Theoret. Appl. Mech., Vol.39, No.1, pp. 55–69, Belgrade 2012

Crystal plasticity: the Hamilton-Eshelby stress in terms of the metric in the intermediate configuration

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to Carmelo Totaro, in memoriam.

Abstract

The Hamilton-Eshelby stress is a basic ingredient in the description of the evolution of point, lines and bulk defects in solids. The link between the Hamilton-Eshelby stress and the derivative of the free energy with respect to the material metric in the plasticized intermediate configuration, in large strain regime, is shown here. The result is a modified version of Rosenfeld-Belinfante theorem in classical field theories. The origin of the appearance of the Hamilton-Eshelby stress (the non-inertial part of the energy-momentum tensor) in dissipative setting is also discussed by means of the concept of relative power.

Keywords: Plasticity, finite deformations, classical field theories

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To describe macroscopic large deformations of a body at a continuum (hydrodynamic) scale, we need a reference configuration coinciding with a regular region $\mathcal{B}$ customarily chosen in a copy[1] of the physical ambient space $\mathbb{R}^3$, and maps $u : \mathcal{B} \times [0, \bar{t}] \to \mathbb{R}^3$ which select in the physical space all current configurations $u(\mathcal{B})$. For reasons of physical plausibility, u is assumed to be (i) one-to-one and (at least piecewise) differentiable with spatial derivative $Du(x)$ indicated by F, (ii) orientation preserving, (iii) and such that, for any smooth function f with compact support over $\mathcal{B} \times \mathbb{R}^3$, the following inequality holds:

$$\int_{\mathcal{B}} f(x, u(x)) \det Du(x) \; dx \leq \int_{\mathbb{R}^3} \sup_{x \in \mathcal{B}} f(x, z) \; dz.$$

The assumption (i) excludes the possible formation of holes and/or fractures. The requirement that u be orientation preserving coincides with imposing the positivity of the determinant of F everywhere in $\mathcal{B}$ (or almost everywhere if one accepts a weak version of (ii)). The last condition (iii) allows self-contact of the boundary of the body along the deformation but excludes self-penetration of the matter (see [2] for the proof of this last property).

When a body may undergo plastic deformation, a common assumption (here named Kröner-Lee decomposition) is that F may admit multiplicative decomposition into elastic part F^e (which is by the way orientation

[1]The choice of $\mathcal{B}$ is essentially a selection of a geometrical setting where we make paragon in pairs between volumes, areas and/or lengths, just to define appropriately measures of deformation in a region – namely $\mathcal{B}$ – that we know, in contrast with the current (deformed) configuration that is unknown a priori. Deformation is a relative concept, relation – that is here comparison, properly – that we make with what we establish be the undeformed shape. For this reason, since $\mathcal{B}$ is only a paragon setting, it is not important that be occupied by the body under examination at a given instant. It is important just that it could be – even only in principle – occupied by the body. Such a point of view allows one to select $\mathcal{B}$ not in the physical space but in an isomorphic copy of it. The option has non-trivial consequences in the definition of non-standard changes in observers – as it will be clear later – and clarifies further on the physical interpretation of the horizontal variations of the energy, that are variations of the reference place, due for example to the redistribution of material inhomogeneities. Regularity of $\mathcal{B}$ is intended here in the sense that $\mathcal{B}$ is an open, connected set, with boundary of non-zero two-dimensional measure, a boundary where the normal n is defined everywhere to within a finite number of corners and edges. These assumptions are of practical nature: (i) by means of $\mathcal{B}$ we describe the macroscopic shapes of macroscopic bodies that we meet in daily experience, (ii) we have to use Gauss theorem in developing calculations, so we need to be in conditions assuring its validity.

preserving too) and plastic part F^p, namely

$$F = F^e F^p. \tag{1}$$

At every point in $\mathcal{B}$, F^p maps tangent vectors to $\mathcal{B}$ into tangent vectors to a plasticized shape which is usually called *intermediate configuration*. Such a configuration is not known a priori, also it cannot be known once the map u is available, obtained, for example, as a solution to some boundary value problem satisfied by the balance equations, completed by the assignment of constitutive structures (state functions). In this sense Kröner-Lee decomposition is only the transcription of a sort of *Gedanken Experimente*. The reasoning is rather clear in the case of crystalline materials. Consider, in fact, an atomic lattice and imagine first to change it in a way such that the elastic invariants of the crystalline cells are altered, then to deform the new lattice in an elastic way, that is, maintaining unaltered the elastic invariants of the lattice. Such invariants can be constructed locally by using the optical axes of the crystalline cell selected time to time (a basis for scalar invariants have been presented in [6]). F^p is the tangent map from $\mathcal{B}$ to the configuration, obtained by altering only irreversibly the atomic lattice. The decomposition is ideal because we are not able, even in principle, to get a decomposition of the map $x \longmapsto u(x)$ which could 'isolate' a part describing just the irreversible rearrangements of the matter along the deformation.

The decomposition (1) does not means that u could be decomposed also into elastic and plastic components. It is accepted, in fact, that $CurlF^p \neq 0$. Moreover, by definition $CurlF = 0$, so that

$$(CurlF^p)^{\bar{K}}_A = -\left(F^{e-1}\right)^{\bar{K}}_i (DF^e)^i_{\bar{J}C} F^{p\bar{J}}_B \mathsf{e}_{BCA},$$

where e is Ricci's permutation index, capital indices with overbar indicate coordinates in the (unknown) intermediate configuration, the other capital letters in index position refer to coordinates in the reference configuration, the index i represents components in the actual configuration. DF^e is the spatial derivative of F^e with respect to coordinates in $\mathcal{B}$ so $DF^e = \left(\bar{D}F^e\right)F^p$, with $\bar{D}F^e$ the spatial derivative with respect to local coordinates in the intermediate configuration. The referential expression of Burgers tensor B is terms of plastic part of the deformation gradient is given by $(\det F^p)^{-1} F^p RotF^p$ which is the tensor of necessary dislocations, so that $B = -(\det F^p)^{-1} F^p F^{e-1} DF^e[F^p]\mathsf{e}$.

In crystalline plasticity, Kröner-Lee decomposition is supplemented by the requirement that the plastic contribution to the deformation be isocoric, so that F^p is taken unimodular, that is $\det F^p = 1$. The assumption fails in geomaterials and even in classes of metals such as some martensite phases (see [9]). By taking into account such a circumstance, I accept here the non-linear constraint $\det F^p > 0$: I admit then possible non-isocoric plastic strains.

Parts of $\mathcal{B}$ play a role here. They are subsets $\mathfrak{b}$ of $\mathcal{B}$ with non-vanishing volume measure, and regularity properties analogous to the ones of $\mathcal{B}$. The overall **free energy** of a generic part $\mathfrak{b}$ is indicated by $\Psi(\mathfrak{b})$ and given by the integral

$$\Psi(\mathfrak{b}) := \int_{\mathfrak{b}} \psi \, dv.$$

The energy density ψ is assumed to be a differentiable function of space and time. The free energy enters the mechanical dissipation inequality (the isothermal version of the second law)

$$\frac{d}{dt}\int_{\mathfrak{b}} \psi \, dv - \int_{\mathfrak{b}} P \cdot \dot{F} \, dv \leq 0,$$

which is the isothermal version of the second law of thermodynamics and includes, besides the rate of the free energy, the internal stress power[2]. P is the first Piola-Kirchhoff stress – it maps normals to generic smooth surfaces in $\mathcal{B}$ (Cauchy's cuts) to tensions attached to the margins of such surfaces in the current configuration $u(\mathcal{B}, t)$ at the instant t. The inequality is assumed valid for any part $\mathfrak{b}$ and any time-rate involved. The arbitrariness of $\mathfrak{b}$ implies then the validity of the local mechanical dissipation inequality

$$\dot{\psi} - P \cdot \dot{F} \leq 0. \tag{2}$$

To exploit it in the standard way as a source of restrictions a priori on the possible state functions (see [7], [4]), it is necessary to presume for (at least an additive part of) P a constitutive structure similar to the one of ψ. In the case of *elastic-perfectly-plastic* behavior, in crystal plasticity

[2]For short-hand notation $P \cdot \dot{F}$ indicates $\left\langle P, \dot{F} \right\rangle$. In fact, $\dot{F}$ is a fellow of the space of linear maps between the tangent space to $\mathcal{B}$ and the ambient space $\mathbb{R}^3$, with positive determinant. P is a fellow of the related dual space. $P \cdot \dot{F}$ is then the natural product, that is the value taken over $\dot{F}$ of P. In components, $P \cdot \dot{F} = P_i^A \dot{F}_A^i$.

the pertinent free energy density is commonly assumed to be of the type $\psi = \psi(F, F^p) = \psi\left(FF^{p-1}\right) = \psi(F^e)$, when the material is homogeneous. This way, free energy is associated only with the deformation of the crystals to within the slips determining the plastic rearrangement of the matter. The choice $\psi(F, F^p) = \psi(F^e)$ implies that we are accepting an invariance requirement for the free energy, namely $\psi(F, F^p) = \psi(FG, F^pG)$ for any $G \in T_x\mathcal{B} \otimes T_x^*\mathcal{B}$, with $\det G \neq 0$, which is tantamount to assume material covariance, then initial isotropy of the body in the reference state when $\det G > 0$. In that case the special orthogonal group in included in the peer group – the symmetry group – of the material. $\psi(F^e)$ is a function related to the intermediate configuration because F^e is defined there. The problem is that the intermediate configuration cannot be individuated globally. In principle it can be recognized only locally, in a neighborhood of the point considered. Kröner-Lee decomposition, in fact, allows in principle just this local knowledge, no more. So, since $\psi(F^e)$ is over the intermediate configuration, because F^e is defined there, to exploit in the local dissipation inequality a choice of the type $\psi = \psi(F^e)$, we have to project the inequality itself in the intermediate configuration. The projection of the local version is then justified only because we can recognize the intermediate configuration – I repeat – only locally. By multiplying (2) by $\det F^p$, and taking into account Kröner-Lee decomposition, we then get

$$(\det F^p)\,\dot{\psi} - (\det F^p)\,PF^{pT} \cdot \dot{F}^e - (\det F^p)\,F^{eT}P \cdot \dot{F}^p \leq 0. \tag{3}$$

The apex T indicates transposition.

Here I assume a modified form of the free energy for *elastic-perfectly-plastic* homogeneous materials, namely I take

$$\psi = \psi(\tilde{g}, F^e), \tag{4}$$

where $\tilde{g}$ is the natural metric over the intermediate configuration. The hypothesis – already present in [5] – is suggested by the common assumption in finite strain elasticity that $\psi = \psi(\hat{g}, F)$ with $\hat{g}$ the material metric, the one pertaining to $\mathcal{B}$ (see e.g. [4]).

In principle $\tilde{g}$ can be chosen even arbitrarily. However, a natural choice (as it is done from now on) is to consider $\tilde{g}$ the push-forward into the intermediate configuration of the material metric $\hat{g}$ by means of F^p. Precisely, $\tilde{g} = F^{p-T}F^{p-1}$, where, by definition, $F^{p-T} := \left(F^{p-1}\right)^T$. In components we get $\tilde{g}_{\bar{A}\bar{B}} = \left(F^{p-T}\right)^C_{\bar{A}}\,\hat{g}_{CD}\left(F^{p-1}\right)^D_{\bar{B}}$.

In (3) the product PF^{pT} involves the push-forward to the intermediate configuration of the contravariant component of P leaving in the reference place, the other component remains in the current configuration. The tensor PF^{pT}, in components $P_i^A F_A^{p\bar{J}}$, is then over the intermediate configuration. It is then possible to assume that also the tensor PF^{pT} be a function of $\tilde{g}$ and F^e, exactly as the free energy is assumed here to be function of the same state variables.

By taking into account (4) and evaluating the time derivative of ψ, from (3) we then get

$$(\det F^p)\left(\frac{\partial\psi}{\partial F^e} - PF^{pT}\right)\cdot\dot{F}^e + \frac{\partial\psi}{\partial\tilde{g}}\cdot\dot{\tilde{g}} - (\det F^p)\,F^{eT}P\cdot\dot{F}^p \leq 0,$$

and it must hold (in non-relativistic setting) for any choice of $\dot{F}^e$. $\dot{F}^p$ cannot be chosen arbitrarily because it is prescribed by a flow rule assigned a priori or derived from some thermodynamics-type principle such as the principle of maximum dissipation (for details see [8]). Analogous reasoning holds for $\dot{\tilde{g}}$, because we consider it be expressed in terms of F^p. The arbitrariness of $\dot{F}^e$ implies

$$P\left(\tilde{g}, F^e\right) = \frac{\partial\psi\left(\tilde{g}, F^e\right)}{\partial F^e}F^{p-T} \tag{5}$$

and the reduced dissipation inequality

$$(\det F^p)\,F^{eT}P\cdot\dot{F}^p \geq \frac{\partial\psi}{\partial\tilde{g}}\cdot\dot{\tilde{g}},$$

which is, essentially, a restriction on the possible choice of the flow rule for $\dot{F}^p$, because $\dot{\tilde{g}}$ is determined by $\dot{F}^{p-1}$.

It is expedient to consider the (plasticized) intermediate configuration as embedded in a space which is a copy of the ambient (physical) space where all actual configurations are selected by the maps u. A single intermediate configuration may be pertinent to *infinitely many* actual configurations. Let $\hat{\mathbb{R}}^3$ the space hosting the intermediate configuration. In such a space an atlas can be defined. Changes in the atlas in $\hat{\mathbb{R}}^3$ can be determined by maps $h : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ that are differentiable and admit an inverse which is also differentiable. In other words, h is a fellow of the group of diffeomorphisms of $\hat{\mathbb{R}}^3$ onto itself. At every point, the spatial derivative of h is indicated by H and admits the inverse H^{-1}. I consider here a family h_s of h's parametrized

by s, with h_0 the identity, and assume differentiability with respect to s, which is, from now on, identified with time. By indicating by w the derivative $\frac{d}{ds}h_s\,|_{s=0}$, which is also written as $\dot{h}_0$, it is immediate to establish that $\dot{H}_0^{-1} = -D\mathsf{w}$, when w is differentiable, D indicating the spatial derivative of w and $\dot{H}_0^{-1}$ the derivative of H_s^{-1} with respect to the parameter, evaluated at zero.

Under the action of h_s, the free energy density changes then as

$$\psi \overset{h_s}{\longmapsto} (\det H_s)\,\psi\left(H_s^{-T}\tilde{g}H_s^{-1}, F^eH_s^{-1}\right).$$

A free energy $\psi(\tilde{g}, F^e)$ is said to be *equivariant* with reference to changes of the atlas in the intermediate configuration when

$$\psi(\tilde{g}, F^e) = (\det H_s)\,\psi\left(H_s^{-T}\tilde{g}H_s^{-1}, F^eH_s^{-1}\right)$$

for any s, which implies

$$\frac{d}{ds}\psi_{h_s}\,|_{s=0} = 0. \tag{6}$$

Equivariance of $\psi(\tilde{g}, F^e)$ implies isotropy of the intermediate configuration.

Theorem 1. *In crystal perfect plasticity, if the free energy $\psi = \psi(\tilde{g}, F^e)$ is equivariant, the standard Hamilton-Eshelby stress $\mathbb{P} := \psi I - F^TP$, with I the second-rank unit tensor, is such that*

$$\mathbb{P} = 2F^{pT}\frac{\partial\psi(\tilde{g}, F^e)}{\partial\tilde{g}}\tilde{g}F^{p-T},$$

so that

$$P = \psi F^{-T} - 2F^{e-T}\frac{\partial\psi(\tilde{g}, F^e)}{\partial\tilde{g}}\tilde{g}F^{p-T}.$$

In fact, from (6) we get

$$\psi(\tilde{g}, F^e)\,tr D\mathsf{w} - \partial_{\tilde{g}}\psi\cdot\left((D\mathsf{w})^T\,\tilde{g} + \tilde{g}D\mathsf{w}\right) - \partial_{F^e}\psi\cdot F^eD\mathsf{w} = 0,$$

that is

$$\left(\psi I - F^{eT}\partial_{F^e}\psi - 2\frac{\partial\psi}{\partial\tilde{g}}\tilde{g}\right)\cdot D\mathsf{w} = 0,$$

for any w, since $\tilde{g}$ is symmetric. Moreover, the arbitrariness of $D\mathsf{w}$ implies also

$$2\frac{\partial\psi}{\partial\tilde{g}}\tilde{g} = \psi I - F^{eT}\partial_{F^e}\psi$$

so that, by taking into account (5), we get

$$\begin{aligned} 2\frac{\partial\psi}{\partial\tilde{g}}\tilde{g} &= \psi I - F^{eT}PF^{pT} = \psi I - \left(FF^{p-1}\right)^{T} PF^{pT} = \\ &= \psi F^{p-T}F^{p} - F^{p-T}F^{T}PF^{pT} = F^{p-T}\left(\psi I - F^{T}P\right)F^{pT}, \end{aligned}$$

which gives immediately the first relation in the theorem. The second one follows from the relation $\mathbb{P} = \psi I - F^{T}P$.

In components $\mathbb{P}$ is given by

$$\mathbb{P}^{A}_{B} = 2\left(F^{pT}\right)^{\bar{K}}_{B}\left(\frac{\partial\psi}{\partial\tilde{g}}\right)^{\bar{L}}_{\bar{K}}\tilde{g}^{\bar{M}}_{\bar{L}}\left(F^{pT}\right)^{B}_{\bar{M}},$$

where the action of of the metric $\tilde{g}$ and its inverse for lowering and/or raising indices in accord with the multiplication by F^{pT} and F^{p-T} is understood. The result also holds true in presence of hardening when the hardening parameter are considered as internal (non-observable) variable so, as such, they are insensible to changes in observers.

In the abstract I have stated that the Hamilton-Eshelby stress (called in continuum mechanics just Eshelby stress, while in calculus of variations Hamilton stress) plays a basic role in the description of the evolution of bulk, line and point defects. In conservative setting, when we have at hands an elastic simple body, $\mathbb{P}$ arises when we evaluate the horizontal variations of the energy $\psi = \psi(F)$ that are variations of the reference place $\mathcal{B}$. $\mathbb{P}$ is then involved in Noether theorem – it is the non-inertial part of the energy-momentum tensor. Eshelby has interpreted the procedure leading to Noether theorem in the case where there is presence of virtually evolving bulk defects in a solid body, clarifying so the role of $\mathbb{P}$ (see [1]). In dissipative setting there are various ways to arrive at an analogous interpretation. The one proposed in [3] seems to require less assumptions and structure than others (see references and comparisons in [3]). Here I vary in a specific aspect what is proposed in [3] because I think that in this way the physical evidence of that proposal can be clarified better.

The point is the representation of the actions due to the evolution of a macroscopic defects. The argument leading to the appearance of $\mathbb{P}$ is based on two considerations:

1. A body with an evolving defect is a mutant body, and mutation can be represented in the reference setting by considering a family of reference

places $\mathcal{B}$, depending on time. Essentially we do not need to consider the whole family, rather its infinitesimal generator that is a vector field $(x,t) \longmapsto w \in \mathbb{R}^3$ over $\mathcal{B}$, a field considered differentiable in space. It describes virtually the incoming alteration of the distribution of material elements over $\mathcal{B}$ during the mutation.

2. An observer is a representation of all geometrical settings that are necessary to describe the morphology of a body and its motion. An observer is then the assignment of an atlas in the ambient physical space where the current configurations (this is standard) are described, another atlas over the space hosting the reference configuration, a third atlas over the time interval, determining the time scale. Each atlas is a representation of the space where it is assigned. Here we can consider just synchronous changes in observers. Once synchronicity[3] of different observers has been accepted, in standard continuum mechanics changes in observers are then only changes of atlas in the ambient physical space; a special class is the one induced by isometries. In the standard view all the observers evaluate the same reference place. Here, changes in observer *may* involve also changes in the atlas in the reference space. So, I consider changes in observers which are isometric in *both* the ambient space and the space hosting the reference place. If $(x,t) \longmapsto v(x,t) \in T_{u(x)}u(\mathcal{B})$ is a vector field over the reference space, taking values on the ambient physical space – it is the Lagrangian representation of a virtual velocity field over $u(\mathcal{B})$ – and is evaluated by some observer, another observer, differing by a time-parametrized family of isometries, will evaluate a new velocity, say v'. Its pull back in the frame of the first observer at the instant t is indicated by $v^* := Q^T(t)\, v'$, where $Q(t) \in SO(3)$ is the value at t of the rotation mapping the velocity vector evaluated by the second observer in the frame of the first one. We then get the standard relation

$$v^*(x,t) = v(x,t) + c(t) + q(t) \times (u(x) - u_0), \tag{7}$$

where c is the translation velocity of the second observer, measured by the first, while $q(t)$ is the relevant rotation velocity at t. The point u_0 is fixed arbitrarily in space. An analogous relation is pertinent

[3]Of course we could consider affine changes in the time scale, but the result would be no more general than what is proposed in the following lines. We are not in relativistic setting.

to isometric changes in observers in the space hosting the reference configuration. For $w(x,t)$, indicated in item 1 above, we get, in fact, for the same reasons justifying the previous relation for v^*,

$$w^*(x,t) = w(x,t) + \mathfrak{c}(t) + \mathfrak{q}(t) \times (x - x_0), \tag{8}$$

where $w^*(x,t)$ has the role in the reference place of the velocity v^* in the physical space. The point x_0 is fixed arbitrarily. Translational ($\mathfrak{c}(t)$) and rotational ($\mathfrak{q}(t)$) velocities in the reference space are not necessarily equal to their counterparts in the ambient (physical) space.

When the body in $\mathcal{B}$ does not undergo any structural mutation, $\mathcal{B}$ is fixed once and for all, and for any part $\mathfrak{b}$ the external power $\mathcal{P}^{ext}_{\mathfrak{b}}(v)$, evaluated along v, of bulk and surface standard actions is given by

$$\mathcal{P}^{ext}_{\mathfrak{b}}(v) := \int_{\mathfrak{b}} b \cdot v \, dx + \int_{\partial\mathfrak{b}} Pn \cdot v \, d\mathcal{H}^2,$$

with b the vector of body interactions, P the Piola-Kirchhoff stress, $\mathcal{H}^2$ the two-dimensional Hausdorff measure. Both b and Pn are co-vectors over $u(\mathcal{B})$, values of fields defined over $\mathcal{B}$. As it is well know, the requirement that $\mathcal{P}^{ext}_{\mathfrak{b}}(v)$ be invariant under (7) implies the validity of standard balance equations, when the fields involved satisfy appropriate conditions of regularity. The velocity v is relative to a fixed place. It can be also identified with $\dot{u}$, the real velocity along the motion $(x,t) \longmapsto u(x,t)$. When macroscopic mutations arise, $\mathcal{B}$ varies in the reference space with initial velocity w, so that it would be natural to evaluate the external power over the relative velocity. However, the simple difference $v - w$ does not make physical sense because v and w are in two different spaces, so that it is necessary to push forward w from the reference space to the physical space by means of the gradient of deformation $F := Du(x)$ because the two spaces are connected by the deformation u. The difference making sense is then $v - Fw$. Additionally, when $\mathcal{B}$ varies as a consequence of a mutation due to nucleation and growth of defects, there is a redistribution of material elements generating energy flow through any boundary inside the body, and also an inhomogeneous distribution of the energy e due to the evolution of defects. I do not presume any explicit constitutive structure of the energy at this stage. I affirm only that ψ is of the type $\psi(x,t,\varsigma)$, where explicit time dependence implies possible aging, ς indicates the list of state variables to be decided

later when the second law of thermodynamics is used to determine restrictions a priori to the state functions of the stress, as mentioned previously. The list ς depends also on space and time. As a function $\psi(\cdot, t, \varsigma)$ of the sole x, the energy takes into account homogeneity due to the redistribution of defects. In such a redistribution, material bonds are also broken and other bonds are formed: the reference configuration then changes. Forces f and couples μ develop power in the process of annihilation and reformation of material bonds due to the mutation. A couple μ can be generated – and it is undetermined – even in conservative setting when isotropy is broken. They are co-vectors over $\mathcal{B}$ where the mutation is represented. It is possible to take into account these new actions, fluxes and inhomogeneity of the energy, and the relative velocity in an extended notion of external power that I call **relative power**, by indicating it by $\mathcal{P}^{rel}_{\mathfrak{b}}(v, w)$. It is essentially the power of actions, evaluated over the relative velocity $v - Fw$, indicated by $\mathcal{P}^{rel-a}_{\mathfrak{b}}(v, w)$, *plus* the power, $\mathcal{P}^{dis}_{\mathfrak{b}}(w)$, developed in the rearrangement of the matter in the reference place – the bulk mutation, then – by the velocity field w. $\mathcal{P}^{rel}_{\mathfrak{b}}(v, w)$ is defined by

$$\mathcal{P}^{rel}_{\mathfrak{b}}(v, w) := \mathcal{P}^{rel-a}_{\mathfrak{b}}(v, w) + \mathcal{P}^{dis}_{\mathfrak{b}}(w)$$

where

$$\mathcal{P}^{rel-a}_{\mathfrak{b}}(v, w) := \int_{\mathfrak{b}} b \cdot (v - Fw)\ dx + \int_{\partial\mathfrak{b}} Pn \cdot (v - Fw)\ d\mathcal{H}^2,$$

$$\mathcal{P}^{dis}_{\mathfrak{b}}(w) := \int_{\partial\mathfrak{b}} (n \cdot w)\,\psi\ d\mathcal{H}^2 - \int_{\mathfrak{b}} (\partial_x \psi + f) \cdot w\ dx + \int_{\mathfrak{b}} \mu \cdot curl w\ dx,$$

where $\partial_x \psi$ is the **explicit derivative** of ψ with respect to x. The definition of $\mathcal{P}^{dis}_{\mathfrak{b}}(w)$ differs from what I have proposed previously in [3]. This new version seems to me more satisfactory from the viewpoint of a physical interpretation of the entities defined. A theorem can be then proven.

Theorem 2. *The following two sets of assertions are equivalent.*

Set 1: $\mathcal{P}^{rel}_{\mathfrak{b}}(v, w)$ *is invariant under isometric changes in observers for any choice of* $\mathfrak{b}$.

Set 2: (*i*) *If the fields* $x \longmapsto b := b(x)$ *and* $x \longmapsto P := P(x)$ *are integrable over* $\mathcal{B}$*, then for every part* $\mathfrak{b}$ *the following integral balances hold:*

$$\int_{\mathfrak{b}} b\ dx + \int_{\partial\mathfrak{b}} Pn\ d\mathcal{H}^2 = 0,$$

$$\int_{\mathfrak{b}} (y-y_0)\times b\ dx + \int_{\partial\mathfrak{b}} (y-y_0)\times Pn\ d\mathcal{H}^2 = 0,$$

$$\int_{\partial\mathfrak{b}} \mathbb{P}n\ d\mathcal{H}^2 - \int_{\mathfrak{b}} F^*b\ dx - \int_{\mathfrak{b}} (\partial_x e + f)\ dx = 0,$$

$$\int_{\partial\mathfrak{b}} (x-x_0)\times \mathbb{P}n\ d\mathcal{H}^2 - \int_{\mathfrak{b}} (x-x_0)\times F^*b\ dx -$$

$$- \int_{\mathfrak{b}} (x-x_0)\times (\partial_x e + f)\ dx + \int_{\mathfrak{b}} 2\mu\ dx = 0.$$

where $\mathbb{P} := \psi I - F^*P.$, *with* I *the second order unit tensor.*

(*ii*) *If the fields* $x \longmapsto P$ *and* $x \longmapsto \mathbb{P}$ *are of class* $C^1(\mathcal{B}) \cap C^0\left(\bar{\mathcal{B}}\right)$ *then*

$$\mathtt{Div}P + b = 0,$$

$$\mathtt{Skw}PF^* = 0,$$

$$\mathtt{Div}\mathbb{P} - F^*b - \partial_x e = f.$$

$$\mathtt{Skw}\mathbb{P} = \mu\times$$

(*iii*) *If the material is homogeneous, no driving force is present, and* $\mu = 0$, *then* $\mathbb{P}$ *is symmetric and, in absence of body forces,*

$$\int_{\partial\mathfrak{b}} \mathbb{P}n\ d\mathcal{H}^2 = 0$$

(*iv*) *An extended version of the virtual power principle holds:*

$$\mathcal{P}^{rel}_{\mathfrak{b}}(v,w) = \mathcal{P}^{rel-inn}_{\mathfrak{b}}(v,w),$$

where

$$\mathcal{P}^{rel-inn}_{\mathfrak{b}}(v,w) := \int_{\mathfrak{b}} (P\cdot Dv + \mathbb{P}\cdot Dw + \mu\cdot curl w)\ dx.$$

f should be prescribed constitutively. It is the bulk force driving a defect. The torque μ remains undetermined. It is just useful to remind us that $\mathbb{P}$ is, in general, non-symmetric, as it appears in Theorem 1. The first two integral balances in Theorem 2 (the standard balances of forces and torques) correspond to the Killing fields of the metric in the ambient (physical) space – it is an additional reason for the choice of selecting the reference place in a copy of the ambient space. The latter two integral

balances[4] correspond to the Killing fields of the material metric $\hat{g}$. When the setting is conservative, so that $\psi = \psi(x, F)$, $P(x, F) = \frac{\partial \psi(x,F)}{\partial F}$, and f vanishes, the relation $\mathcal{P}_{\mathfrak{b}}^{rel}(v, w) = \mathcal{P}_{\mathfrak{b}}^{rel-inn}(v, w)$ reduces to the integral version of the pointwise balance appearing in Noether theorem. In presence of inertia, there is an inertia contribution in the bulk force b because it can be decomposed additively into inertial and non-inertial components, and a tensor contribution which is given by the kinetic energy time the unit second-rank tensor – the resulting tensor added to $\mathbb{P}$ generates the standard energy-momentum tensor. Comparisons with other procedures producing the balances of configurational actions and explaining their role can be found in [3]. Reasons of physical significance, evidence, eventually elegance can allow one to discriminate and prefer one line of reasoning to the other.

The proof of Theorem 2 is really simple. Essentially, just the implication $Set1 \Longrightarrow Set2$ has to be proven, the opposite appearing along the proof as evident. In fact, invariance of $\mathcal{P}_{\mathfrak{b}}^{rel}(v, w)$ with respect to isometric changes in observers in both ambient and reference space reads as $\mathcal{P}_{\mathfrak{b}}^{rel}(v, w) = \mathcal{P}_{\mathfrak{b}}^{rel}(v^*, w^*)$ for any choice of the translational velocities $c(t)$ and $\mathfrak{c}(t)$ the rotational ones $q(t)$ and $\mathfrak{q}(t)$. Since $\mathcal{P}_{\mathfrak{b}}^{rel}$ is linear in the rates, the previous equality furnishes

$$\mathcal{P}_{\mathfrak{b}}^{rel}(c + q \times (u(x) - u_0), \mathfrak{c} + \mathfrak{q} \times (x - x_0)) = 0.$$

So, the arbitrariness of c, $\mathfrak{c}$, q, and $\mathfrak{q}$ implies the integral balances. The rest is algebra.

Acknowledgements

This work has been developed within the activities of the research group in 'Theoretical Mechanics' of the 'Centro di Ricerca Matematica Ennio De Giorgi' of the Scuola Normale Superiore in Pisa. The support of the GNFM-INDAM is acknowledged.

[4]They are also called *balances of configurational actions*.

References

[1] Eshelby J. D. (1975), The elastic energy-momentum tensor, *J. El.*, **5**, 321-335.

[2] Giaquinta M., Modica G., Souček J. (1998), *Cartesian currents in the calculus of variations*, voll. I and II, Springer-Verlag, Berlin.

[3] Mariano P. M. (2009), The relative power and its invariance, *Rendiconti Lincei*, **20**, 227-242.

[4] Marsden J. E., Hughes T. R. J. (1994), *Mathematical foundations of elasticity*, Dover, New York.

[5] Miehe C. (1998), A constitutive frame of elastoplasticity at large strains based on the notion of a plastic metric, *Int. J. Solids Struct.*, **35**, 3859-3897.

[6] Parry G. P., Šilhavý M. (1999), Elastic scalar invariants in the theory of defective crystals, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 455, 4333–4346.

[7] Šilhavý M. (1997), *The mechanics and thermodynamics of continuous media*, Springer-Verlag, Berlin.

[8] Simo J. C., Hughes T. R. J. (1998), *Computational inelasticity*, Springer-Verlag, Berlin.

[9] Spitzig W. A., Richmond O. (1984), The effect of pressure on the flow stress of metals, *Acta Metallurgica*, **32**, 457-463.

Submitted in June 2011

Plastičnost kristala: Hamilton-Eshelby-jev napon u metrici medjukonfiguracije

U radu je uspostavljena veza izmedju Hamilton-Eshelby-evog napona i gradijenta slobodne energije u odnosu na materijalnu metriku plastično deformisane medjukonfiguracije u uslovima velikih deformacija. Dobijeni rezultat je modifikovana verzija Rosenfeld-Belinfante-ove teoreme iz klasične teorije polja. Prisustvo Hamilton-Eshelby-evog napona u uslovima disipacije energije je diskutovano korišćenjem koncepta relativne snage.

doi:10.2298/TAM1201055M Math.Subj.Class.: 74C20, 70S99