

# Stochastic finite elements: where is the physics?

Martin Ostoja-Starzewski \*

## Abstract

The micromechanics based on the Hill-Mandel condition indicates that the majority of stochastic finite element methods hinge on random field (RF) models of material properties (such as Hooke's law) having no physical content, or even at odds with physics. At the same time, that condition allows one to set up the RFs of stiffness and compliance tensors in function of the mesoscale and actual random microstructure of the given material. The mesoscale is defined through a Statistical Volume Element (SVE), i.e. a material domain below the Representative Volume Element (RVE) level. The paper outlines a procedure for stochastic scale-dependent homogenization leading to a determination of mesoscale one-point and two-point statistics and, thus, a construction of analytical RF models.

**Keywords:** random media, random fields, mesoscale, anisotropy, stochastic finite elements, multiscale methods, uncertainty quantification

## 1 Motivation behind the stochastic finite elements (SFE)

With the advent of "multiscale methods", the contemporary solid mechanics begins to recognize the hierarchical structure of materials, but hardly their statistical nature. The latter aspect has been present in computational mechanics through the so-called stochastic finite elements (SFE). There has not been much connection between the two: the multiscale methods remain mostly deterministic, while the

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\*Department of Mechanical Science and Engineering, also Institute for Condensed Matter Theory, and Beckman Institute, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

SFE has been oblivious to multiscale issues such as micromechanics, homogenization and upscaling. Clearly, there is a need to connect both fields so as to develop a multiscale stochastic mechanics.

Now, a mechanical/aerospace/civil engineer wants to be able to predict transient displacements, velocities and stresses. For example, in the realm of geomechanics, prediction of dynamic, not just quasi-static, responses is crucial to safe placement and operation of the ground-based infrastructure (buildings, bridges, etc.). This critical need is highlighted by accidents and disasters where large, usually subterranean energies are being released from, say, gas pipe explosions, earthquakes, and mine and tunnel collapse. Conventional analyses of such events are based on rather simple models of continuum mechanics set up on deterministic, homogeneous fields of mass density and material properties. Such simplifying assumptions are in stark contrast to the highly heterogeneous nature of solids, soils and rocks where the separation of scales can hardly be justified. Clearly, a new generation of models is needed: formulation of constitutive equations for multiscale stochastic materials and development of solution methods for static and transient dynamic responses.

These challenges are expounded in Fig. 1, where a wavefront propagates from a buried source (e.g. gas main). As the wavefront evolves, its thickness broadens by one or two orders of magnitude, it attenuates geometrically and viscoplastically. However, the evolution is non-deterministic because the microstructure being encountered is spatially random; in fact, there may be a (dry or water-saturated) porous medium on very fine length scales, accompanied by a network of large cracks on a larger length scale. As the wavefront broadens, the material properties that are to enter the mechanics model should be smeared out over ever larger length scales [1]. In this composite figure we also illustrate a possibility that the wavefront encounters a geological stratification beyond which the structure is that of a compacted granular medium. Thus, we need a method for homogenizing random heterogeneous microstructures in function of the wave length and the length scale of a typical grain size  $d$  is that of stochastic (rather than deterministic) wave propagation. As already established in the context of shock and acceleration waves in one-dimensional (1-D) random media [e.g. 2,3], a deterministic evolution – based on a straightforward averaging of material parameters – differs from that found in a stochastic model with material randomness present.

## 2 A brief review of phenomenological SFE

Figures 1 and 2 point to situations where the separation of scales

$$\left. \begin{array}{l} d \ll \\ d < \end{array} \right\} L \ll L_{macro}, \quad (2.1)$$

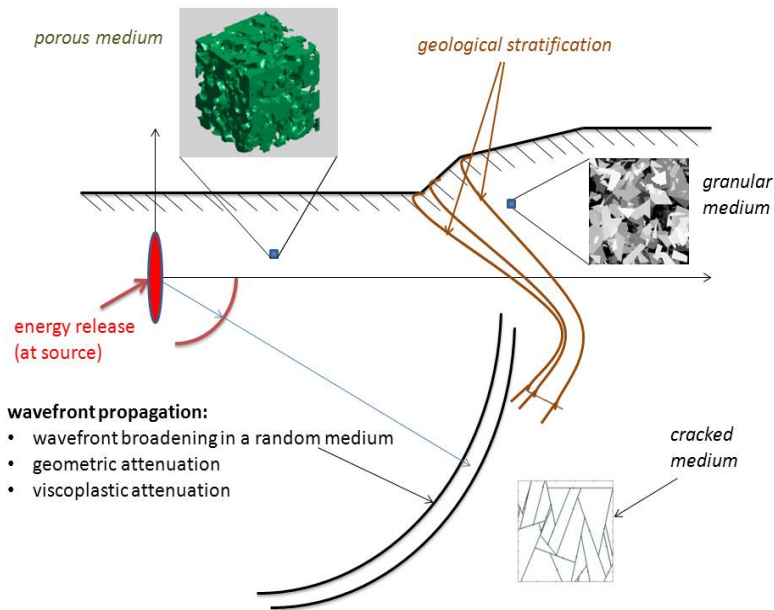


Figure 1: A schematic showing evolution of a wavefront in a random geological medium with stratification and three different microstructures: porous, cracked, and granular.

does not necessarily hold, which implies that we need a strategy to deal with material properties below the *representative volume element* (RVE). This equation as well as Fig. 2 shows three levels: (a) the *microscale*  $d$ , (b) the *mesoscale*  $L$ , and (c) the *macroscale*  $L_{macro}$  of the entire body domain for which we wish to solve a boundary value problem. The mesoscale refers to the domain over which we introduce a constitutive law, such as the wavefront thickness in Fig. 1. It is a tacit assumption in conventional solid mechanics that  $L$  is sufficiently large to allow the homogenization of the random microstructure and sufficiently small to play the role of an infinitesimal element (i.e. RVE) relative to  $L_{macro}$ . If that separation of scales is not justified, only a random continuum can be used. As a result, we need to establish some methods to deal with solution of macroscopic boundary value problems having the mesoscale SVE as input. Such problems are necessarily stochastic, and this leads us to a formulation of random fields (RFs) of material properties from the *statistical volume element* (SVE) information, and their input into numerical methods leading then to so-called *stochastic finite element* (SFE) and *stochastic finite difference methods*.

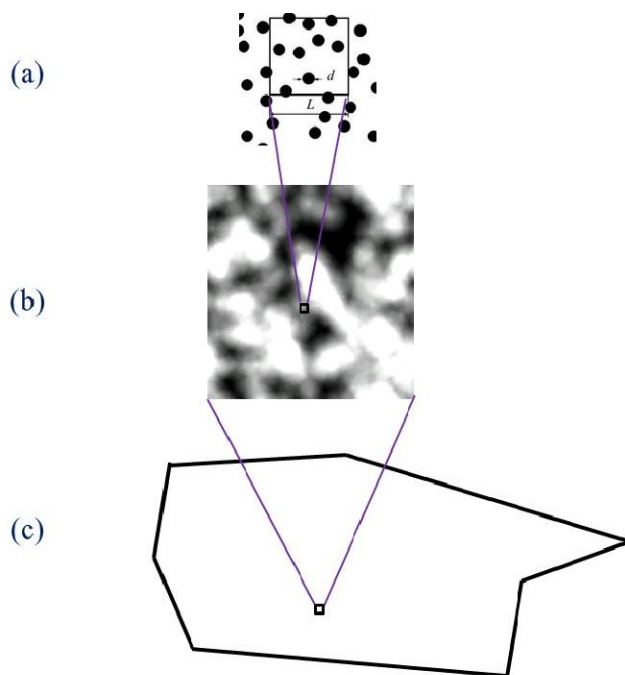


Figure 2: (a) A Boolean model of a random material; (b) a mesoscale continuum approximation, modeling smoothly inhomogeneous medium by placing a mesoscale window in the microstructure of (a); (c) a macroscopic body.

The strategy of conventional SFE is different: in that they basically proceed as follows: (i) assume a RF of constitutive coefficients, (ii) use it as input into the global FE scheme, usually based on the minimum potential energy principle, and (iii) derive the global response either for the first two moments or for the ensemble in the Monte Carlo sense [4-13]; see a critical review in [14]. In the following we will briefly review the basic tenets of the SFE, but first we observe that, given a deterministic field equation in mechanics

$$\mathcal{L}\mathbf{u} = \mathbf{f}, \quad (2.2)$$

randomness may enter through either the operator (i.e., material properties), the forcing function (temporal in nature), or the boundary and/or initial conditions. However, the choice of randomness of forcing  $\mathbf{f}$  in time is fundamentally different from the randomness of the field operator  $\mathcal{L}$  in physical space. The point is that

$\mathbf{f}(\omega, t)$  in (2.2) implicitly involves some *local averaging* in the time domain

$$f_{\Delta t}(\omega, t) = \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} f(\omega, t') dt', \quad (2.3)$$

which is needed to smear out fluctuations in, say, wind forcing on a structure, over time scales too short to have any influence on the oscillator. Commonly the subscript  $\Delta t$  in  $f$  on the left hand-side is suppressed, and we simply write  $f(\omega, t)$ . On the other hand, the local averaging in physical space is not consistent with the concepts of micromechanics, and, as shown in Chapter 7, should be replaced by stochastic homogenization, which, by offering three optional boundary conditions, leads to a non-uniqueness of continuum approximation. Now, if local averaging is applied to a stiffness (respectively, compliance) tensor field, it yields a Voigt-type (Reuss-type) estimate of stiffness (compliance) for some spatial domain of the microstructure. As discussed in Section 3.1, the local (or spatial) averaging should be applied to the energy density, thus yielding the Hill-Mandel condition as a basis for scale-dependent constitutive laws.

Most of the SFE studies are based on a direct generalization of Hooke's law to RFs restricted to the case of weak fluctuations in material properties, whereby the stiffness matrix is expressed as the sum of the mean  $[K]$  and the random noise  $K'(\omega)$

$$[K(\omega)] = \langle [K] \rangle + \epsilon [K'(\omega)], \quad \omega \in \Omega, \quad \epsilon \ll 1. \quad (2.4)$$

**Perturbation method** This approach consists in a replacement of the random system by a (theoretically infinite) number of identical deterministic systems each of which depends on the solution for the lower order equations. Thus, to second order, for the static problem  $- [K(\omega)] \{U\} = \{f\}$  – the solution is expressed as the sum

$$\{U\} = \{U_0\} + \epsilon \{U_1\} + \epsilon^2 \{U_2\}, \quad \epsilon \ll 1. \quad (2.5)$$

This leads to a system of equations

$$\begin{aligned} \{U_0\} &= \langle [K] \rangle^{-1} \{f\}, \\ \{U_1\} &= - \langle [K] \rangle^{-1} [K'(\omega)] \{U_0\}, \\ \{U_2\} &= - \langle [K] \rangle^{-1} [K'(\omega)] \{U_1\}. \end{aligned} \quad (2.6)$$

**Neumann series method** This method [15], is based on a Neumann series for the inverse of the random operator  $[K(\omega)]$ , which takes the following form

$$\begin{aligned} [K(\omega)] &= (I - P(\omega) + P^2(\omega) - P^3(\omega) + \dots) \langle [K] \rangle^{-1} \\ P(\omega) &= \langle [K] \rangle^{-1} [K'(\omega)]. \end{aligned} \quad (2.7)$$

The approach was introduced as an avenue for a speedier way of solving the stochastic problem by a Monte Carlo simulation. To that end, also a Cholesky decomposition of  $[K(\omega)]$  is implemented.

**Weighted integral method** This method, in the setting of an elastic plate problem [16], begins with a locally isotropic RF of, say, Young's modulus and assign it to all the finite elements according to

$${}^i E(\omega, \mathbf{x}) = \langle {}^i E \rangle + [1 + {}^i f(\omega, \mathbf{x})] \quad \langle {}^i f \rangle = 0. \quad (2.8)$$

Next, the stiffness matrix of each element of  ${}^i V$  is calculated as

$$[{}^i K(\omega)] = \int_{{}^i V} [{}^i B]^T [C(\omega)] [{}^i B] d\mathbf{x} = [{}^i K_0] + {}^i X_0(\omega) [\Delta {}^i K_0(\omega)], \quad (2.9)$$

where  $[{}^i K_0]$  and  $[\Delta {}^i K_0(\omega)]$  are deterministic matrices, while  ${}^i X_0(\omega)$  is a random variable given as

$${}^i X_0(\omega) = \int_{{}^i V} f(\omega, \mathbf{x}) d\mathbf{x}. \quad (2.10)$$

From a micromechanics standpoint, this approach gives a Voigt-type estimate (bound) for the effective stiffness of the  $i$ -th finite element. Similarly, if applied to the compliance, it would yield a Reuss-type estimate of flexibility. The micromechanics tells us that such estimates are crude and, in principle, both of these should be used to bound the overall material response.

**Spectral method** It is well known that, in a representation of a random function by a Fourier series, the coefficients of the expansion become, in general, correlated. In order to retain the uncorrelatedness while obtaining the desired orthogonality of random coefficients, a Karhunen-Loève expansion [17] is introduced. This idea has been employed in [7] to represent the spatial variability of the RF of Young's modulus  $E$  such as in (2.8). This method claims not to be limited to weak fluctuations and to avoid the inconsistencies between various other methods involved in the inversion of the random stiffness matrix  $[K(\omega)]$ . Also, it is designed to do away with the problem of dealing with a large number of random variates resulting from a pointwise representation of the RF  $E(\omega, \mathbf{x})$ . However, besides having convergence problems and no link to the microstructure, this approach admits realizations of fields having negative (!) stiffnesses on finite domains [18,44].

### 3 From micromechanics to SFE

#### 3.1 The Hill-Mandel condition

The last decade has seen active growth of two disciplines: multiscale mechanics (MM) and uncertainty quantification (UQ). The MM models overwhelmingly take a deterministic form. The UQ is in a dire need of stochastic mechanics models linked to microstructures which are intrinsically random and, therefore, cause statistical scatter in response. The standard tool in the latter area consists of (i) RF modeling

of materials and (ii) stochastic finite elements (SFE), which typically start with the following Ansatz:

(a) Let's take Young's modulus  $E$  as a RF with smooth realizations over the macroscopic material domain  $D$ , perhaps with Poisson's ratio  $\nu$  as another RF or just a constant:

$$\{E(\mathbf{x}, \omega); \mathbf{x} \in D, \omega \in \Omega\}, \quad \{\nu(\mathbf{x}, \omega); \mathbf{x} \in D, \omega \in \Omega\}. \quad (3.1)$$

Here  $\omega$  stands for an elementary event in the probability space  $\Omega$ , obviously indexing one realization.

(b) Use the vector RF  $(E, \nu)$  as input into the SFE model, which involves a partitioning of  $D$  into elements ( $e$ ):

$$D = \cup_{e=1}^N D_e \quad (3.2)$$

and prescribing the  $(E, \nu)$  fields over each and every finite element domain  $D_e$  to its stiffness matrix  $K_i$  either by using local (i.e. volume) averaging or some other *ad hoc* (i.e. not micromechanically justified) scheme. Next, the conventional SFE methods employ a minimum potential energy principle without any reference to the definition of the representative volume element (RVE) and its size for a particular material system.

Also note that  $(E, \nu)$  is not a well-posed RF because  $E$  and  $\nu$  are entries of the fourth-rank stiffness tensor field, whose eight-rank tensor correlation (or covariance) function should correctly transform under rotations according to the  $SO(3)$  group. Additionally,  $\nu$  is a secondary elastic constant as opposed to  $E$ . Indeed, the micromechanics tells us that  $(\kappa, \mu)$  – i.e. the bulk and shear moduli – is a much better pair when working with isotropic materials.

Let us observe:

**1.** A smooth realization of the scalar RF  $E$  with  $\nu$  constant does not correspond to any physical material, except for the non-random case when  $\nu$  is constant.

**2.** Taking both  $E$  and  $\nu$  as RFs with smooth realizations still does not correspond to any specific material, and, assuming a spatial gradient in properties  $(E, \nu)$ , contradicts the assumption of isotropy.

**3.** One may rectify this "isotropy problem" by adopting a RF of the stiffness tensor:

$$\{C_{ijkl}(\mathbf{x}, \omega); \mathbf{x} \in D, \omega \in \Omega\}, \quad (3.3)$$

but, then, one has to ask: Is any stiffness tensor field realization  $C_{ijkl}(\omega)$  of such a RF consistent with the Hill (or Hill-Mandel [19-22]) condition? If the  $C_{ijkl}$  RF is chosen according to the Hill-Mandel condition, then, as discussed below, it is anisotropic with probability one, non-unique, and scale-dependent: the larger is the scale of domain relative to the microscale (e.g. grain size), the smaller is its coefficient of variation.

4. Suppose such a scale-dependent RF consistent with micromechanics has been set up, a stochastic (or flexibility) stiffness matrix for any domain can readily be formulated.

This upscaling is explained as follows: The Hill-Mandel condition is an expression of **equivalence of an energetic-type and a mechanical-type formulation of the constitutive response** of any heterogeneous material at any given point  $\mathbf{x}$  (for any specific realization  $B(\omega) \in \mathcal{B}$  of the random medium occupying the domain  $D$  centered at  $\mathbf{x}$ ):

$$\overline{\sigma(\omega): \varepsilon(\omega)} = \overline{\sigma(\omega)} : \overline{\varepsilon(\omega)}, \quad (3.4)$$

where an overbar stands for the volume averaging. For an unbounded space domain ( $\delta \rightarrow \infty$ ), (4) is trivially satisfied and the randomness vanishes, Fig. 2. Here

$$\delta = L/d \quad (3.5)$$

is a *mesoscale*, with  $L$  being the size of domain  $D$  and  $d$  the typical microscale (grain size). As is well known, the necessary and sufficient condition for (4) to hold is

$$\int_{\partial B_\delta} (\mathbf{t} - \bar{\sigma} \cdot \mathbf{n}) \cdot (\mathbf{u} - \bar{\varepsilon} \cdot \mathbf{x}) dS = 0, \quad (3.6)$$

which dictates the loading of  $B(\omega)$  on its boundary  $\partial B_\delta$ , under the condition of strong ellipticity of all the phases. Clearly, (6) is satisfied by three different types of boundary conditions on the mesoscale: *uniform displacement* (or essential, Dirichlet) ( $d$ )

$$\mathbf{u}(\mathbf{x}) = \varepsilon^0 \cdot \mathbf{x} \quad \forall \mathbf{x} \in \partial B_\delta; \quad (3.7)$$

or *uniform traction* (or natural, Neumann) ( $t$ )

$$\mathbf{t}(\mathbf{x}) = \sigma^0 \cdot \mathbf{n} \quad \forall \mathbf{x} \in \partial B_\delta; \quad (3.8)$$

or *uniform displacement-traction* (also called *orthogonal-mixed*) ( $dt$ )

$$(\mathbf{u}(\mathbf{x}) - \varepsilon^0 \cdot \mathbf{x}) \cdot (\mathbf{t}(\mathbf{x}) - \sigma^0 \cdot \mathbf{n}) = 0 \quad \forall \mathbf{x} \in \partial B_\delta. \quad (3.9)$$

Here we employ  $\varepsilon^0$  and  $\sigma^0$  to denote constant tensors, prescribed *a priori* and note, from the strain average and stress average theorems:  $\varepsilon^0 = \bar{\varepsilon}$  (for perfect interfaces) and  $\sigma^0 = \bar{\sigma}$ .

*Note:* Each of these boundary conditions results in a different *mesoscale* (or *apparent*) stiffness, or compliance tensor for the statistical volume element (SVE). These terms are used to make a distinction from the deterministic *macroscale* (or *effective, overall,...*) properties that are typically denoted by  $^{eff}$  in conventional micromechanics [23,24]; while rigorously obtained, these properties (i.e.  $\mathbf{C}_\infty^{eff}$ ) lack a quantitative specification of the RVE size, especially when one turns to nonlinear or inelastic materials.



Now, for any given realization  $B_\delta(\omega)$  of the random medium  $\mathcal{B}$ , (2.7) gives a mesoscale stiffness tensor  $\mathbf{C}_\delta^d(\omega)$  with the constitutive law

$$\bar{\sigma} = \mathbf{C}_\delta^d(\omega) : \varepsilon^0, \quad (3.10)$$

which allows setting up of a stiffness matrix of the mesoscale finite element  $e$  directly coinciding with the domain  $B_\delta$

$$\mathbf{K}(\omega) = \int_{D_e} \mathbf{B}^T \cdot \mathbf{C} \cdot \mathbf{B} \, dV. \quad (3.11)$$

Upon collecting all  $\mathbf{K}$ 's into a global stiffness matrix, one can set up a finite element scheme from the minimum potential energy formulation. The tensors obey the scaling hierarchy

$$\langle \mathbf{S}_1^t \rangle^{-1} \leq \dots \leq \langle \mathbf{S}_{\delta'}^t \rangle^{-1} \leq \langle \mathbf{S}_\delta^t \rangle^{-1} \leq \dots \leq \mathbf{C}^{eff} \leq \dots \leq \langle \mathbf{C}_\delta^d \rangle \leq \langle \mathbf{C}_{\delta'}^d \rangle \leq \dots \leq \langle \mathbf{C}_1^d \rangle$$

$\forall \delta' = \delta/2.$

(3.12)

Here  $\mathbf{S}_1^t \equiv \mathbf{S}_{\delta=1}^t$  and  $\mathbf{C}_1^d \equiv \mathbf{C}_{\delta=1}^d$ , while  $\langle \cdot \rangle$  denotes the ensemble averaging, while  $\mathbf{S}_\delta^t(\omega) [\neq [\mathbf{C}_\delta^d(\omega)]^{-1}]$  with probability one] is a mesoscale compliance tensor resulting from (2.8) and defined according to

$$\bar{\varepsilon} = \mathbf{S}_\delta^t(\omega) : \sigma^0, \quad (3.13)$$

which leads to a flexibility matrix of the mesoscale finite element

$$\mathbf{L}(\omega) = \int_{D_e} \mathbf{B}^T \cdot \mathbf{S} \cdot \mathbf{B} \, dV. \quad (3.14)$$

Upon collecting all  $\mathbf{L}$ 's into a global flexibility matrix, one can set up a finite element scheme from the minimum complementary energy formulation. This and the preceding scheme lead to two bounds on the global response, characterized by a competition of two opposing trends: the larger is the mesoscale, the weaker is the random noise in  $\mathbf{L}$  and  $\mathbf{F}$ , but the coarser is the FE mesh. By the same token, the finer is the FE mesh, the further apart are the tensors  $\mathbf{C}_\delta^d$  and  $(\mathbf{S}_\delta^t)^{-1}$ , which then preclude a convergence of both FE schemes as would be the case in the absence of any noise. Thus, there is an optimal FE size, or mesoscale  $\delta$ , as has been shown in anti-plane response of two-phase strongly random media [25] ... even at the percolation point [26].

*Note:* Another possibility is to employ the third boundary condition, (2.9), involving a combination of (2.4) and (2.5), to get a stiffness tensor  $\mathbf{C}_\delta^{dt}(\omega)$  which falls somewhere between  $\mathbf{C}_\delta^d(\omega)$  and  $[\mathbf{S}_\delta^t(\omega)]^{-1}$ . One can then set up either a stiffness or flexibility matrix, but the global FE solution would have no bounding character, albeit with weaker scaling. A typical example is shown in Fig. 3.

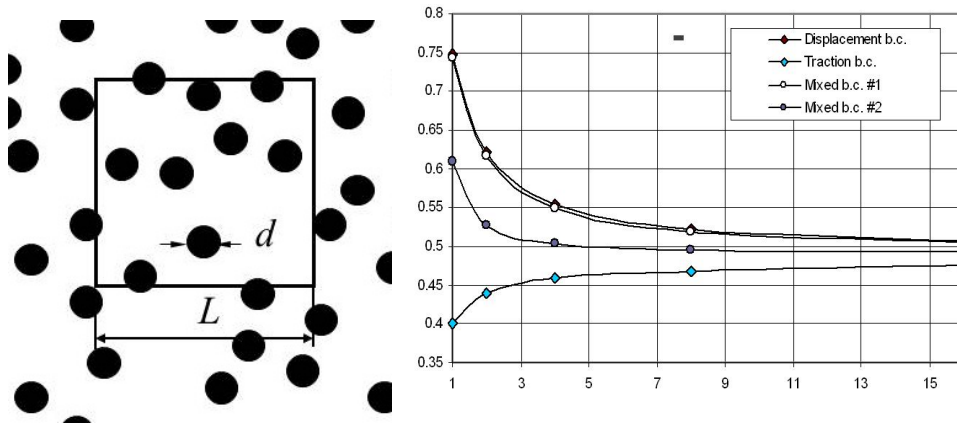


Figure 3: (a) Sample of a random matrix-inclusion composite. (b) Typical scaling of stiffnesses resulting from uniform displacement, traction and two different mixed-orthogonal boundary conditions.

*Note:* If the microstructural geometry and physical properties are described by the isotropic statistics, then, at any finite  $\delta$ , one expects some anisotropy of all the mesoscale tensors with probability one, with isotropy that can be attained for  $\delta \rightarrow \infty$  [27,28]. Thus, the RFs have anisotropic realizations [29-31].

*Note:* Three properties are required for the derivation of such mesoscale bounds:

- (i) Statistical homogeneity and ergodicity of the microstructure.
- (ii) The Hill-Mandel condition leading to admissible boundary conditions, along with the ellipticity of all the constituent phases assuring the validity of the mean strain and stress theorems.
- (iii) Variational principles of elasticity.

*Note:* While these mesoscale bounds are averages of first invariants of the stiffness (under (3.17)) and compliance tensors (under (3.18)), the RFs need to be specified in terms of the one-point and (at least) two-point statistics of these tensors, e.g. [29,32] and Fig. 5 below. Once known, such statistics allow a rapid generation of stochastic stiffness and flexibility matrices for the SFE problem at hand. It follows that the mesoscale bounds and the RFs based on them can be shown to hold for other than linear elastic materials, providing the properties (i-iii) are appropriately generalized.

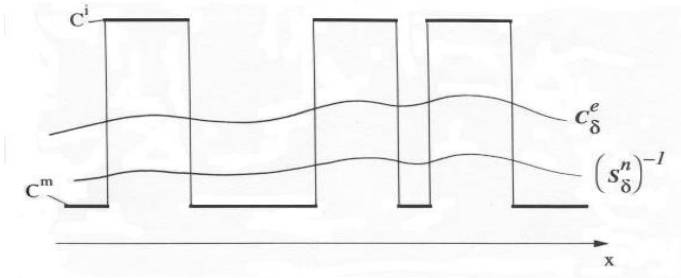


Figure 4: The setup of random fields: from a piecewise-constant realization of a composite to two approximating continua at a finite mesoscale [21]. Here the superscript  $e$  stands for essential (i.e. displacement  $d$ ), while  $n$  for natural (i.e. traction  $t$ ) boundary conditions.

### 3.2 Towards mesoscale random fields of elastic materials

We are interested in determining tensor RFs with continuum realizations, set up on mesoscale (above the scale of heterogeneities),

$$\mathcal{C}_\delta = \{ \mathbf{C}_\delta(\mathbf{x}, \omega) ; \mathbf{x} \in B, \omega \in \Omega \}. \tag{3.15}$$

For the sake of a simple exposition, let us first consider the anti-plane elasticity and work with the second-rank tensor  $C_{ij} = C_{i3j3}$  ( $i, j = 1, 2$ ) as a special case of the 4th rank stiffness tensor. To be able to set up a global stiffness matrix for mesoscale finite elements implies that one has to have a RF description of the material's stiffness field, for  $\delta = 10$ , analogous to a RF of velocity in statistical turbulence,

$$\rho_{ij}^{kl}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\langle [C_{ij}(\mathbf{x}_1) - \langle C_{ij}(\mathbf{x}_1) \rangle] [C_{kl}(\mathbf{x}_2) - \langle C_{kl}(\mathbf{x}_2) \rangle] \rangle}{\sigma_{ij}(\mathbf{x}_1) \sigma_{kl}(\mathbf{x}_2)}. \tag{3.16}$$

Here  $\sigma_{ij}(\mathbf{x}_1)$  and  $\sigma_{kl}(\mathbf{x}_2)$  are the standard deviations of  $C_{ij}(\mathbf{x}_1)$  and  $C_{kl}(\mathbf{x}_2)$ , respectively; these  $\sigma$ 's are not to be confused with the use of  $\sigma$  for the Cauchy stress tensor. We focus on wide-sense stationary (WSS) random media and so the WSS property carries over to mesoscale RF  $\mathcal{C}_\delta$  and  $\rho$  depends simply on the vector  $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ .

In view of (2nd eqn above), we will have a fourth-rank tensor field as a function not only of the distance  $\mathbf{x}$  between both locations, but also of the mesoscale (resolution), the geometric distribution of two constituent phases (such as the volume fraction  $v_{black}$ ), and the contrast of both phases ( $\alpha = C_{black}/C_{white}$ ). Thus,

$$\rho_{ij}^{kl} = \rho_{ij}^{kl}(\mathbf{x}, \delta, v_{black}, \alpha). \tag{3.17}$$

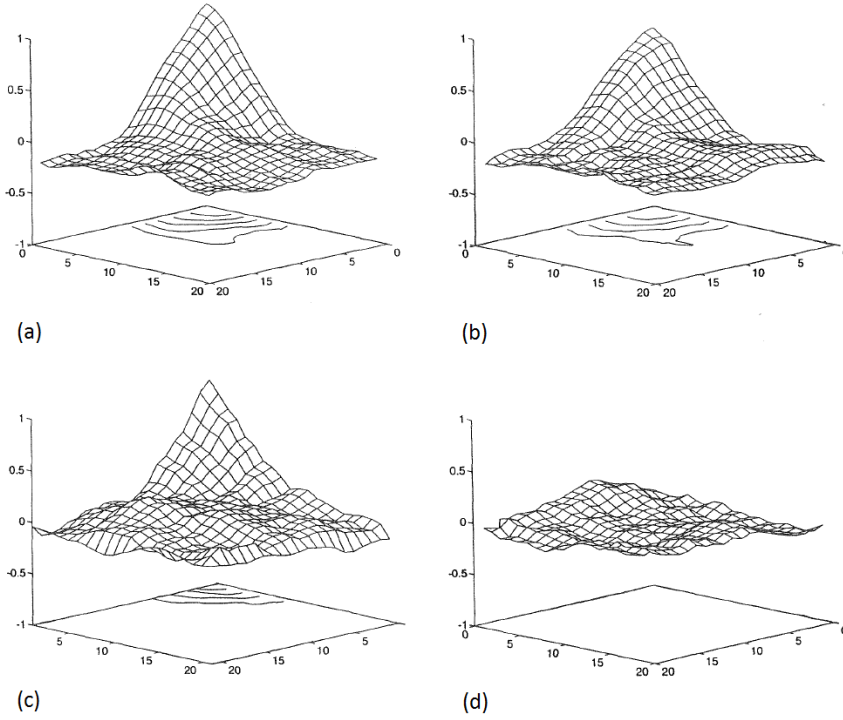


Figure 5: Graphs of the correlation coefficients  $\rho_{ij}^{kl}$  of mesoscale compliances  $\mathbf{S}_\delta^t$ , at  $\delta = 10$ : (a)  $\rho_{11}^{11}$ ; (b)  $\rho_{12}^{12}$ ; (c)  $\rho_{11}^{12}$ ; (d)  $\rho_{11}^{22}$ . The material system is a random chessboard [29].

By placing one mesoscale window at the origin of the coordinate system and another at  $\mathbf{x}$ , then finding both windows' mesoscale stiffness tensors, and repeating the process in Monte Carlo sense for many locations, we can find (17) for a specific  $\delta$ , see Fig. 8.4 in [11]. The next step will be to run the procedure for many different  $\delta$ 's, and then construct the best fit to all the results by starting from the already available correlation functions of scalar RFs, such as [21]

$$\rho(x) = \exp[-Ax^\alpha], \quad A > 0, \quad 0 < \alpha \leq 2; \quad (3.18)$$

$$\rho(x) = [1 + x^2/l^2]^{-a}, \quad a < 1/2; \quad (3.19)$$

$$\rho(x) = \frac{\exp[-Ax^\alpha]}{1 + Bx^\alpha}, \quad A, B > 0, \quad 0 < \alpha, \beta \leq 2; \quad (3.20)$$

$$\rho(x) = \exp\left[-\sum_{s=1}^r Ax^{\alpha_s}\right], \quad A_s > 0, \quad 0 < \alpha_s \leq 2, \quad s = 1, \dots, r; \quad (3.21)$$

$$\rho(x) = \left[-\prod_{s=1}^r (1 + B_s x^{\beta_s}) l_s\right], \quad B_s > 0, \quad 0 < \beta_s \leq 2, \quad l_s = 1, 2, \dots; \quad (3.22)$$

$$\rho(x) = \frac{(\cosh Bx^\alpha)^s}{1 + Ax^\alpha}, \quad A + B(2l - s) > 0, \quad 0 < \alpha \leq 2, \quad s = 1, \dots, r; \quad (3.23)$$

where  $x = |\mathbf{x}|$ .

To summarize, for any chosen mesoscale  $\delta$ , there are two approximating tensor-valued RFs, continuous-valued with continuous parameter  $\mathbf{x} \in \mathbb{E}^2$  or  $\mathbb{E}^3$ . They describe the random material, in an approximate way, by two sets of realizations:  $\{\mathbf{C}_\delta^d(\omega, \mathbf{x}); \omega \in \Omega, \mathbf{x} \in \mathbb{R}^2\}$  and  $\{\mathbf{S}_\delta^t(\omega, \mathbf{x}); \omega \in \Omega, \mathbf{x} \in \mathbb{R}^2\}$ , see Fig. 4 for a conceptual picture. At every location  $\mathbf{x}$ , these fields provide rigorous bounds on the material response in the sense of the Hill-Mandel condition. As illustration, Fig. 5 shows the correlation functions of mesoscale compliance tensors  $\mathbf{S}_\delta^t$  in anti-plane elasticity of random disk-matrix composites.

## 4 Open challenges

While the scaling laws for several material systems such as those of Fig. 3(b) (usually only in 2-D settings) are already known, the goal is to determine the one-point *and* two-point statistics (i.e. correlation and structure functions) and build the analytical RF models as outlined above. The work should generally proceed along these lines:

1. Adopt a specific 3D model of a random material (e.g. consider Fig. 1). That is, calibrate a mathematical morphology model via image analysis on several available specimens [33, 34]. Note that the advantage of the model is that it allows an arbitrarily large number of specimens  $B_\delta(\omega) = \mathcal{B}_\delta$ , for any given mesoscale  $\delta$ , to be generated very rapidly.
2. Set up the Hill-Mandel condition and corresponding uniform boundary conditions.
3. Determine the hierarchies of scale-dependent bounds and scaling laws on poroelastic response.
4. Assess the one- and two-point statistics of mesoscale response tensors, covering a range of scales, volume fractions and contrasts in material properties.
5. Develop analytical RF models of tensors in point 4., i.e. having anisotropic realizations on finite mesoscales, yet tending to isotropic realizations as  $\delta \rightarrow \infty$ , and possessing generally anisotropic correlation structures.

6. Determine the one- and two-point statistics of mesoscale yield functions.
7. Generalize the above results to other than random linear elastic microstructures: large motions [35], coupled fields [36,37], saturated media [38], plasticity (poroelasticity and thermo-poroelasticity) [39-42], or, say, dynamic loadings on mesoscale [43], etc.
8. Motivated by Fig.1, generalize the WSS model to wider RF classes:
  - (a) Intrinsically stationary (locally homogeneous) RFs.
  - (b) Quasi-stationary RFs.
9. Synthesize the results of in ready-to-use formulas, maps and software modules that could be used by solid mechanicians working in all the fields that require RF properties of materials and structures (e.g. multiscale mechanics, wave propagation, biomechanics, geomechanics, etc.).

## 5 Closure

All the researchers employing multiscale mechanics in a wide variety of fields are in need of a much improved understanding of materials' responses accounting for spatial randomness. At the same time, the stochastic finite element (SFE) methods, which belong to the very basic tools of uncertainty quantification (UQ) are in need of RF models consistent with mechanics. The review we presented here approaches this subject matter from the standpoint of multiscale mechanics of random heterogeneous materials, based on scale-dependent homogenization of stochastic micromechanics (as embodied in the Hill-Mandel condition), combined with modern RF theories.

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### **Stohastički konačni elementi: gde je tu fizika?**

Mikromehanika zasnovana na Hill-Mandel-ovom uslovu ukazuje da većina stohastičkih metoda konačnih elemenata počiva na RF (random field) modelima materijalnih svojstava, koji su često u neslaganju sa fizikom problema. Istovremeno, ovaj uslov omogućava da se formiraju RF tenzori krutosti i fleksibilnosti u funkciji mikrostrukture analiziranog materijala na mezonivou. Mezonivo je definisan na bazi statističkog zapreminskog elementa (SZE), koji je ispod materijalne skale reprezentativnog zapreminkog elementa (RZE). U radu je formulisana procedura homogenizacije, zavisna od stohastičkog nivoa, koja vodi odgovarajućoj na mezonivou statistici i izgradnji analitičkih RF modela.