

Three-dimensional fundamental solution in transversely isotropic thermoelastic diffusion material

Rajneesh Kumar* Vijay Chawla†

Abstract

The aim of the present investigation is to study the fundamental solution for three dimensional problem in transversely isotropic thermoelastic diffusion medium. After applying the dimensionless quantities, two displacement functions are introduced to simplify the basic three-dimensional equations of thermoelastic diffusion with transverse isotropy for the steady state problem. Using the operator theory, we have derived the general expression for components of displacement, mass concentration, temperature distribution and stress components. On the basis of general solution, three dimensional fundamental solutions for a point heat source in an infinite thermoelastic diffusion media is obtained by introducing four new harmonic functions. From the present investigation, a special case of interest is also deduced to depict the effect of diffusion.

Keywords: Thermoelastic diffusion, fundamental solution, Transverse isotropic, heat source.

1 Introduction

Fundamental solutions or Green's functions play an important role in both applied and theoretical studied on the physics of solids. Fundamental solutions can be used to construct many analytical solutions solving boundary

*Department of Mathematics, Kurukshetra University Kurukshetra-136119, Haryana, India, e-mail: _rajneesh_kuk@rediffmail.com

†Department of Mathematics, Kurukshetra University Kurukshetra-136119, Haryana, India, e-mail:vijay.chawla@ymail.com

value problems of practical problems when boundary conditions are imposed. They are essential in boundary element method (BEM) as well as the study of cracks, defects and inclusion. Many researchers have been investigated the Green's function for elastic solid in isotropic and anisotropic elastic media, notable among them are Freedholm[1], Lifshitz and Rezensveig [2], Elliott[3], Kroner[4], Synge [5] , Lejcek [6] , Pan and Chou [7] and Pan and Yuan [8].

When thermal effects are considered, Sharma [9] investigated the fundamental solution for transversely isotropic thermoelastic material in an integral form. Chen et al. [10] derived the three dimensional general solution for transversely isotropic thermoelastic materials. Hou et al. [11, 12] investigated the Green's function for two and three-dimensional problem for a steady Point heat source in the interior of a semi-infinite thermoelastic materials. Also, Hou et.al [13] investigated the two dimensional general solutions and fundamental solutions for orthotropic thermoelastic materials.

Diffusion is defined as the spontaneous movement of the particles from a high concentration region to the low concentration region and it occurs in response to a concentration gradient expressed as the change in the concentration due to change in position. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. Today, thermal remains a practical process to separate isotopes of noble gases (e.g. xexon) and other light isotopes (e.g. carbon) for research purpose.

When diffusion effects are considered, Nowacki [14, 15, 16, 17] developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Sherief et al. [18] developed the generalized theory of thermoelastic diffusion with one relaxation time which allows finite speeds of propagation of waves. Kumar and Kansal [20] derived the basic equations for generalized thermoelastic diffusion. Kumar and Chawla [21] discussed the surface wave propagation in a elastic layer lying over a thermoelastic diffusion half-space. Also, Kumar and Chawla [21] investigated the Fundamental solution in orthotropic thermoelastic diffusion material. However, the important Fundamental solution for three-dimensional problem for a steady point heat source in transversely isotropic thermoelastic diffusion material has not been discussed so far.

Keeping in view of these applications, we studied the three dimensional fundamental solution in transversely isotropic thermoelastic diffusion material. Using the operator theory, the general expression for components of displacement, mass concentration, temperature distribution and stress com-

ponents has been derived. On the basis of general solution, three dimensional fundamental solution for a point heat source in an infinite thermoelastic diffusion media is presented by introducing four newly harmonic functions. A special case of interest is also deduced.

2 Basic equations

Following Sherief and Saleh [18], the basic governing equations for homogeneous anisotropic generalized thermoelastic diffusion solid in the absence of body forces, heat and mass diffusion sources are

i Constitutive relations:

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl} + a_{ij} T + b_{ij} C, \quad (1)$$

ii Equations of motion:

$$c_{ijkl} \varepsilon_{kl,j} + a_{ij} T_{,j} + b_{ij} C_{,j} = \rho \ddot{u}_i, \quad (2)$$

iii Equation of heat conduction:

$$\rho C_E \dot{T} + a T_0 \dot{C} - a_{ij} T_0 \dot{\varepsilon}_{ij} = K_{ij} T_{,ij}, \quad (3)$$

iv Equation of mass diffusion:

$$-\alpha_{ij}^* b_{km} \varepsilon_{km,ij} - \alpha_{ij}^* b C_{,ij} + \alpha_{ij}^* a T_{,ij} = -\dot{C} \quad (4)$$

Here, c_{ijkl} ($= c_{kmi j} = c_{jikm} = c_{ijmk}$) are elastic parameter; a_{ij} ($= a_{ji}$), b_{ij} ($= b_{ji}$) are respectively, the tensor of thermal and diffusion moduli. ρ is the density and C_E is the specific heat at constant strain, a, b are, respectively, coefficient describing the measure of thermoelastic diffusion effects and of diffusion effects, T_0 is the reference temperature assumed to be such that $\left| \frac{T}{T_0} \right| \ll 1$. K_{ij} ($= K_{ji}$), σ_{ij} ($= \sigma_{ji}$) and $\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$ denote the components of thermal conductivity, stress and strain tensor respectively. $T(x, y, z, t)$ is the temperature change from the reference temperature T_0 and C is the mass concentration. u_i are components of displacement vector. α_{ij}^* ($= \alpha_{ji}^*$) are diffusion parameters.

In the above equations, the symbol $(,)$ followed by a suffix denotes differentiation with respect to spatial coordinate and a superposed dot (" $\dot{}$ ") denotes the derivative with respect to time respectively.

3 Formulation of the problem

We consider a homogenous transversely isotropic thermoelastic diffusion medium. Let us take Oxyz as the frame of reference in Cartesian coordinates.

For three dimensional problem, we assume the displacement vector, temperature distribution and mass concentration are, respectively, of the form

$$\vec{u} = (u, v, w), T(x, y, z, t), C(x, y, z, t). \quad (5)$$

Moreover, we are discussing steady problem

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial T}{\partial t} = \frac{\partial C}{\partial t} = 0. \quad (6)$$

We define the dimensionless quantities as:

$$(x', y', z', u', v', w', r', d') = \frac{\omega_1^*}{v_1} (x, y, z, u, v, w, r, d),$$

$$(T', C') = \frac{1}{c_{11}} (a_1 T, b_1 C),$$

$$\sigma'_{ij} = \frac{\sigma_{ij}}{a_1 T_0}, \quad H' = \frac{a_1}{c_{11} K_1} H.$$

where

$$v_1^2 = b_1, \quad \omega_1^* = \frac{a C_{11}}{K_1} \quad (7)$$

The equations (2)-(4) for transversely isotropic thermoelastic diffusion material, with the aid of (5) - (6) and applying the dimensionless quantities defined by (7) on resulting equations, after suppressing the primes, we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \delta_2 \frac{\partial^2}{\partial y^2} + \delta_1 \frac{\partial^2}{\partial z^2} \right) u + \left(\delta_3 \frac{\partial^2}{\partial x \partial y} \right) v + \left(\delta_4 \frac{\partial^2}{\partial x \partial z} \right) w - \left(\frac{\partial}{\partial x} \right) T - \left(\frac{\partial}{\partial x} \right) C = 0, \quad (8)$$

$$\left(\delta_3 \frac{\partial^2}{\partial x \partial y} \right) u + \left(\delta_2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \delta_1 \frac{\partial^2}{\partial z^2} \right) v + \left(\delta_4 \frac{\partial^2}{\partial z \partial y} \right) w - \left(\frac{\partial}{\partial y} \right) T - \left(\frac{\partial}{\partial y} \right) C = 0, \quad (9)$$

$$\left(\delta_4 \frac{\partial^2}{\partial x \partial z}\right) u + \left(\delta_4 \frac{\partial^2}{\partial z \partial y}\right) v + \left(\delta_1 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \delta_5 \frac{\partial^2}{\partial z^2}\right) w - \varepsilon_1 \left(\frac{\partial}{\partial z}\right) T - \gamma_1 \left(\frac{\partial}{\partial z}\right) C = 0, \quad (10)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T + \varepsilon_2 \left(\frac{\partial^2}{\partial z^2}\right) T = 0, \quad (11)$$

$$\begin{aligned} & \frac{\partial}{\partial x} \left[q_1^* \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + q_2^* \frac{\partial^2}{\partial z^2} \right] u + \frac{\partial}{\partial y} \left[q_1^* \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + q_2^* \frac{\partial^2}{\partial z^2} \right] v + \\ & \frac{\partial}{\partial z} \left[q_3^* \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + q_4^* \frac{\partial^2}{\partial z^2} \right] w + \left[q_5^* \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + q_6^* \frac{\partial^2}{\partial z^2} \right] T - \\ & \left[q_7^* \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + q_8^* \frac{\partial^2}{\partial z^2} \right] C = 0, \end{aligned} \quad (12)$$

where

$$\begin{aligned} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) &= \frac{1}{c_{11}} (c_{44}, c_{66}, c_{12} + c_{66}, c_{13} + c_{44}, c_{33}), \\ \varepsilon_1 &= \frac{a_3}{a_1}, \quad \gamma_1 = \frac{b_3}{b_1}, \quad \varepsilon_2 = \frac{K_3}{K_1}, \end{aligned}$$

$$\begin{aligned} (q_1^*, q_2^*, q_3^*, q_4^*) &= \frac{1}{c_{11}} (\alpha_1^* \omega_1^* b_1, \alpha_3^* \omega_1^* b_1, \alpha_1^* \omega_1^* b_3, \alpha_3^* \omega_1^* b_3), (q_5^*, q_6^*) \\ &= \frac{1}{a_1} (\alpha_1^* \omega_1^* a, \alpha_3^* \omega_1^* a), (q_7^*, q_8^*) \\ &= \frac{1}{b_1} (\alpha_1^* \omega_1^* b, \alpha_3^* \omega_1^* b), \end{aligned}$$

$$\begin{aligned} a_1 &= (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3, \\ a_3 &= 2c_{13}\alpha_1 + c_{33}\alpha_3, \quad b_1 = (c_{11} + c_{12})\alpha_{1c} + c_{13}\alpha_{3c}, \\ b_3 &= 2c_{13}\alpha_{1c} + c_{33}\alpha_{3c}, \quad c_{66} = \frac{c_{11} - c_{12}}{2}. \end{aligned}$$

4 Static general solution

Two displacements functions Ψ and G are introduced as follows

$$u = \frac{\partial \Psi}{\partial y} - \frac{\partial G}{\partial x}, \quad v = -\frac{\partial \Psi}{\partial x} - \frac{\partial G}{\partial y}. \quad (13)$$

Using the displacements functions Ψ and G in equations (8)-(12), we obtain

$$\left[\delta_2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \delta_1 \frac{\partial^2}{\partial z^2} \right] \Psi = 0 \quad (14)$$

$$D \begin{Bmatrix} G \\ w \\ C \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (15)$$

where D is the differential operator matrix given by

$$\begin{bmatrix} \Delta + \delta_1 \frac{\partial^2}{\partial z^2} & -\delta_4 \frac{\partial}{\partial z} & 1 & 1 \\ -\delta_4 \Delta \frac{\partial}{\partial z} & \delta_1 \Delta + \delta_5 \frac{\partial^2}{\partial z^2} & -\gamma_1 \frac{\partial}{\partial z} & -\varepsilon_1 \frac{\partial}{\partial z} \\ -\left(q_1^* \Delta^2 + q_2^* \Delta \frac{\partial^2}{\partial z^2} \right) & q_3^* \Delta \frac{\partial}{\partial z} + q_4^* \frac{\partial^3}{\partial z^3} & -\left(q_7^* \Delta + q_8^* \frac{\partial^2}{\partial z^2} \right) & q_5^* \Delta + q_6^* \frac{\partial^2}{\partial z^2} \\ 0 & 0 & 0 & \Delta + \varepsilon_3 \frac{\partial^2}{\partial z^2} \end{bmatrix}$$

Equation (15) is a homogeneous set of differential equations in G, w, C, T . The general solution by the operator theory is as follows

$$u = A_{i1}F, \quad w = A_{i2}F, \quad C = A_{i3}F, \quad T = A_{i4}F \quad (i = 1, 2, 3, 4). \quad (16)$$

The determinant of the matrix D is given as

$$|D| = \left(\bar{a} \frac{\partial^6}{\partial z^6} + \bar{b} \Delta \frac{\partial^4}{\partial z^4} + \bar{c} \Delta^2 \frac{\partial^2}{\partial z^2} + \bar{d} \Delta^3 \right) \times \left(\Delta + \varepsilon_3 \frac{\partial^2}{\partial z^2} \right), \quad (17)$$

where $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ and Δ are given in Appendix A. The function F in equation (16) satisfies the following homogeneous equation

$$|D| F = 0. \quad (18)$$

It can be seen that if $i = 1, 2, 3$ are taken in equation (16), three general solution are obtained in which $T = 0$. These solutions are identical to those

without thermal fact and are not discussed here. Therefore if $i = 4$ should be taken in equation (16), the following solution is obtained

$$u = \frac{\partial \Psi}{\partial y} - \left(\bar{a}_1 \Delta^2 + \bar{b}_1 \Delta \frac{\partial^2}{\partial z^2} + \bar{c}_1 \frac{\partial^4}{\partial z^4} \right) \frac{\partial F}{\partial x}, \quad (19a)$$

$$v = -\frac{\partial \Psi}{\partial x} - \left(\bar{a}_1 \Delta^2 + \bar{b}_1 \Delta \frac{\partial^2}{\partial z^2} + \bar{c}_1 \frac{\partial^4}{\partial z^4} \right) \frac{\partial F}{\partial y}, \quad (19b)$$

$$w = \left(\bar{a}_2 \Delta^2 + \bar{b}_2 \Delta \frac{\partial^2}{\partial z^2} + \bar{c}_2 \frac{\partial^4}{\partial z^4} \right) \frac{\partial F}{\partial z}, \quad (19c)$$

$$C = \left(\bar{a}_3 \Delta^3 + \bar{b}_3 \Delta^2 \frac{\partial^2}{\partial z^2} + \bar{c}_3 \Delta \frac{\partial^4}{\partial z^4} + \bar{d}_4 \frac{\partial^6}{\partial z^6} \right) F, \quad (19d)$$

$$T = \left(\bar{a} \frac{\partial^6}{\partial z^6} + \bar{b} \Delta \frac{\partial^4}{\partial z^4} + \bar{c} \Delta^2 \frac{\partial^2}{\partial z^2} + \bar{d} \Delta^3 \right) F, \quad (19e)$$

where $\bar{a}_i, \bar{b}_i, \bar{c}_i (i = 1, 2, 3)$ and \bar{d}_4 are given in Appendix B.

In cylindrical coordinates (r, θ, z) , the general solution can be easily obtained. In fact, the expression for w, T and C are identical to that to that in equation (19 c,d,e), while those for radial and circumferential displacements u_r and u_θ are, respectively

$$u_r = \frac{\partial \Psi}{r \partial \theta} - \left(\bar{a}_1 \Delta^2 + \bar{b}_1 \Delta \frac{\partial^2}{\partial z^2} + \bar{c}_1 \frac{\partial^4}{\partial z^4} \right) \frac{\partial F}{\partial r}, \quad (20a)$$

$$u_\theta = -\frac{\partial \Psi}{\partial r} - \left(\bar{a}_1 \Delta^2 + \bar{b}_1 \Delta \frac{\partial^2}{\partial z^2} + \bar{c}_1 \frac{\partial^4}{\partial z^4} \right) \frac{\partial F}{r \partial \theta}. \quad (20b)$$

Here $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$ is the Laplacian in polar coordinates. The general solutions of equations of (18) in terms of F can be rewritten as

$$\prod_{j=1}^4 \left(\Delta + \frac{\partial^2}{\partial z_j^2} \right) F = 0, \quad (21)$$

where $z_j = s_j z$, $s_4 = \sqrt{\frac{K_1}{K_3}}$ and $s_j (j = 1, 2, 3)$ are three roots (with positive real part) of the following algebraic equation

$$\bar{a} s^6 - \bar{b} s^4 + \bar{c} s^2 - \bar{d} = 0. \quad (22)$$

As known from the generalized Almansi theorem [Ding et al. [23]], the function F can be expressed in terms of four harmonic functions

1. $F = F_1 + F_2 + F_3 + F_4$ for distinct s_j ($j = 1, 2, 3, 4$),
2. $F = F_1 + F_2 + F_3 + zF_4$ for $s_1 \neq s_2 \neq s_3 = s_4$,
3. $F = F_1 + F_2 + zF_3 + z^2F_4$ for $s_1 \neq s_2 = s_3 = s_4$,
4. $F = F_1 + zF_2 + z^2F_3 + z^3F_4$ for $s_1 = s_2 = s_3 = s_4$,

where F_j satisfies the following harmonic equation

$$\left(\Delta + \frac{\partial^2}{\partial z_j^2} \right) F_j = 0 \quad (j = 1, 2, 3, 4). \quad (23)$$

The general solution for the case of distinct roots, can be derived as follows

$$\begin{aligned} u &= \frac{\partial \Psi}{\partial y} - \sum_{j=1}^4 p_{1j} \frac{\partial^5 F_j}{\partial x \partial z_j^4}, & v &= -\frac{\partial \Psi}{\partial x} - \sum_{j=1}^4 p_{1j} \frac{\partial^5 F_j}{\partial y \partial z_j^4} \\ w &= \sum_{j=1}^4 s_j p_{2j} \frac{\partial^5 F_j}{\partial z_j^5}, & C &= \sum_{j=1}^4 p_{3j} \frac{\partial^6 F_j}{\partial z_j^6}, & T &= \sum_{j=1}^4 p_{44} \frac{\partial^6 F_4}{\partial z_4^6}. \end{aligned} \quad (24)$$

where

$$\begin{aligned} p_{kj} &= \bar{a}_k - \bar{b}_k s_j^2 + \bar{c}_k s_j^4 \quad (k = 1, 2), \\ p_{3j} &= -\bar{a}_3 + \bar{b}_3 s_j^2 - \bar{c}_3 s_j^4 + \bar{d}_4 s_j^6, \\ p_{44} &= -\bar{d} + \bar{c} s_j^2 - \bar{b} s_j^4 + \bar{a} s_j^6. \end{aligned}$$

In the similar way general solution for the other three cases can be derived.

Equation (24) can be further simplified by taking

$$p_{1j} \frac{\partial^4 F_j}{\partial z_j^4} = \psi_j \quad (j = 1, 2, 3, 4), \quad (25)$$

and writing $\psi_0 = \psi$, as follows:

$$\begin{aligned} u &= \frac{\partial \Psi_0}{\partial y} - \sum_{j=1}^4 \frac{\partial \psi_j}{\partial x}, & v &= -\frac{\partial \Psi_0}{\partial x} - \sum_{j=1}^4 \frac{\partial \psi_j}{\partial y}, & w &= \sum_{j=1}^4 s_j P_{1j} \frac{\partial \psi_j}{\partial z_j}, \\ C &= \sum_{j=1}^4 P_{2j} \frac{\partial^2 \psi_j}{\partial z_j^2}, & T &= P_{34} \frac{\partial^2 \psi_4}{\partial z_4^2}, \end{aligned} \quad (26)$$

where

$$P_{1j} = p_{2j}/p_{1j}, \quad P_{2j} = p_{3j}/p_{1j}, \quad P_{34} = p_{44}/p_{14}.$$

The function ψ_j satisfies the harmonic equations

$$\left(\Delta + \frac{\partial^2}{\partial z_j^2} \right) \psi_j = 0 \quad (j = 0, 1, 2, 3, 4). \quad (27)$$

in which

$$z_0 = s_0 z, \quad s_0 = \sqrt{\frac{\delta_2}{\delta_1}}.$$

In cylindrical coordinates (r, θ, z) , the expression for w, T, C will remain the same as given in equation (26), while the components of displacement in cylindrical coordinates are

$$u_r = \frac{\partial \Psi_0}{r \partial \theta} - \sum_{j=1}^4 \frac{\partial \psi_j}{\partial r}, \quad u_\theta = -\frac{\partial \Psi_0}{\partial r} - \sum_{j=1}^4 \frac{\partial \psi_j}{r \partial \theta}. \quad (28)$$

Let us introduce the following notations for the components both in Cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z)

$$\begin{aligned} U &= u + iv = e^{i\theta}(u_r + iu_\theta), \\ \sigma_1 &= \sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\theta\theta}, \\ \sigma_2 &= \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = e^{2i\theta}(\sigma_{rr} - \sigma_{\theta\theta} + 2i\sigma_{r\theta}), \\ \tau_z &= \sigma_{xz} + i\sigma_{yz} = e^{i\theta}(\sigma_{zr} + i\sigma_{z\theta}). \end{aligned}$$

Upon using these notations, the general solution (26) in Cartesian coordinates (x, y, z) can be simplified as

$$\begin{aligned} U &= -\Gamma_1 \left(i\Psi_0 + \sum_{j=1}^4 \Psi_j \right), \quad w = \sum_{j=1}^4 s_j P_{1j} \frac{\partial \psi_j}{\partial z_j}, \\ C &= \sum_{j=1}^4 P_{2j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad T = P_{34} \frac{\partial^2 \psi_4}{\partial z_4^2}, \\ \sigma_1 &= 2 \sum_{j=1}^4 (c_{66} - r_j s_j^2) \Delta \Psi_j, \quad \sigma_2 = -2c_{66}^* \Gamma_1^2 \left(i\Psi_0 + \sum_{j=1}^4 \Psi_j \right), \\ \sigma_{zz} &= -\sum_{j=1}^4 r_j \Delta \Psi_j, \quad \tau_z = \Gamma_1 \left[\sum_{j=1}^4 s_j r_j \frac{\partial \Psi_j}{\partial z_j} - i s_0 c_{44}^* \frac{\partial \Psi_0}{\partial z_0} \right]. \end{aligned} \quad (29)$$

where

$$\Gamma_1 = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y},$$

and

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} && \text{in Cartesian coordinates } (x, y, z), \\ \Delta &= \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{r^2 \partial \theta^2} && \text{in cylindrical coordinates } (r, \theta, z), \end{aligned}$$

as well as

$$\begin{aligned} r_j &= \frac{c_{11}^* + c_{13}^* P_{1j} s_j^2 - c_{11}^* P_{2j} - c_{11}^* P_{34}}{s_j^2} = c_{44}^* (1 - P_{1j}) \\ &= -c_{13}^* - c_{33}^* s_j^2 P_{1j} + \varepsilon_1 c_{11}^* P_{34} + \gamma_1 c_{11}^* P_{2j}, \end{aligned} \quad (30a)$$

$$(c_{11}^*, c_{13}^*, c_{33}^*, c_{44}^*, c_{66}^*) = \frac{1}{a_1 T_0} (c_{11}, c_{13}, c_{33}, c_{44}, c_{66}). \quad (30b)$$

For non-torsional axisymmetric problem $\Psi_0 = 0$ and $\Psi_j (j = 1, 2, 3, 4)$ are independent of θ , so that $u_\theta = 0$ and $\sigma_{z\theta} = \sigma_{r\theta} = 0$. The general solution in cylindrical coordinates (r, θ, z) can be simplified to the following form

$$\begin{aligned} u_r &= - \sum_{j=1}^4 \frac{\partial \psi_j}{\partial r}, \quad w = \sum_{j=1}^4 s_j P_{1j} \frac{\partial \psi_j}{\partial z_j}, \\ C &= \sum_{j=1}^4 P_{2j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad T = P_{34} \frac{\partial^2 \psi_4}{\partial z_4^2}, \\ \sigma_{rr} &= 2c_{66}^* \sum_{j=1}^4 \frac{1}{r} \frac{\partial \psi_j}{\partial r} - \sum_{j=1}^4 s_j^2 r_j \frac{\partial^2 \psi_j}{\partial z_j^2}, \\ \sigma_{\theta\theta} &= -2c_{66}^* \sum_{j=1}^4 \frac{1}{r} \frac{\partial \psi_j}{\partial r} - \sum_{j=1}^4 (s_j^2 r_j - 2c_{66}^*) \frac{\partial^2 \psi_j}{\partial z_j^2}, \\ \sigma_{zz} &= \sum_{j=1}^4 r_j \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad \sigma_{zr} = \sum_{j=1}^4 s_j r_j \frac{\partial^2 \psi_j}{\partial r \partial z_j}. \end{aligned} \quad (31)$$

For torsional axisymmetric problem $\Psi_j = 0 (j = 1, 2, 3, 4)$ and Ψ_0 is independent of θ , so that $u_r = u_z = 0, T = 0$ and $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{rz} = 0$. The

general solution can be simplified as

$$u_\theta = -\frac{\partial \Psi_0}{\partial r}, \quad \sigma_{r\theta} = 2c_{66}^* \left(\frac{1}{2} \frac{\partial^2}{\partial z_0^2} - \frac{\partial^2}{\partial r^2} \right) \Psi_0, \quad \sigma_{z\theta} = -s_0 c_{44}^* \frac{\partial^2 \Psi_0}{\partial r \partial z_0}. \quad (32)$$

5 Solution for a point heat source in infinite transversely isotropic thermodiffusion elastic material

As shown in Fig.1 We consider an infinite transversely isotropic thermodiffusion elastic material, whose isotropic plane is perpendicular to z -axis. A point heat source H is applied at the origin of cylindrical coordinate frame (r, θ, z) or Cartesian coordinate frame (x, y, z) . The general solution given by equation (31) is derived in this section.

For non-torsional axisymmetric problem, assume the harmonic function as follows

$$\Psi_0 = 0, \quad \Psi_j = A_j [\text{sign}(z)z_j \log R_j^* - R_j], \quad (j = 1, 2, 3, 4), \quad (33)$$

where $R_j^* > 1$ and A_j are arbitrary constants to be determined, $\text{sign}()$ is the signum function, and

$$R_j^* = R_j + \text{sign}(z)z_j, \quad R_j = \sqrt{r^2 + z_j^2} = \sqrt{x^2 + y^2 + z_j^2}, \quad (j = 1, 2, 3, 4). \quad (34)$$

Substituting equation (33) into equation (31), we obtain

$$u_r = \sum_{j=1}^4 A_j \frac{r}{R_j^*}, \quad w = \sum_{s_j} P_{1j} A_j \text{sign}(z) \log(R_j^*), \quad (35a)$$

$$C = \sum_{j=1}^4 P_{2j} \frac{A_j}{R_j}, \quad T = P_{34} \frac{A_4}{R_4}, \quad (35b)$$

$$\sigma_{rr} = 2c_{66}^* \sum_{j=1}^4 \frac{A_j}{R_j^*} - \sum_{j=1}^4 s_j^2 r_j \frac{A_j}{R_j}, \quad \sigma_{zz} = \sum_{j=1}^4 r_j \frac{A_j}{R_j}, \quad (35c)$$

$$\sigma_{\theta\theta} = 2c_{66}^* \sum_{j=1}^4 \frac{A_j}{R_j^*} - \sum_{j=1}^4 (s_j^2 r_j - 2c_{66}^*) \frac{A_j}{R_j}, \quad (35d)$$

$$\sigma_{zr} = \sum_{j=1}^4 s_j r_j A_j \frac{\text{sign}(z)r}{R_j R_j^*}. \quad (35e)$$

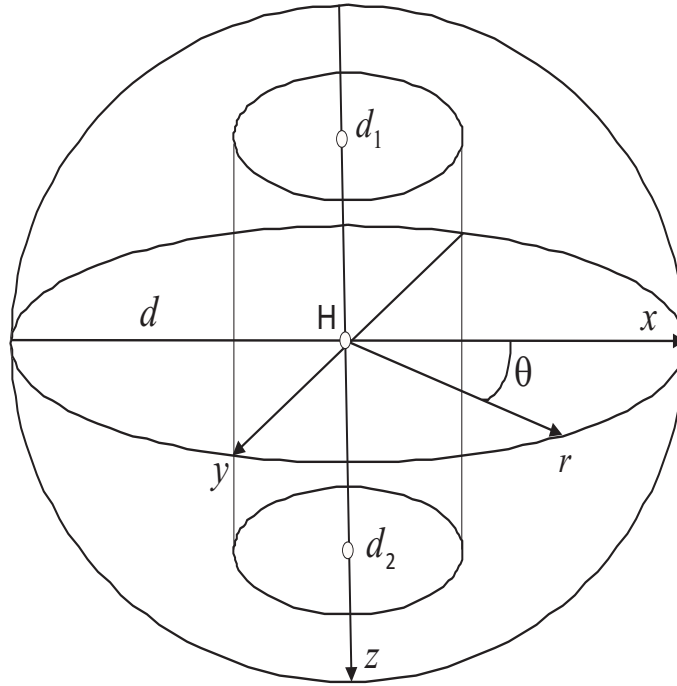


Figure 1: Geometry of the problem

The continuity on plane $z = 0$ for w and τ_{zr} leads to the following expressions

$$\sum_{j=1}^4 s_j P_{1j} A_j = 0, \quad (36)$$

$$\sum_{j=1}^4 s_j r_j A_j = 0. \quad (37)$$

Substituting r_j from equation (30a) into equation(37) yields

$$\sum_{j=1}^4 s_j c_{44}^* (1 - P_{1j}) A_j = 0. \quad (38)$$

By virtue of (36), equation (38) can be simplified to

$$\sum_{j=1}^4 s_j A_j = 0. \quad (39)$$

When the mechanical, concentration and, thermal equilibrium for a cylinder of $d_1 \leq z \leq d_2$ ($d_1 < 0 < d_2$) and $0 \leq r \leq d$ are considered (cf. Fig.1), following three equations can be obtained

$$\int_0^{2\pi} \int_0^d [\sigma_{zz}(r, \theta, d_2) - \sigma_{zz}(r, \theta, d_1)] r dr d\theta + d \int_0^{2\pi} \int_{d_1}^{d_2} \sigma_{zr}(d, \theta, z) dz d\theta = 0, \quad (40a)$$

$$\int_0^{2\pi} \int_0^d \left[\frac{\partial C}{\partial z}(r, \theta, d_2) - \frac{\partial C}{\partial z}(r, \theta, d_1) \right] r dr d\theta + d \int_0^{2\pi} \int_{d_1}^{d_2} \frac{\partial C}{\partial r}(d, \theta, z) dz d\theta = 0, \quad (40b)$$

$$-\varepsilon_2 \int_0^{2\pi} \int_0^d \left[\frac{\partial T}{\partial z}(r, \theta, d_2) - \frac{\partial T}{\partial z}(r, \theta, d_1) \right] r dr d\theta - d \int_0^{2\pi} \int_{d_1}^{d_2} \frac{\partial T}{\partial r}(d, \theta, z) dz d\theta = H. \quad (40c)$$

Some useful integrals are listed as follows

$$\int \frac{1}{R_j} r dr = \int \frac{r}{\sqrt{r^2 + z_j^2}} dr = R_j, \int \text{sign}(z) \frac{r}{R_j R_j^*} dz = -\frac{1}{s_j} \frac{r}{R_j^*}, \quad (41a)$$

$$\int \frac{\partial T}{\partial z} r dr = s_4 P_{34} A_4 \int \frac{z_4}{R_4}, \quad (41b)$$

$$\int \frac{\partial T}{\partial r} dz = A_4 \frac{P_{34}}{s_4} \text{sign}(z) \frac{r}{R_4 R_4^*}, \quad (41c)$$

$$\int \frac{\partial C}{\partial z} r dr = s_j P_{2j} A_j \frac{z_j}{R_j}, \quad (41d)$$

$$\int \frac{\partial C}{\partial r} dz = A_j \frac{P_{2j}}{s_j} \text{sign}(z) \frac{r}{R_j R_j^*}. \quad (41e)$$

It is noticed that the integrals (41c,e) are not continuous at $z = 0$, so that following expressions should be used

$$\int_{d_1}^{d_2} \frac{\partial T}{\partial r} dz = \int_{d_1}^{0^-} \frac{\partial T}{\partial r} dz + \int_{0^+}^{d_2} \frac{\partial T}{\partial r} dz, \quad (42)$$

$$\int_{d_1}^{d_2} \frac{\partial C}{\partial r} dz = \int_{d_1}^{0^-} \frac{\partial C}{\partial r} dz + \int_{0^+}^{d_2} \frac{\partial C}{\partial r} dz. \quad (43)$$

Substituting equation (35 b,c) into equation (40 a) and using the integrals (41 a), we obtain

$$\sum_{j=1}^4 r_j A_j I_1 = 0, \quad (44)$$

where

$$I_1 = [R_j(r, d_2) - R_j(r, d_1)]_{r=0}^{r=d} - \left[\frac{d^2}{R_j^*(d, z)} \right]_{z=d_1}^{z=d_2},$$

i.e.

$$I_1 = \left[\sqrt{r^2 + s_j^2 d_2^2} - \sqrt{r^2 + s_j^2 d_1^2} \right]_{r=0}^{r=d} - \left[\frac{d^2}{z_j + \sqrt{d^2 + z_j^2}} \right]_{z=d_1}^{z=d_2} = 0.$$

This shows that the equations (40a) and (44) are satisfied automatically.

Substituting the value of C from (35 a) into equation (40 b) and using the integrals (41 d,e) and (43), we obtain

$$\sum_{j=1}^4 P_{2j} A_j \eta_j = 0, \quad (45)$$

where

$$\begin{aligned} \eta_j &= s_j^2 \left[\frac{s_j d_2}{R_j(r, d_2)} - \frac{s_j d_1}{R_j(r, d_1)} \right]_{r=0}^{r=d} \\ &+ \left[\frac{\text{sign}(z) d^2}{R_j(d, z) R_j^*(d, z)} \right]_{z=d_1}^{0^-} + \left[\frac{\text{sign}(z) d^2}{R_j(d, z) R_j^*(d, z)} \right]_{z=0^+}^{z=d_2}. \end{aligned}$$

On simplifying, we obtain

$$\begin{aligned} \eta_j &= \left[\frac{s_j^2 [s_j d_2 (d^2 + s_j^2 d_2^2)^{1/2} + s_j d_2] + d^2}{d^2 + s_j^2 d_2^2 + s_j d_2 (d^2 + s_j^2 d_2^2)^{1/2}} \right] \\ &+ \left[\frac{s_j^2 [-s_j d_1 (d^2 + s_j^2 d_1^2)^{1/2} + s_j d_1] + d^2}{d^2 + s_j^2 d_1^2 + s_j d_1 (d^2 + s_j^2 d_1^2)^{1/2}} \right] - 2. \end{aligned}$$

Making use of equation (35b) in the equation (40c) with $s_4 = \sqrt{K_1/K_3}$ and integrals (41b,c), and (42), we obtain

$$-A_4 I_3 = \frac{H}{2\pi P_{34} \sqrt{K_3/K_1}}, \quad (46)$$

where

$$\begin{aligned}
 I_3 &= \left[\frac{s_4 d_2}{R_4(r, d_2)} - \frac{s_4 d_1}{R_4(r, d_1)} \right]_{r=0}^{r=d} + \left[\frac{\text{sign}(z) d^2}{R_4(d, z) R_4^*(d, z)} \right]_{z=d_1}^{z=0^-} \\
 &+ \left[\frac{\text{sign}(z) d^2}{R_4(d, z) R_4^*(d, z)} \right]_{z=0^+}^{z=d_2} = -2.
 \end{aligned} \tag{47}$$

Thus, A_4 can be determined from equation (46) and (47), as follows

$$A_4 = \frac{H}{4\pi P_{34} \sqrt{K_3/K_1}}. \tag{48}$$

Applying A_4 in equations (36), (39) and (45), we get a system of three non-homogeneous equations with three unknowns i.e. A_1, A_2 and A_3 which can be easily determined by giving numerical values of unknown parameters.

6 A special case of negligible diffusion

In the absence of diffusion effects, equations (34 a) - (34 c) yield:

$$\begin{aligned}
 u_r &= \sum_{j=1}^3 A_j \frac{r}{R_j^*}, w = \sum_{j=1}^3 s_j P_{1j} A_j \text{sign}(z) \log(R_j^*), T = P_{23} \frac{A_4}{R_4}, \\
 \sigma_{rr} &= 2c_{66}^* \sum_{j=1}^3 \frac{A_j}{R_j^*} - \sum_{j=1}^3 s_j^2 w_j \frac{A_j}{R_j}, \sigma_{zz} = \sum_{j=1}^3 r_j \frac{A_j}{R_j}, \\
 \sigma_{\theta\theta} &= 2c_{66}^* \sum_{j=1}^3 \frac{A_j}{R_j^*} - \sum_{j=1}^4 (s_j^2 w_j - 2c_{66}^*) \frac{A_j}{R_j}, \sigma_{zr} = \sum_{j=1}^3 s_j r_j A_j \frac{\text{sign}(z)r}{R_j R_j^*},
 \end{aligned} \tag{49}$$

where $z_j = s_j z$, $s_3 = \sqrt{\frac{K_1}{K_3}}$ and $s_j (j = 1, 2)$ are two roots (with positive real part) of the following equation

$$a'^4 - b'^2 + c' = 0,$$

and

$$\begin{aligned}
 a' &= \delta_1 \delta_5, b' = \delta_5 + \delta_1^2 - \delta_4^2, c' = \delta_1, \\
 P_{ij} &= \frac{p'_{2j}}{p'_{1j}}, P_{23} = \frac{p'_{33}}{p'_{13}},
 \end{aligned}$$

$$p'_{kj} = a'_k - b'_k s_j^2 \quad (k = 1, 2),$$

$$\begin{aligned} p'_{33} &= a'_3 - b'_3 s_j^2 + c'_3 s_j^4, a'_1 = -\delta_1, b'_1 = \delta_5 - \delta_4 \varepsilon_1, \\ b'_1 &= \delta_4 - \varepsilon_1, b'_2 = \delta_1 \varepsilon_1, a'_3 = \delta_1, \\ b'_3 &= (\delta_4^2 - \delta_1^2) - \delta_5, c'_3 = \delta_1 \delta_5. \end{aligned}$$

The above results are similar to those obtained by Hou et al.[22] for an infinite transversely isotropic thermoelastic material..

7 Conclusion

The three dimensional fundamental solution in transversely isotropic thermoelastic diffusion media has been derived. By using the operator theory, we have derived the general expression for components of displacement, mass concentration, temperature distribution and stress components. On the basis of general solution, three dimensional fundamental solution for a point heat source in an infinite thermoelastic diffusion media are presented by introducing four newly harmonic functions. Since all the components are expressed in terms of elementary functions, so it is convenient to use them. A special case of interest is also deduced from the general expression, in the absence of diffusion effect.

Acknowledgement

One of the authors Mr. Vijay Chawla is thankful to Kurukshetra University, Kurukshetra for financial support in terms of University Research Scholarship.

Appendix A

$$\begin{aligned}
\bar{a} &= \delta_1(\delta_5 q_8^* + 2\gamma_1 q_4^*), \\
\bar{b} &= (\delta_1^2 - \delta_4^2)q_8^* + \delta_5(q_2^* + q_8^*) + \delta_1 q_3^*(1 + \gamma_1) + \\
&\quad + q_4^*(\gamma_1 - \delta_4) + \delta_1(q_3^* - \delta_5 q_7^*) - \delta_4 q_2^* \gamma_1, \\
\bar{c} &= (\delta_4^2 - \delta_1^2)q_7^* + q_3^*(1 + \gamma_1) - \delta_4(\gamma_1 q_7^* + q_3^*) + \\
&\quad \delta_5(q_1^* - q_7^*) + \delta_1(q_2^* + q_8^*), \\
\bar{d} &= \delta_1(q_7^* + q_1^*),
\end{aligned}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Appendix B

$$\begin{aligned}
\bar{a}_1 &= (q_5^* - q_7^*)\delta_1, \\
\bar{b}_1 &= \delta_1(q_8^* - q_6^*) + \delta_5(q_5^* - q_7^*) + \varepsilon_1(\delta_4 q_7^* - q_3^*) - \gamma_1 \delta_4 q_5^* \bar{c}_1 \\
&= (\gamma_1 q_6^* + \varepsilon_1 q_8^*)\delta_4 + (q_8^* - q_6^*)\delta_5 - q_4^* \varepsilon_1 \bar{a}_2 \\
&= (q_1^* + q_5^*)\gamma_1 + \varepsilon_1(q_1^* - q_7^*) + \delta_4(q_7^* - q_5^*), \\
\bar{b}_2 &= \delta_1(\gamma_1 q_5^* - \varepsilon_1 q_7^*) + \varepsilon_1(q_2^* + q_8^*) - \gamma_1(q_2^* + q_6^*) + \delta_4(q_6^* - q_8^*), \\
\bar{c}_2 &= \delta_1(\varepsilon_1 q_8^* - \gamma_1 q_6^*), \\
\bar{a}_3 &= (q_1^* - q_5^*)\delta_1, \\
\bar{b}_3 &= (\delta_4^2 - \delta_1^2)q_5^* + \delta_5(q_1^* - q_5^*) + \delta_1(q_2^* + q_6^*) - \delta_4 \varepsilon_1 q_1^* \bar{c}_3 \\
&= (\delta_1^2 - \delta_4^2)q_6^* + \delta_5(q_2^* + q_6^*) - \delta_4(\varepsilon_1 q_2^* + 1) - \delta_1 \delta_5 q_5^*, \\
\bar{d}_4 &= \delta_1 \delta_5 q_6^*
\end{aligned}$$

References

- [1] I. Fredholm, Sur Les Equations de L'Equilibre D'un Corps Solide Elastique, Acta Mathematica., 23(1900), 1-42.
- [2] I.M. Lifshitz, L.N. Rozentsveig, Construction of Green tensor for the fundamental equation of elasticity theory in the case of an unbounded elastically anisotropic medium, Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki, 17 (1947), 783-791.
- [3] H.A. Elliott, Three dimensional stress distribution in hexagonal anisotropic crystals, Proc. Cambridge Philos. Soc., 44(1948), 522-533.
- [4] E. Kroner, Das fundamentale Integral der anisotropen elastischen Differentialgleichungen, Z.Phys. 136(1953), 402-410.
- [5] J.L. Synge, The Hypercircle in Mathematical Physics, Cambridge University Press, London, UK, 1957.
- [6] J. Lejcek, The Green's function of the theory of elasticity in an isotropic hexagonal medium, Czech. J. Phys., B19 (1969), 799-803.
- [7] Y.C. Pan, T.W. Chou, Point forces solution for an infinite transversely isotropic solid, ASME J. Appl. Mech., 43(1976), 608-612.
- [8] E. Pan, F.G. Yuan, Three dimensional Green's function in anisotropic biomaterials, Int. J. Solid Struct., 37(2000), 5329-5351.
- [9] B. Sharma, Thermal stresses in transversely isotropic semi-infinite elastic solids, ASME J. Appl. Mech., 23(1958), 86-88.
- [10] W.Q. Chen, H.J. Ding, D.S. Ling, Thermoelastic field of transversely isotropic elastic medium containing a penny-shaped crack: exact fundamental solution, Int. J. Solid Struct., 41(2004), 69-83.
- [11] P.F. Hou, A.Y.T. Leung, C.P. Chen, Green functions for semi-infinite transversely isotropic thermoelastic materials, ZAMM Z. Angew. Math. Mech., 1(2008), 33-41.
- [12] P.F. Hou, L. Wang, T. Yi, 2D Green's functions for semi-infinite orthotropic thermoelastic plane, Appl. Math. Model., 33(2009), 1674-1682.
- [13] P.F. Hou, H. Sha, C.P. Chen, 2D general solution and fundamental solution for orthotropic thermoelastic materials, Engineer. Anal. Bound. Elem., 45 (2008), 392-408.
- [14] W. Nowacki, Dynamical problem of thermodiffusion in solid - 1, Bulletin of Polish Academy of Sciences Series, Science and Tech., 22(1974), 55-64.
- [15] W. Nowacki, Dynamical problem of thermodiffusion in solid-11, Bulletin of Polish Academy of Sciences series, Science and Tech., 22(1974), 129-135.
- [16] W. Nowacki, Dynamical problem of thermodiffusion in solid-111, Bulletin of Polish Academy of Sciences Series, Science and Tech., 22(1974), 275-276.

- [17] W. Nowacki, Dynamic problems of thermodiffusion in solids, Proc. Vib. Prob., 15(1974), 105-128.
- [18] H.H. Sherief, H. Saleh, A half space problem in the theory of generalized Thermoelastic diffusion, int. J. of Solid and Struct. 42(2005), 4484-4493.
- [19] R. Kumar, T. Kansal, Propagation of Lamb waves in transversely isotropic Thermoelastic diffusive plate, Int. J. Solid Struct., 45(2008), 5890-5913.
- [20] R. Kumar, V. Chawla , Surface wave propagation in an Elastic Layer Lying over a Thermodiffusive Elastic Half-Space with Imperfect Boundary, Mech. Advan. Mater. Struct., 18(2011),352-363.
- [21] R. Kumar, V. Chawla, A Study of Fundamental Solution in Orthotropic Thermodiffusive Elastic Media, Int. Commun. Heat Mass Trans., 38(2011), 456-462.
- [22] P.F.Hou, A.Y.T.leung and C.P.Chen, fundamental solution for transversely isotropic thermoelastic materials,Int.J.Solids struct. 45(2008), 392-408.
- [23] Ding,H.J., Chen,B. and Liang,J., General solutions for coupled equations for piezoelectric media, Int.J.Solids struct. 33(1996), 2283-2298.

Submitted in February 2012, revised in April 2012.

3D fundamentalno rešenje u transversalno izotropnom termoelastičnom materijalu sa difuzijom

Proučava se fundamentalno rešenje za 3D problem u poprečno izotropnoj termoelastičnoj sredini sa difuzijom. Posle primene bezdimenzionih promenljivih uvode se dve funkcije pomeranja u cilju uprošćenja osnovnih jednačina termoelastične difuzije za problem stacionarnog stanja. Korišćenjem teorije operatora izvedeni su opšti izrazi za komponente pomeranja, masene koncentracije, rasporeda temperature i komponentata napona. Na osnovu opšteg rešenja, uvodjenjem četiri nove harmonijske funkcije dobijena su 3D fundamentalna rešenja za tačkasti izvor toplote unutar beskonačne termoelastične sredine. Jedan specijalan slučaj je takodje izveden za ilustraciju efekta difuzije.