Steady Stokes flow past dumbbell shaped axially symmetric body of revolution-an analytic approach

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Abstract

In this paper, the problem of steady Stokes flow past dumbbell-shaped axially symmetric isolated body of revolution about its axis of symmetry is considered by utilizing a method (Datta and Srivastava, 1999) based on body geometry under the restrictions of continuously turning tangent on the boundary. The relationship between drag and moment is established in transverse flow situation. The closed form expression of Stokes drag is then calculated for dumbbell-shaped body in terms of geometric parameters b, c, d and a with the aid of this linear relation and the formula of torque obtained by (Chwang and Wu, part 1, 1974) with the use of singularity distribution along axis of symmetry. Drag coefficient and moment coefficient are defined in various forms in terms of dumbbell parameters. Their numerical values are calculated and depicted in respective graphs and compared with some known values.

Keywords: Stokes flow, dumbbell shaped axially symmetric body, drag, torque.

1 Introduction

In physical and biological science, and in engineering, there is a wide range of problems of interest like sedimentation problem, lubrication processes etc.

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concerning the flow of a viscous fluid in which a solitary or a large number of bodies of microscopic scale are moving, either being carried about passively by the flow, such as solid particles in sedimentation, or moving actively as in the locomotion of micro-organisms. In the case of suspensions containing small particles, the presence of the particles will influence the bulk properties of the suspension, which is a subject of general interest in Rheology. In the motion of micro-organisms, the propulsion velocity depends critically on their body shapes and modes of motion, as evidenced in the flagellar and ciliary movements and their variations. A common feature of these flow phenomena is that the motion of the small objects relative to the surrounding fluid has a small characteristic Reynolds number Re. Typical values of Re may range from order unity, for sand particles settling in water, for example, down to 10^{-2} to 10^{-6}, for various micro-organisms. In this low range of Reynolds numbers, the inertia of the surrounding fluid becomes insignificant compared with viscous effects and is generally neglected and the Navier-Stokes equations of motion reduce to the Stokes equations as a first approximation. The zero Reynolds number flow is called Stokes flow.

All these motions are characterized by low Reynolds numbers and are described by the solution of the Stokes equations. Although the Stokes equations are linear, to obtain exact solutions to them for arbitrary body shapes or complicated flow conditions is still a formidable task. There are only relatively few problems in which it is possible to solve exactly the creeping motion equations for flow around a single isolated solid body. Stokes in [1] calculated the flow around a solid sphere undergoing uniform translation through a viscous fluid whilst Oberbeck in [2] solved the problem in which an ellipsoid translates through liquid at a constant speed in an arbitrary direction. Edwards in [3], applying the same technique, obtained the solution for the steady motion of a viscous fluid in which an ellipsoid is constrained to rotate about a principal axis. The motion of an ellipsoidal particle in a general linear flow of viscous fluid at low Reynolds number has been solved by Jeffery [4], whose solution was also built up using ellipsoidal harmonics. The analysis described by Jeffery extended further by Taylor in [5]. Goldstein in [6] obtained a force on a solid body moving through viscous fluid. Lighthill in [7] studied the problem of squirming motion of nearly spherical deformable bodies through liquids at very small Reynolds number. Hill and Power [8] have obtained arbitrarily closed approximations of drag by proving a complimentary pair of extremum principles for a Newtonian viscous fluid in quasi-static flow.

Payne and Pell [9] used the methods of generalized axially symmetric

potential theory to calculate the flow past a class of axisymmetric bodies, including the lens, ellipsoid of revolution, spindle, and two separated spheres. Brenner [10] gave the general expression for Stokes resistance over an arbitrary particle. Brenner and Cox [11] obtained the expression of resistance to a particle of arbitrary shape in translational motion at small Reynolds number. Tuck [12] developed a method for a simple problem in potential theory and is applied to a problem in Stokes flow, yielding a procedure for obtaining the Stokes drag on a blunt slender body of arbitrary shape. Acrivos and Taylor [13] presented the general solution of the creeping flow equations for the motion of an arbitrary particle in an unbounded fluid in terms of spherical coordinates. They derived the force exerted on the particle, and the particular case of a slightly but otherwise arbitrarily deformed sphere was treated by them. Brenner [14, 15, 16, 17, 18] further presented a theoretical calculation of the low Reynolds number resistance of a rigid, slightly deformed sphere to translational and rotational motions in an unbounded fluid. In which, he derived explicit expressions, to the first-order in the small parameter characterizations, which relates the Stokes resistance dyadic with the torque dyadic and the location of the centre of hydrodynamic stress of the particle to its geometry. Cox [19] generalized the result of resistance of a particle of arbitrary shape in translation at small Reynolds number given by Brenner and Cox [11]. He also obtained quantitative results for both a spheroid and a dumbbell shaped body in pure translation and also for a translating rotating sphere and for a dumbbell shaped body in pure rotation. O'Brien [20] expressed the change in drag with change in shape from a sphere in terms of a form factor. Taylor [21] studied the motion of axi-symmetric bodies in viscous fluids. Batchelor [22] has studied Stokes flow past a slender body of arbitrary(not necessarily circular) cross-section. Morrison [23] derived the force on an accelerating body in an axisymmetric slow viscous flow which is valid for any axisymmetrical body, irrespective of the conditions at the surface. Naruse [24] studied the low Reynolds number flow of an incompressible fluid past a body by solving the Navier-Stokes equations, on the basis of the method of matched asymptotic expansions. It is shown that, when the shape of the body is symmetric with respect to a point, the force on the body is determined to the order of Re squared times log Re, where Re denotes the Reynolds number.

In a series of work over low Reynolds number hydromechanics, Chwang and Wu [25, 26], have explored the fundamental singular solution of the Stokes equation to obtain solutions for several specific body shapes translating and

rotating in a viscous fluid. Regarding the distribution range of the singularities, it was pointed out that some results for plane-symmetric bodies in a potential flow may also be valid in all types of Stokes flow. By providing the exact solution of the Stokes equation in an elegant, closed form, the singularity method proved to be a useful alternative to the more standard methods of solution. Unfortunately, one cannot, in a straight forward manner, generalize this approach to the systems of many particles or to particles in the vicinity of a wall. Alawneh and Kanwal [27] obtained closed form solutions for various boundary value problems in mathematical physics by considering suitable distributions of the Dirac delta function and its derivatives on lines and curves. They have derived equations for the n-dimensional dumbbell shaped body.

In a series of paper, Gluckman et al. [28, 29] developed a new numerical method for treating the slow viscous motion past finite assembles of particles of arbitrary shape, termed the multipole representation technique. The approach is based on the theory that the solution for any object conforming to a natural coordinate system in a particle assemblage can be approximated by a truncated series of multi-lobular disturbances in which the accuracy of the representation is systematically improved by the addition of higher-order multipoles. For example, for a system of spherical particles the solution is found in terms of Legendre functions. Youngren and Acrivos [13] used the boundary-element method to calculate hydrodynamic forces and torques acting on spheroidal and cylindrical particles in a uniform and simple shear flow. They expressed the solution of Stokes equations in the form of linear integral equations for the Stokeslet distribution over the particle surface. The required density of the Stokeslets, identical with the surface stress forces can be obtained numerically by reducing the integral equations to a system of linear algebraic equations. The technique has been successfully tested against the analytical solutions for spheroidal particles in a shear flow. Fischer et al. [31] calculated the total force exerted on the isolated rigid obstacle in three dimensional space and in two dimensions placed in the stationary flow of an incompressible viscous fluid with the help of matched asymptotic expansions. Wu [32] proposed a new method of the line distribution of discrete singularities and continuous singularities to solve the Stokes flow that passes the arbitrary non-slender prolate axisymmetrical body. Wu applied this method to calculate the drag factor and the pressure distribution for the Cassini prolate oval as an example of the non-slender prolate arbitrary body. Wu and Qing [33] have proposed the same singularity method to treat the creeping

motion of the arbitrary prolate axisymmetrical body. They obtained analytic expressions in closed form and numerical results for the prolate spheroid and Cassini oval for the flow field. Tun [34] calculated the drag factor and the pressure distribution for the Cassini oval as an example of the nonslender prolate arbitrary axisymmetric body using the same method of line distribution of discrete singularities and continuous singularities. Zhu and Wu [35] discussed the problem of Stokes flow of the arbitrary oblate axisymmetrical body by using the same method of singularity distribution. They have obtained the drag factor for oblate Cassini oval. Dabros [36] attempted to find the hydrodynamic forces and velocities of arbitrary shaped particles, placed in an arbitrary flow field, particularly in the vicinity of the wall, using a singular point solution as the base function. Kim [37] wrote a note on Faxen laws for non-spherical particles and gave the formula of hydrodynamics resistance experienced by particle deviates from spherical shape.

Leith [38] extended the Stokes law on a sphere to a non-spherical object by allocating the interaction of the fluid with the object into its interaction with two analogous spheres, one with the same projected area and one with the same surface area as the object. He used this approach to characterized dynamic shape factor for objects whose shape factors are already exists in the literature. He reported the shape factor for a sphere, cylinders, prisms, spheroids and double conicals. Yuan and Wu [39] obtained the analytic expressions in closed form for flow field by distributing continuously the image Sampsonlets with respect to the plane and by applying the constant density, the linear and the parabolic approximation. They calculated the drag factor of the prolate spheroid and the Cassini oval for different slender ratios and different distances between the body and the plane. Power and Miranda [40] have successfully given the Fredholm integral equation representation of second kind for Stokes resistance problems i.e. when the velocity of particle is known, and the forces and moments are to be found. They represented the velocity as a double layer integral to which they added a Stokeslet and a Rotlet, both located at the centre of the body. Equating the representation to the given velocity resulted in a Fredholm integral equation of the second kind in the double layer density, and a numerical solution became possible after relating the Stokeslet and Rotlet strengths(force and moment) to the unknown double-layer density. Karrila and Kim [72] showed the completion of the double layer representation by Power and Mirinda [40] to be one of many possible completions. They suggested the same representation for the velocity as did by Power and Mirinda [40] and discuss various completions, suggesting one

which is advantageous to an iterative numerical process for multiparticle systems. Both these completions are successful because, as observed by Power and Mirinda [40] and previously by Ladyzhenskaya [42], the double layer representation alone is able to represent flow fields that correspond to the total force and total moment equal to zero. A much more extensive review over numerical methods may be found in the paper of Weinbaum and Ganatos [43] and to the paper of Karrila and Kim [41]. Hsu and Ganatos [44] have studied the motion of a rigid body in a viscous fluid bounded by a plane wall. Chester [45] considered the motion of a body through a viscous fluid at low Reynolds number. He derived general formulae for the force and couple acting on a body of arbitrary shape and implemented over to reduce some special cases. Liron and Barta [46] have presented a new singular boundary-integral equation of the second kind for the stresses on a rigid particle in motion in Stokes flow. They also produced the forces and moments on the particle with the help of generalized Faxen law. Hubbard and Douglas [47] presented simple and accurate method of estimating the translational hydrodynamic friction on rigid Brownian particles of arbitrary shape. Keh and Tseng [48] presented a combined analytical and numerical study for Stokes flow caused by an arbitrary body of revolution and calculated drag on prolate and oblate Cassini ovals. Lowenberg [49] computed the Stokes resistance, added mass, and Basset force numerically for finite-length, circular cross-section cylinders using a boundary integral formulation. In this study, he found analytical formulas for the Stokes force, added mass, Basset force of spheroids which contrasted with the numerical results for cylinders of the aspect ratio in the range: $0.01 \leq a/b \leq 100$. He concluded that for some of these parameters, significant differences persist for disk and rod shaped particles. Feng and Wu [50] gave the convergent results for prolate Cassini ovals by using a method of combined analytic-numerical method. Douglas et al. [51] calculated the translational friction coefficient and the capacitance of a variety of objects with a probabilistic method involving hitting the probed objects with random walks launched from an enclosing spherical surface. Zhou and Pozrikidis [52] implemented the method of fundamental solutions to compute Stokes flow past or due to the motion of solid particles. The computed locations and strengths of the singularities have been compared by them with those corresponding to exact discrete and continuous singularity representations, and the computed force and torque exerted on the particles are compared with exact values available from analytical solutions. Brenner [53] studied the hydrodynamic Stokes resistance on non-spherical particles.

Datta and Srivastava [54] developed a new approach to evaluate the Stokes drag force in a simple way on an axially symmetric body with some geometrical constraints placed in axial flow and transverse flow under the no-slip boundary conditions. The results of drag on both the flow situations were successfully tested not only for sphere, prolate and oblate spheroid but also for other bodies like deformed sphere, cycloidal and egg-shaped bodies of revolution with acceptable limit of error. This method has been described in the section 2 as the same is exploited here to study the problem of Stokes flow around dumbbell shaped axially symmetric body. Datta and Srivastava [55] obtained the optimum drag profile in axi-symmetric Stokes flow under the restrictions of constant volume and constant cross section area by exploiting DS conjecture given by Datta and Srivastava [54]. Palaniappan and Ramkissoon [56] provided a complete survey of drag formula over axi-symmetric particle in Stokes flow. Tsai et al. [57] have provided the practical and numerical implementations of the method of fundamental solutions for three-dimensional exterior Stokes problems with quiet far-field condition. This numerical scheme has been checked by them for sphere and rotating dumbbell shaped body. Srivastava [58] obtained the optimum volume profile in axi-symmetric Stokes flow by exploiting the DS conjecture given by Datta and Srivastava [54]. Scolan and Etienne [59] have discussed some aspects of the force and moment computations in incompressible and viscous flows on bodies of arbitrary shaped by using the projection techniques developed by Quartapelle and Napolitano [60] without explicitly calculating the pressure. Bowen and Masliyah [61] obtained an approximate solution to the equation of motion governing Stokes flow past a number of isolated closed bodies of revolution by the least square fitting of a truncated series expression for the stream function to known boundary conditions. They found reasonably accurate ($\pm 5\%$) estimate for the Stokes resistance on body shapes, such as cylinders and cones, for which the solutions are exceedingly difficult. They applied the computed drag values in determining the limitations of the various empirical expressions used to predict the drag resistance of these geometrically simple bodies. Blake et al. [62] have considered some of the properties of the S-transform as well as exploiting the special properties associated with Legendre polynomials to generate a range of slender body shape with fixed Stokes drag. Radha et al. [63] have described a new approximate method to discuss uniform flow past rigid bodies of two different shapes using a complete general solution [64, 65] of Stokes equations in an incompressible viscous fluid. They have proposed that with this new method approximate values of

physical quantities like drag experienced by a rigid body could be obtained in a simple way in accurate manner. Sherief et al. [66] have investigated the translational motion of an arbitrary body of revolution in a micropolar fluid by using a combined analytical-numerical method. They have evaluated the drag exerted on a prolate spheroid for various values of the aspect ratio and for different values of the micropolarity parameters. They further applied this technique to the prolate Cassini ovals for justifying good convergence. Srivastava [67] presented the optimum cross-section profile in axisymmetric flow by utilizing the same method proposed by Datta and Srivastava [54].

For the detailed study over the concerned topic, reader is advised to go through the books of Lamb [68], Batchelor [69], Happel and Brenner [70], Langlois [71], Kim and Karrila [72], Pozrikidis [73, 74], Kohr and Pop [75].

In most of these investigations the main result of physical interest is the drag experienced by body and torque on body rotating about its axis of revolution. In the analysis presented in this paper, we targeted to study the salient features of class of dumbbell axi-symmetric bodies by applying the method proposed by Datta and Srivastava [54] and the results given by Chwang and Wu [25] described in the section 2 briefly.

1.1   Applications

Stokes flow of an arbitrary body is of interest in biological phenomena and chemical engineering. In fact, the body with simple form such as sphere or ellipsoid is less encountered in practice. The body, which is presented in science and technology, often takes a complex arbitrary form. For example, under normal condition, the erythrocyte is a biconcave disk in shape, which can easily change its form and present different contour in blood motion due to its deformability. In second half of twentieth century, a considerable progress has been made in treating the Stokes flow of an arbitrary body.

The study of isolated dumbbell shaped axially symmetric body has great importance in biological and engineering applications. Such type of shapes plays a vital role in swift swimming or self propulsion of microorganisms and human bodies in which pushing or pulling is based on dumbbell([76], [77]), aerobic and spore forming bacteria which plays an important part in cell division and spore germination ([78], [79]), shape of red blood cells in healthy or infected mammalian and human bodies([80], [81, 82]). In circulating blood, the red blood cells are severely deformed. The study of red blood cell geometry and deformability throws light on the mechanical properties of cells and cell membranes, and thus is of basic importance to biology and rheology. It

is also of value clinically, because change of shape and size and strength of red blood cells may be indicative of disease like Malaria for example. Evans and Fung [83] gave the pictorial presentation of infected red blood cells in the arteries of human body in the shapes of dumbbell(bi-concave) described exactly in the present analysis. According to Fung [84], red blood cell, during the flow, get deformed either due to the translation or rotation. But in either situation, it took the shape of dumbbell of various aspect ratios discussed in the present analysis. In this way, the author claims at this stage, being mathematician apart from biologist, can say that this analysis may be very helpful in finding the drag and couple of infected blood cells which may be very helpful in the prevention of disease and drug delivery system.

2 Body geometry and method

Let us consider the axially symmetric body of characteristic length L placed along its axis(x-axis, say) in a uniform stream U of viscous fluid of density ρ_1 and kinematic viscosity ν. When Reynolds number UL/ν is small, the motion is governed by Stokes equations (Happel and Brenner, 1964, cf. [70]),

$$0 = \left(\frac{1}{\rho_1}\right) grad\, p + \nu\nabla^2 \mathbf{u}\ , \qquad div\, \mathbf{u}\ = 0, \tag{2.1}$$

subject to the no-slip boundary condition.

We have taken up the class of those axially symmetric bodies which possesses continuously turning tangent, placed in a uniform stream U along the axis of symmetry (which is x-axis), as well as constant radius 'b' of maximum circular cross-section at the mid of the body. This axi-symmetric body is obtained by the revolution of meridional plane curve (depicted in Figure 1) about axis of symmetry which obeys the following limitations:

i. tangents at the points A, on the x-axis, must be vertical,

ii. tangents at the points B, on the y-axis, must be horizontal, and

iii.the semi-transverse axis length 'b' must be fixed.

The point P on the curve is may be represented by the Cartesian coordinates (x, y) or polar coordinates (r, θ) respectively, PN and PM are the length of tangent and normal at the point P. The symbol R stands for the intercepting length of normal between the point on the curve and point on axis of symmetry and symbol α is the slope of normal PM which can be vary from 0 to π.
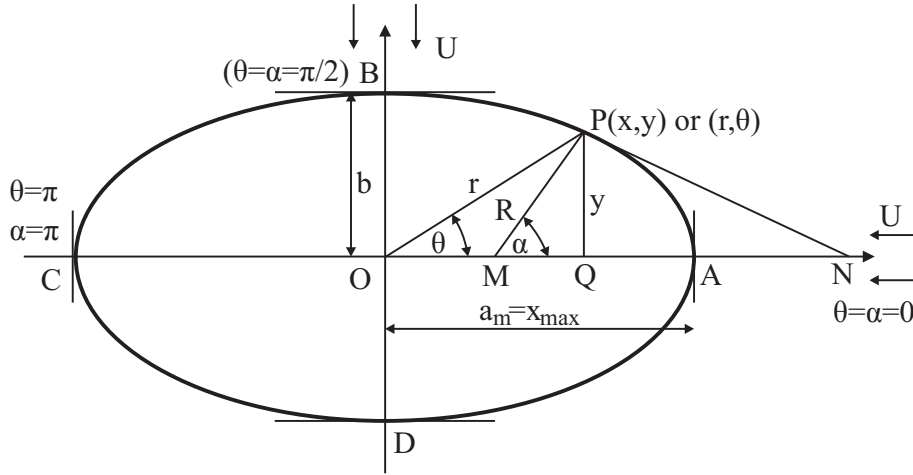
Figure 1: Geometry of axially symmetric body

Axial flow

The expression of Stokes drag on such type of axially symmetric bodies placed in axial flow(uniform flow parallel to the axis of symmetry) is given by [54]

$$F_{||} = \frac{1}{2} \frac{\lambda b^2}{h_{||}}, \quad \text{where} \quad \lambda = 6\pi\mu U_{||} \tag{2.2}$$

and

$$h_{||} = \left(\frac{3}{8}\right) \int_0^\pi R sin^3 \alpha \, d\alpha \tag{2.3}$$

where the suffix $||$ has been introduced to assert that the force is in the axial direction.

Sometimes it will be convenient to work in Cartesian co-ordinates. Therefore, referring to the Fig. 1, for the profile geometry, we have

$$y = R Sin\alpha, \quad tan\alpha = -\left(\frac{dy}{dx}\right)^{-1} = -\frac{dx}{dy} = -x'. \tag{2.4}$$

Using above transformation, we may express (2.3) as

$$h_{||} = -\frac{3}{4} \int_0^a \frac{yy''}{\left(1 + y'^2\right)^2} dx, \tag{2.5}$$

where $2\,a_m$ represents the axial length of the body and dashes represent derivatives with respect to x. In the sequel, it will be found simpler to work with y as the independent variable. Thus, $h_{||}$ assumes the form

$$h_{||} = \quad -\frac{3}{4}\int_0^b \frac{yx'^2x''}{\left(1+x'^2\right)^2}dy, \tag{2.6}$$

where dashes represent derivatives with respect to y.

Transverse flow

The expression of Stokes drag on such type of axially symmetric bodies placed in transverse flow(uniform flow perpendicular to the axis of symmetry) is given by [54]

$$F_\perp = \left(\frac{1}{2}\right)\frac{\lambda b^2}{h_y}, \qquad \text{where} \qquad \lambda = 6\pi\mu U_\perp, \tag{2.7}$$

and

$$h_\perp = \left(\frac{3}{16}\right)\int_0^\pi R\left(2sin\alpha - sin^3\alpha\right)d\alpha. \tag{2.8}$$

According to the same manner as we did in axial flow, equation (2.8) may also be written in Cartesian form as (in both cases having x and y treated as independent)

$$h_\perp = -\frac{3}{8}\int_0^a \frac{yy''\left[1 + 2\left(y'\right)^2\right]}{\left[1+\left(y'\right)^2\right]^2}dx, \tag{2.9}$$

and

$$h_\perp = -\frac{3}{8}\int_0^b \frac{yx''\left[2 + \left(x'\right)^2\right]}{\left[1+\left(x'\right)^2\right]^2}dy, \tag{2.10}$$

In (2.9) and (2.10), the dashes represents derivative with respect to x and y respectively. where the suffix $\perp$ has been placed to designate the force due to the external flow along the y-axis, the transverse direction.

The proposed conjecture is, of course, subject to restrictions on the geometry of the meridional body profile y(x) of continuously turning tangent implying that $y\prime(x)$ is continuous together with $y\prime\prime(x) \neq 0$, thereby avoiding corners or sharp edges or other kind of nodes and straight line portions,

$y = ax + b$, $x_1 \leq x \leq x_2$. Also, it should be noted here that the method holds good for convex axially symmetric bodies which possesses fore-aft symmetry about the equatorial axis perpendicular to the axis of symmetry(polar axis). Apart from this argument, It is interesting to note here that the proposed conjecture is applicable also to those axi-symmetric bodies which fulfill the condition of continuously turning tangent but does not possess fore-aft symmetry like egg shaped body [54]. This conjecture is much simpler to evaluate the numerical values of drag than other existing numerical methods like Boundary Element Method (BEM), Finite Element Analysis (FEA) etc. as it can be applied to a large set of convex axi-symmetric bodies possessing fore-aft symmetry about maximal radius situated in the middle of the body for which analytical solution is not available or impossible to evaluate.

Since both axial and transverse flows have been considered in a free stream results of the force at an oblique angle of attack may be resolved into its components to get the required result. The present analysis can be extended to generate a drag formula for axi- symmetric bodies for more complex flows like paraboloidal flow for which free stream may be represented by average velocity [26]. Authors are working in this direction and also searching the avenues of this analysis for non-linear Stokes flow.

The proposed analysis can be extended to calculate the couple on a body rotating about its axis of symmetry. Description of the same is given below.

Torque on rotating axially symmetric body

The moment on a sphere of radius 'b' with angular velocity Ω is given by the formula [70]

$$
\begin{aligned}
M &= 6\pi\mu b^3 \Omega \int_{\alpha=0}^{\pi} sin^3\alpha d\alpha \\
&= \int_0^{\pi} dm \qquad \text{(say)}
\end{aligned}
\tag{2.11}
$$

where

$$dm = 6\pi\mu b^3 \Omega \, sin^3\alpha d\alpha. \tag{2.12}$$

Comparing it with the elemental force df(with R replaced by b) as given by [54]

$$df = \frac{3}{4}\lambda R sin^3\alpha d\alpha \tag{2.13}$$

keeping in mind that two forces constitute a couple, we have (by equation (2.12))

$$dm = 6\pi\mu b^3 \Omega \sin^3\alpha d\alpha = 6\pi\mu b \left(b^2 \sin^3\alpha d\alpha\right)$$
$$= \left(2\pi\mu b^2 \Omega\right)\left(\frac{2}{3}\frac{df}{\pi\mu U}\right) \tag{2.14}$$
$$= \left(\frac{4}{3}\right)\left(\frac{b^2\Omega}{U}\right) df,$$

on integrating this differential equation over upper part of axis of symmetry (Figure 1.) from $\alpha=0$ to $\alpha = \pi$ or $\theta=0$ to $\theta = \pi$, we can have the expression of couple in terms of axial drag as

$$M_{\parallel} = \frac{4}{3}\frac{b^2\Omega}{U}F_{\parallel}. \tag{2.15}$$

3 Closed form drag expression for dumbbell shaped axially symmetric body

We consider a class of dumbbell-shaped bodies in rotation about axis of symmetry (z-axis) that can be represented by a pair of isolated rootlets of equal strength (Figure 2) similar to that used in [25] as

$$\Gamma(z) = (1/2)\Gamma_0 \quad \delta(x + c) + (1/2)\Gamma_0\delta(x - c), \tag{3.1}$$

for which the no-slip boundary condition, requiring $u_\theta(z, \rho_0(z)) = \Omega\rho_0(z)$, becomes

$$R_1^{-3} + R_2^{-3} = \frac{2\Omega}{\Gamma_0}, (\rho = \rho_0(z), |z| \le a), \tag{3.2}$$

where

$$R_1 = \left[(z + c)^2 + \rho^2\right]^{\frac{1}{2}}, R_2 = \left[(z - c)^2 + \rho^2\right]^{\frac{1}{2}}. \tag{3.3}$$

At the terminal points $(z = \pm a, \rho_0 = 0)$ and at the dumbbell neck $(z = 0, \rho_0 = d)$ condition (3.2) reduces, respectively, to

$$\left(a^2 - c^2\right)^3 = a\left(a^2 + 3c^2\right)\left(c^2 + d^2\right)^{\frac{3}{2}}, \tag{3.4}$$

and

$$\left(c^2 + d^2\right)^{-\frac{3}{2}} = \frac{\Omega}{\Gamma_0}. \tag{3.5}$$
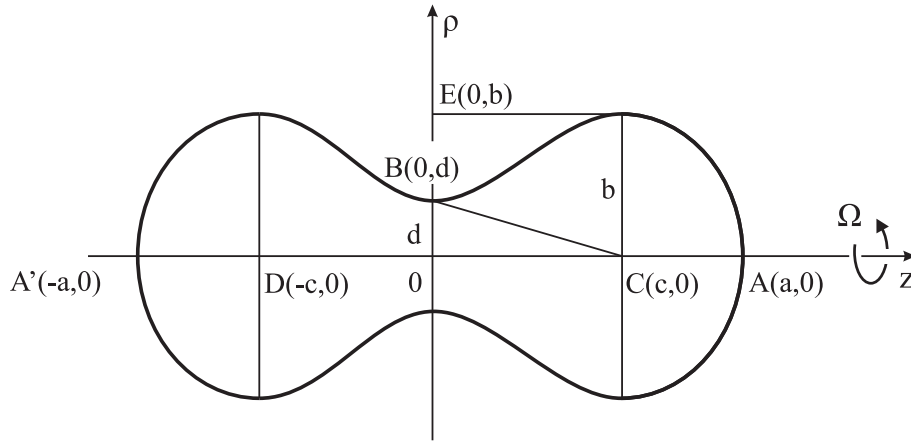
Figure 2: A class of dumbbell-shaped bodies in rotation that can be represented by a pair of isolated rotlets

Elimination of Ω/Γ_0 between (3.4) and (3.5) provides

$$a\left(a^2 + 3c^2\right)\left(a^2 - c^2\right)^{-3} = \frac{\Omega}{\Gamma_0}, \tag{3.6}$$

which provides a relationship between the geometric parameters d, c and a. This relationship may be re-written in the various forms:

$$\left(1 - \frac{c^2}{a^2}\right)^3 = \left(1 + 3\frac{c^2}{a^2}\right)\left(\frac{c^2}{a^2} + \frac{d^2}{a^2}\right)^{\frac{3}{2}}, \tag{3.7a}$$

$$\left(\frac{a^2}{b^2} - \frac{c^2}{b^2}\right)^3 = \left(\frac{a}{b}\right)\left(\frac{a^2}{b^2} + 3\frac{c^2}{b^2}\right)\left(\frac{c^2}{b^2} + \frac{d^2}{b^2}\right)^{\frac{3}{2}}, \tag{3.7b}$$

and

$$\left(\frac{a^2}{d^2} - \frac{c^2}{d^2}\right) = \left(\frac{a}{d}\right)\left(\frac{a^2}{d^2} + 3\frac{c^2}{d^2}\right)\left(1 + \frac{c^2}{d^2}\right)^{\frac{3}{2}}. \tag{3.7c}$$

The same elimination between (3.2) and (3.4) results in an algebraic equation which determines the shape function $\rho_0/a = f(z/a, c/a)$, which depends on one geometric parameter, c/a. This algebraic equation may be written in the form

$$\frac{1}{\left[\left(\frac{z}{a} + \frac{c}{a}\right)^2 + \frac{\rho_0^2}{a^2}\right]^{\frac{3}{2}}} + \frac{1}{\left[\left(\frac{z}{a} - \frac{c}{a}\right)^2 + \frac{\rho_0^2}{a^2}\right]^{\frac{3}{2}}} = 2\frac{\left(1 + 3\frac{c^2}{a^2}\right)}{\left(1 - \frac{c^2}{a^2}\right)^3}. \tag{3.8}$$

The torque [25] on rotating dumbbell-shaped body about its axis of symmetry(z-axis) with angular velocity Ω, is

$$M_z \quad = \quad 8\pi\mu\Omega \left(c^2 + d^2\right)^{\frac{3}{2}}, \tag{3.9}$$

$$= \quad 8\pi\mu\Omega a^3 \left(\frac{c^2}{a^2} + \frac{d^2}{a^2}\right)^{\frac{3}{2}}. \tag{3.10}$$

The torque coefficient C_M in that paper [25] has been defined with reference to $8\,\pi\mu\Omega a\,b^2$, where 'b' is the maximum radial extent, that is $b = max[\rho_0(z)]$

$$C_{Mz} = \frac{M}{8\pi\mu\Omega ab^2} = \frac{\left(c^2 + d^2\right)^{\frac{3}{2}}}{ab^2}, \tag{3.11}$$

$$= \frac{\left(\frac{c^2}{a^2} + \frac{d^2}{a^2}\right)^{\frac{3}{2}}}{\frac{b^2}{a^2}}, \tag{3.12a}$$

$$= \frac{\left(\frac{c^2}{b^2} + \frac{d^2}{b^2}\right)^{\frac{3}{2}}}{\frac{a}{b}}, \tag{3.12b}$$

$$= \frac{\left(1 + \frac{c^2}{d^2}\right)^{\frac{3}{2}}}{\left(\frac{a}{d}\right)\left(\frac{b}{d}\right)^2}. \tag{3.12c}$$

Let F_z be the Stokes drag experienced by dumbbell body (described in Figure 2) in transverse flow situation and M_z be the torque on rotating dumbbell-shaped body about its axis of symmetry(z-axis), then by using the linear relationship between drag and torque in section 2(eq. 2.15), we get, by using (3.9),

$$F_z = 6\pi\mu U \frac{\left(c^2 + d^2\right)^{\frac{3}{2}}}{b^2}. \tag{3.13}$$

This closed form solution of axial Stokes drag on dumbbell body is found to be new and never seen in the literature. In equation (3.13), U is uniform stream velocity along the axis of symmetry.

Now, we can define the drag coefficient C_F in various forms with reference to $6\pi\mu Ua$, $6\pi\mu Ub$, $6\pi\mu Ud$, i.e. drag on sphere with radius *a, b,* and *d* as

$$C_{F_z} = \frac{\left(\frac{c^2}{a^2} + \frac{d^2}{a^2}\right)^{\frac{3}{2}}}{\frac{b^2}{a^2}}, \tag{3.14}$$

$$= \frac{\left(\frac{c^2}{b^2} + \frac{d^2}{b^2}\right)^{\frac{3}{2}}}{\frac{a}{b}}, \tag{3.15}$$

$$= \frac{\left(1 + \frac{c^2}{d^2}\right)^{\frac{3}{2}}}{\left(\frac{a}{d}\right)\left(\frac{b^2}{d^2}\right)}, \tag{3.16}$$

$$= \frac{\left[\frac{c^2}{a^2} + \frac{d^2}{a^2}\right]^{\frac{3}{2}}}{\left(\frac{b}{a}\right)^3} \tag{3.17}$$

$$= \left[\frac{c^2}{b^2} + \frac{d^2}{b^2}\right]^{\frac{3}{2}}, \tag{3.18}$$

$$= \frac{\left(1 + \frac{c^2}{d^2}\right)^{\frac{3}{2}}}{\left(\frac{b}{d}\right)^3}, \tag{3.19}$$

$$= \frac{\left[\frac{c^2}{a^2} + \frac{d^2}{a^2}\right]^{\frac{3}{2}}}{\left(\frac{d}{a}\right)\left(\frac{b^2}{a^2}\right)}, \tag{3.20}$$

$$= \frac{\left[\frac{c^2}{b^2} + \frac{d^2}{b^2}\right]^{\frac{3}{2}}}{\left(\frac{d}{b}\right)}, \tag{3.21}$$

$$= \frac{\left[1 + \frac{c^2}{d^2}\right]^{\frac{3}{2}}}{\left(\frac{b}{d}\right)^2}. \tag{3.22}$$

Now, the moment coefficient C_M, apart from that defined in (3.12) by [25] can be defined in various other ways with reference to $8\pi\mu\Omega a^3$, $8\pi\mu\Omega b^3$, $8\pi\mu\Omega d^3$ i.e. the moment or torque on rotating sphere with radius a, b and d respectively, as

$$C_{M_z} = \left[\frac{c^2}{a^2} + \frac{d^2}{a^2}\right]^{\frac{3}{2}}, \tag{3.23}$$

$$= \frac{\left[\frac{c^2}{b^2} + \frac{d^2}{b^2}\right]^{\frac{3}{2}}}{\left(\frac{a}{b}\right)^3}, \tag{3.24}$$

$$= \frac{\left[1 + \frac{c^2}{d^2}\right]^{\frac{3}{2}}}{\left(\frac{a}{d}\right)^2}, \tag{3.25}$$

$$= \frac{\left[\frac{c^2}{a^2} + \frac{d^2}{a^2}\right]^{\frac{3}{2}}}{\left(\frac{b}{a}\right)^3}, \tag{3.26}$$

$$= \left[\frac{c^2}{b^2} + \frac{d^2}{b^2}\right]^{\frac{3}{2}}, \tag{3.27}$$

$$= \frac{\left[1 + \frac{c^2}{d^2}\right]^{\frac{3}{2}}}{\left(\frac{b}{d}\right)^3}, \tag{3.28}$$

$$= \frac{\left[\frac{c^2}{a^2} + \frac{d^2}{a^2}\right]^{\frac{3}{2}}}{\left(\frac{d}{a}\right)^3}, \tag{3.29}$$

$$= \frac{\left[\frac{c^2}{b^2} + \frac{d^2}{b^2}\right]^{\frac{3}{2}}}{\left(\frac{d}{b}\right)^3}, \tag{3.30}$$

$$= \left[1 + \frac{c^2}{d^2}\right]^{\frac{3}{2}}. \tag{3.31}$$

On utilizing the fact that the sum of maximum radius 'b' and half of the focal distance 'c' is equal to semi axial length 'a' i.e.

$$b + c = a \Rightarrow \frac{b}{a} + \frac{c}{a} = 1, \tag{3.32}$$

which is the extension of fact used in [25] that the sum of two geometric parameters b/a and c/a is nearly unity. This restriction may be easily achieved for dumbbell shaped body(Figure 2) by locating the rootlets of equal strength

exactly at those points on axis of symmetry which are the centers of hemi-spheres having radius 'b', the maximum radial extent.

By using relation (3.32) and the relationship (3.7A), the expressions of drag coefficient (3.14) and moment coefficient (3.12) may be re-written in single parameter c/a and found to be same as

$$C_{F_z} = \frac{\left[1 - \frac{c^2}{a^2}\right]^3}{\left[1 + 3\frac{c^2}{a^2}\right]\left[1 - \frac{c}{a}\right]^2}, \qquad (3.33)$$

$$C_{M_z} = \frac{\left[1 - \frac{c^2}{a^2}\right]^3}{\left[1 + 3\frac{c^2}{a^2}\right]\left[1 - \frac{c}{a}\right]^2}, \qquad (3.34)$$

the various other forms of drag coefficient and moment coefficient may also be obtained with the use of relation (3.32) and geometric relationship (3.7A-C) as well the previously defined non-dimensional values of drag [(3.15) to (3.22)] and torque [(3.23) to (3.31)].

4   Parametric analysis

The azimuthal component, u_θ, of velocity for flow field around rotating dumb-bell shaped body of revolution may be written, by using (3.2), as

$$u_\theta = \Omega\rho(z)$$

$$= \frac{\Gamma_0}{2}\left[\frac{1}{\{(z+c)^2 + \rho^2\}^{3/2}} + \frac{1}{\{(z-c)^2 + \rho^2\}^{3/2}}\right]\rho(z) \qquad (4.1)$$

by using (3.5), it can be written in parameters c and d as

$$= \frac{\Gamma_0}{2}\left[\frac{1}{\{(z+c)^2 + \rho^2\}^{3/2}} + \frac{1}{\{(z-c)^2 + \rho^2\}^{3/2}}\right]\rho(z), \qquad (4.2)$$

by using (3.4), it can be written in parameters a and c as

$$= \frac{(a^2 - c^2)^3\,\Omega}{2a\,(a^2 + 3c^2)}\left[\frac{1}{\{(z+c)^2 + \rho^2\}^{3/2}} + \frac{1}{\{(z-c)^2 + \rho^2\}^{3/2}}\right]\rho(z), \qquad (4.3)$$

The last expression may be also written in parameter c/a as follows:

$$= \frac{\left(1 - \frac{c^2}{a^2}\right)^3 \Omega}{2\left(1 + 3\frac{c^2}{a^2}\right)} \left[\frac{1}{\left\{\left(\frac{z}{a} + \frac{c}{a}\right)^2 + \frac{\rho^2}{a^2}\right\}^{3/2}} + \frac{1}{\left\{\left(\frac{z}{a} - \frac{c}{a}\right)^2 + \frac{\rho^2}{a^2}\right\}^{3/2}} \right] \rho(z), \quad (4.4)$$

for one piece dumbbell body, $0 \leq$ c/a ≤ 0.556, and for class of dumbbell shaped bodies, -1 $\leq$ z/a ≤ 1 and -1 $\leq$ ρ/a ≤ 1. Tsai et al. [57] verified their numerical technique for quiet far-field conditions and carried numerical experiments for a $= 1$, c $= 0.4$, $\Gamma_0 = 1$, d $= 0.619123$, and $\Omega = 2.497031$. They obtained the azimuthal velocity contour(Figure 10, page 321, [57]) around the corresponding rotating dumbbell body. As b $=$ max[ρ_0(z)], the numerical value of b cannot be less than the value of neck radius $\rho_0(0) = d$, for isolated one piece dumbbell body(Figure 2). So, for single dumbbell body, we must have relationship between parameters d $\leq$ b $\leq$ c $\leq$ a or d/a $\leq$ b/a $\leq$ c/a $\leq$ 1. Analysis does not cover the description for the situation when the body splits into the two equal spheres of same radius for $c/a > 0.556$. The present analysis holds only for $c/a \leq 0.556$ which is valid only for one piece dumbbell body. All the calculations has been done by imposing the restriction (3.32) in which the sum of b and c is kept exactly equal to a. In the paper [25], the sum of b and c is nearly unit and has been achieved by placing the rootlet on axis of symmetry at which $b = max[\rho_0(z)]$. All the numerical values of drag and torque corresponding to various forms have been given in tables 1-3 and their variations are depicted in figures 3-8. All the values given in tables 1-3 are based on geometrical forms (3.7A-C) connecting dumbbell parameters a, b, c and d. In table 1, c/a and d/a are calculated from relation (3.7A), b/a from restriction (3.32), non-dimensional values of drag from forms (3.14), (3.17), (3.20) and non-dimensional values of torque from forms (3.23), (3.26), (3.29), (3.12a). In table 2, c/b and d/a are calculated from relation (3.7B), a/b from restriction (3.32), non-dimensional values of drag from forms (3.15), (3.18), (3.21) and non-dimensional values of torque from forms (3.24), (3.27), (3.30), (3.12b). In table 3, c/d and b/d are calculated from relation (3.7C), a/d from restriction (3.32), non-dimensional values of drag from forms (3.16), (3.19), (3.22) and non-dimensional values of torque from forms (3.25), (3.28), (3.31), (3.12c).

Table 1: Numerical values of drag, torque with respect to dumbbell parameters b/a, c/a and d/a based on relationship (3.7A) for one piece isolated dumbbell body

b/a (3.32)	c/a (3.7A)	d/a (3.7A)	C_{F_z}			C_{M_z}			
			(3.14)	(3.17)	(3.20)	(3.23)	(3.26)	(3.29)	(3.12a)
0.4455	0.5545	0.0028	0.8590	21.8090	297.7704	0.1704	21.8090	∞	0.8590
0.4505	0.5495	0.1089	0.8661	1.9227	7.9539	0.1757	1.9227	136.1180	0.8661
0.4604	0.5396	0.2004	0.8997	1.9542	4.4897	0.1907	1.9542	23.6970	0.8997
0.4803	0.5197	0.3015	0.9401	1.9575	3.1183	0.2168	1.9575	7.9137	0.9401
0.5100	0.4900	0.4027	0.9809	1.9233	2.4358	0.2551	1.9233	3.9068	0.9809
0.5448	0.4552	0.5039	1.0549	1.9364	2.0936	0.3131	1.9364	2.4473	1.0549
0.5912	0.4088	0.6002	1.0956	1.8533	1.8255	0.3829	1.8533	1.7712	1.0956
0.6458	0.3542	0.7062	1.1824	1.8309	1.6743	0.4931	1.8309	1.4001	1.1824
0.7120	0.2880	0.8025	1.2226	1.7171	1.5235	0.6198	1.7171	1.1992	1.2226
0.8014	0.1986	0.9037	1.2333	1.5390	1.3648	0.7921	1.5390	1.0733	1.2333
1.0000	0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2: (Numerical values of drag, torque with respect to dumbbell parameters a/b, c/b and d/b based on relationship (3.7B))

a/b	c/b	d/b	C_{F_z}			C_{M_z}			
(3.32)	(3.7B)	(3.7B)	(3.15)	(3.18)	(3.21)	(3.24)	(3.27)	(3.30)	(3.12b)
2.2446	1.2446	0.0064	0.8589	1.9279	297.7206	0.8589	1.9279	∞	0.8589
2.2197	1.2197	0.2417	0.8660	1.9224	7.9528	0.8660	1.9224	136.0999	0.8660
2.1720	1.1720	0.4352	0.8997	1.9541	4.4894	0.8997	1.9541	23.6957	0.8997
2.0820	1.0820	0.6277	0.9011	1.9573	3.1181	0.9011	1.9573	7.9132	0.9011
1.9607	0.9607	0.7896	0.9808	1.9230	2.4354	0.9808	1.9230	3.9058	0.9808
1.8355	0.8355	0.9249	1.0549	1.9363	2.0935	1.0549	1.9363	2.4471	1.0549
1.6914	0.6914	1.0152	1.0956	1.8531	1.8253	1.0956	1.8531	1.7710	1.0956
1.5484	0.5484	1.0935	1.1823	1.8307	1.6741	1.1823	1.8307	1.4000	1.1823
1.4044	0.4044	1.1271	1.2226	1.7170	1.5234	1.2226	1.7170	1.1991	1.2226
1.2478	0.2478	1.1276	1.2333	1.5390	1.3648	1.2333	1.5390	1.0733	1.2333
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3: (Numerical values of drag, torque with respect to dumbbell parameters a/d, c/d and b/d based on relationship (3.7B))

| a/d	c/d	b/d	C_{F_z}			C_{M_z}			
(3.32)	(3.7C)	(3.7C)	(3.16)	(3.19)	(3.22)	(3.25)	(3.28)	(3.31)	(3.12c)
346.6204	192.2010	154.4194	0.8906	1.9282	297.7700	0.1704	1.9282	∞	0.8906
9.1827	5.0459	4.1368	0.8661	1.9227	7.9539	0.1757	1.9227	136.1179	0.8661
4.9900	2.6926	2.2974	0.8997	1.9542	4.4897	0.1907	1.9542	23.6970	0.8997
3.3167	1.7237	1.5930	0.9401	1.9575	3.1183	0.2168	1.9575	7.9137	0.9401
2.4832	1.2167	1.2664	0.9809	1.9233	2.4358	0.2551	1.9233	3.9068	0.9809
1.9845	0.9033	1.0811	1.0549	1.9364	2.0936	0.3131	1.9364	2.4473	1.0549
1.6661	0.6811	0.9850	1.0956	1.8533	1.8255	0.3829	1.8533	1.7712	1.0956
1.4160	0.5015	0.9144	1.1824	1.8309	1.6743	0.4931	1.8309	1.4001	1.1824
1.2461	0.3588	0.8872	1.2226	1.7171	1.5235	0.6198	1.7171	1.1992	1.2226
1.1065	0.2197	0.8867	1.2333	1.5390	1.3648	0.7921	1.5390	1.0733	1.2333
1.0000	0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Figure 3: Variation of drag coefficients (all three forms) with respect to dumbbell parameters d/a, b/a and c/a (based on table 1)

5 Conclusion

The expression of Stokes drag F_z presented in (3.13) on axially symmetric dumbbell isolated one piece body translating through uniform flow velocity U is seems to be new and never seen in the literature. The drag F_z is non-dimensionalize here with respect to $6\pi\mu Ua$, $6\pi\mu Ub$, $6\pi\mu Ud$, i.e. drag on sphere having radius a, b, and d, which is the main reason for getting the nine forms (3.14-3.22). Also, the expression of torque C_M on rotating dumbbell body about z-axis with angular velocity Ω is non-dimensionalize here with respect to $8\pi\mu\Omega a^3$, $8\pi\mu\Omega b^3$, $8\pi\mu\Omega d^3$ i.e. the torque on rotating sphere with radius a, b and d respectively which is the main reason for getting nine forms (3.23-3.31). According to the author's point of view, the present analysis is very useful in the study of prevention of infected blood cells deformed from its actual disk shape to dumbbell(or biconcave) caused due to some reasons or the other. It can be done by finding the aspect ratio of the infected blood
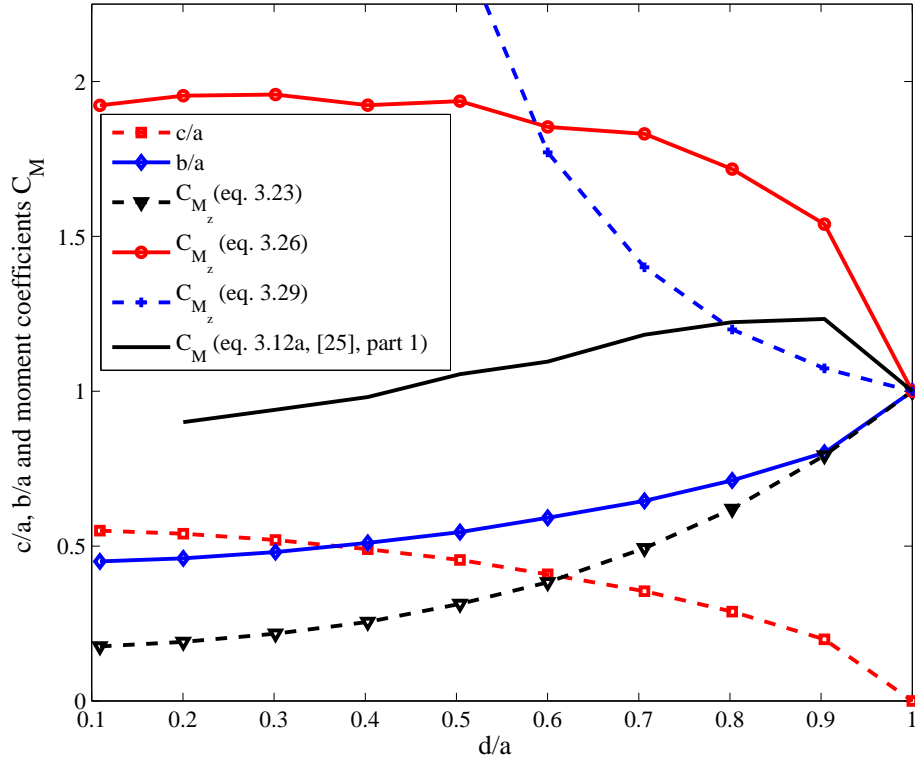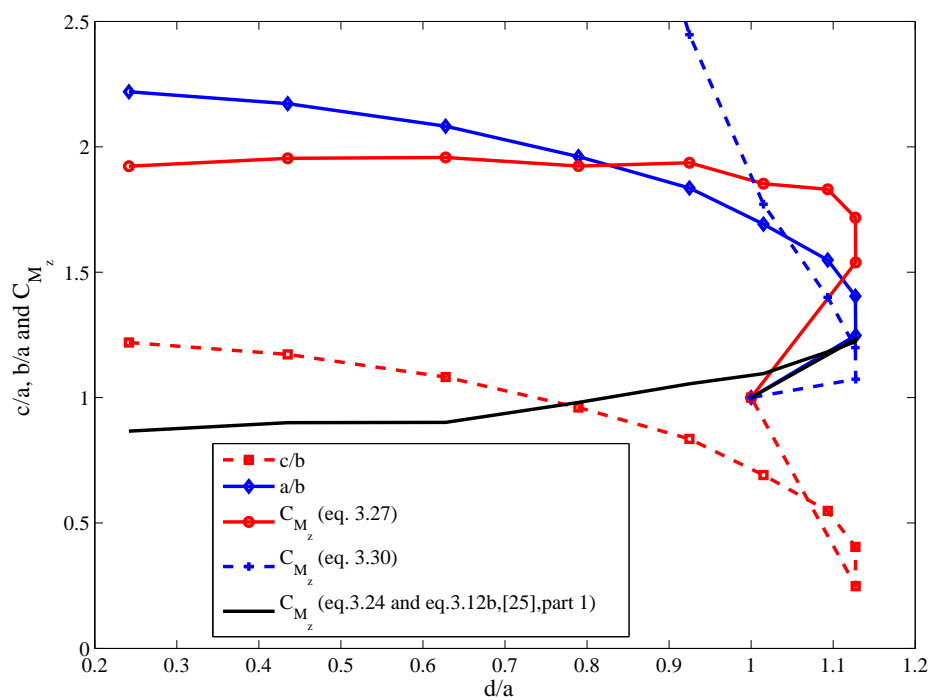
Figure 4: Variation of moment coefficients (all four forms) with respect to dumbbell parameters d/a, b/a and c/a (based on table 1)

cell in form of dumbbell by microscopic analysis. Then the corresponding values of drag and torque value can be evaluated from the tables provided in this paper. Prevention can be owned in terms of supply of accurate medicine may be injected in the infected body through drug delivery systems.

Acknowledgment
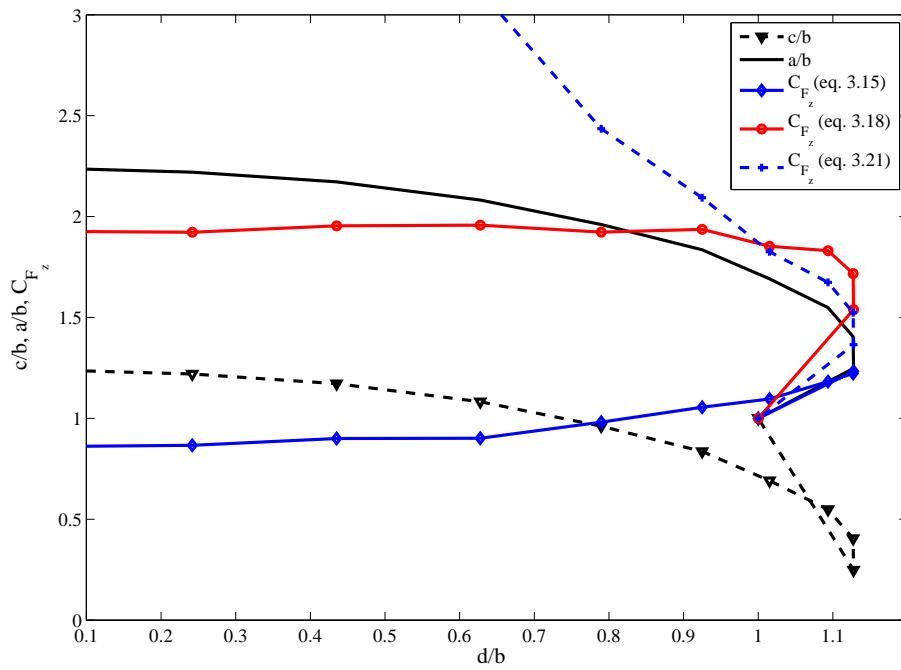
Figure 5: Variation of drag coefficients (all three forms) with respect to dumbbell parameters d/b, c/b and a/b (based on table 2)

Figure 6: Variation of moment coefficients (all four forms) with respect to dumbbell parameters d/b, c/b and a/b (based on table 2)
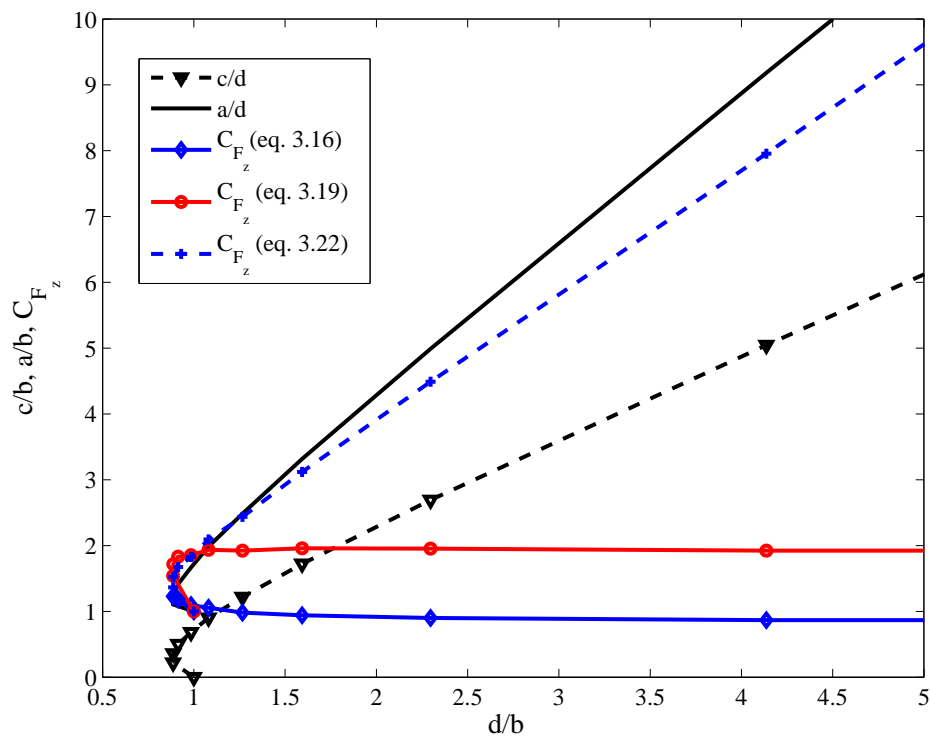
Figure 7: Variation of drag coefficients (all three forms) with respect to dumbbell parameters b/d, c/d and a/d (based on table 3)
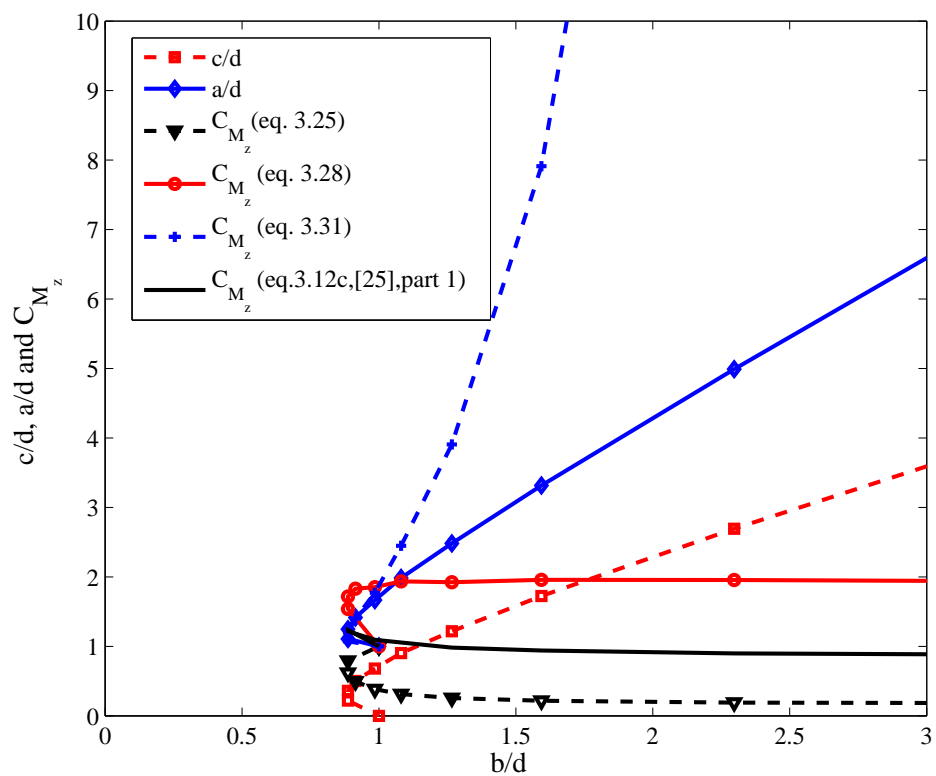
Figure 8: Variation of moment coefficients (all four forms) with respect to dumbbell parameters b/d, c/d and a/d (based on table 3)

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Stacionarno Stoksovo tečenje preko osno simetričnog obrtnog tela oblika tega - analitički pristup

Problem stacionarnog Stoksovog tečenja preko osno simetričnog obrtnog tela oblika tega je razmatran metodom [Datta and Srivastava, 1999] zasnovanom na geometriji tela pri ograničenju neprekidnog okretanja tangente na granici. Relacija izmedju otpora i momenta je ustanovljena za slučaj poprečnog tečenja. Eksplicitni izraz za Stoksov otpor je tada izračunat za tela oblika tega u zavisnosti od geometrijskih parametara b, c, d i a dok je formula za spreg dobijena u (Chwang and Wu, deo 1, 1974) pomoću distribucije singularnosti duž ose simetrije. Koeficijent otpora i koeficijent momenta su definisani u različitim oblicima u zavisnosti od geometrijskih parametara tega. Njihove brojne vrednosti su izračunate i nacrtane na priloženim graficima i uporedjeni sa nekim poznatim veličinama.