\rho}_0}{\rho_0}\Pi + \frac{\partial\Pi}{\partial E^e}:\dot{E}^e + \frac{\partial\Pi}{\partial A_i}:\dot{A}_i + \frac{\partial\Pi}{\partial K_i}\dot{K}_i \\
&= \frac{\partial\Pi}{\partial E^e}:\dot{E}^e + \frac{\partial\Pi}{\partial A_i}:[<\lambda>\bar{A}_i^c + W_0^p A_i - A_i W_0^p] \\
&\quad + <\lambda>(\frac{\partial\Pi}{\partial K_i}\bar{K}_i^c - \Pi tr N_0^p) \\
&= \frac{\partial\Pi}{\partial E^e}:\dot{E}^e + <\lambda>[\frac{\partial\Pi}{\partial A_i}:\bar{A}_i^c + \frac{\partial\Pi}{\partial K_i}\bar{K}_i^c - \Pi tr N_0^p] \qquad (33) \\
&\quad + (A_i\frac{\partial\Pi}{\partial A_i} - \frac{\partial\Pi}{\partial A_i}A_i):W_0^p \\
&= \frac{\partial\Pi}{\partial E^e}:\dot{E}^e + <\lambda>[\frac{\partial\Pi}{\partial A_i}:\bar{A}_i^c + \frac{\partial\Pi}{\partial K_i}\bar{K}_i^c - \Pi tr N_0^p] \\
&\quad - (E^e\frac{\partial\Pi}{\partial E^e} - \frac{\partial\Pi}{\partial E^e}E^e):W_0^p - (\Pi W_0^p - W_0^p\Pi)
\end{aligned}
$$

It is easy to see now that transferring the term $\Pi W_0^p - W_0^p\Pi$ at the left hand side of Eq.(33) and observing that $(E^e(\partial\Pi/\partial E^e) - (\partial\Pi/\partial E^e)E^e):W_0^p = (W_0^p E^e - E^e W_0^p):(\partial\Pi/\partial E^e)$, one can rewrite Eq.(33) as

$$
\overset{\nabla}{\Pi} = \frac{\partial\Pi}{\partial E^e}:\overset{\nabla}{E}^e + <\lambda>[\frac{\partial\Pi}{\partial A_i}:\bar{A}_i^c + \frac{\partial\Pi}{\partial K_i}\bar{K}_i^c - \Pi tr N_0^p] \qquad (34)
$$

With the foregoing definitions of $\bar{A}_i^c$ and $\bar{K}_i^c$ one can straightforwardly show that Eq.(34) is identical to Eq.(24), hence, the same steps can subsequently be taken to derive Eqs.(25a,b), QED. It should be observed that it is not really surprising that this alternative method of obtaining the expression for the loading index λ using the identities expressed by Eqs. (29) and (30) for isotropic functions, yields the same answer as the previous method which was based on the property of isotropic functions to have their rates expressed in terms of corotational rates in regards to *any* spin; the reason is that the derivation of the former (i.e. the identities) was based on the latter property about corotational rates, as shown in Dafalias (1998).

5   Conclusion

The main contribution of this work is to re-examine the formulation of the constitutive relations framework under finite plastic deformations when the

tensor-valued internal variables entering the yield surface are embedded into the plastically deforming continuum in a convected way, thus, extending the previous case of rotational embedding associated with the concepts of constitutive and plastic spins. The choice of the embedding in association with the plastic deformation, reflects what has been called Physico-Geometrical coupling in Dafalias (2001) for embedding with elastic deformation. Exploiting certain properties of scalar-valued and tensor-valued isotropic functions of scalar-valued and tensor-valued variables, it was possible to derive explicit expressions for the loading index or plastic multiplier λ in two different but equivalent ways, by solving the consistency condition for it. Additional novelties of lesser importance are the explicit introduction of scalar-valued internal variables and of relative, rather than only absolute, tensor-valued variables of weight w_i for either convected or rotationally embedded internal variables. It was interesting that no matter what kind of rotational or convected embedding one considers, the formulation ends up with Jaumann corotational rates at the intermediate configuration for either stress or internal variables in regards to the plastic material spin W_0^p, which can be arbitrarily chosen at the intermediate configuration reflecting the arbitrary orientation of the latter, including the choice $W_0^p = 0$ for the so-called spinless configuration Dafalias (1998). In the formulation general elastoplastic coupling is assumed, one reason for the rather complex but explicit form of the final equations, while the nature of the work is necessarily rather technical.

In addition to the aforementioned contributions, the opportunity was given to discuss and clarify again two important points in regards to formulations of finite elastoplastic deformations: 1.The erroneous interpretations attributed in the past to the notion of plastic spin, usually confusing it with the distinctly different plastic material spin (antisymmetric part of the plastic velocity gradient at the intermediate configuration), and 2. The equation derived in Dafalias (1998), which within the framework of the multiplicative decomposition of the deformation gradient provides a purely elastic and purely plastic decomposition of rates of deformation, as opposed to the practice of a direct decomposition of the total rate of deformation into an elastic and a plastic part, with the latter not being purely plastic. In many respects the present work is an extension of and supplement to the previous work by Dafalias (1998), going beyond the plastic spin in considering the convected embedding of tensor-valued internal variables with the plastic deformation.

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Konačne elastoplastične deformacije: izvan plastičnog spina

U radu se analizira primjena unutrašnjih promjenljivih tenzorskog karaktera u konstitutivnoj analizi plastičnih deformacija, sa posebnim akcentom na vezu izmedju geometrijske karakterizacije deformacije i fizičkih svojstava opisanih unutrasnjim promenljivim. U ranijim radovima uobičajeno se pretpostavljalo da su unutrasnje promjenljive tokom deformacije korotacione sa materijalnom substrukturom, ali ne i sa kontinuumom. Ovo je rezultiralo uvodjenjem koncepta konstitutivnog i plastičnog spina za svaku unutrasnju tenzorsku promenljivu. Opisani pristup je prosiren u ovom radu sa korotacione na konvektivnu prirodu ponasanja unutrasnjih promenljivih. Koristeći odgovarajuća svojstva izotropnih tenzorskih funkcija, princip materijalne objektivnosti i uslov plastične konsistentnosti, izvedeni su novi izrazi za indeks plastičnog opterećenja.