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## ON THE RESIDUAL MOTION IN DAMPED VIBRATING SYSTEMS

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**Abstract.** *In this paper, linear vibrating systems, in which the inertia and stiffness matrices are symmetric positive definite and the damping matrix is symmetric positive semi-definite, are studied. Such a system may possess undamped modes, in which case the system is said to have residual motion. Several formulae for the number of independent undamped modes, associated with purely imaginary eigenvalues of the system, are derived. The main results formulated for symmetric systems are then generalized to asymmetric and symmetrizable systems. Several examples are used to illustrate the validity and application of the present results.*

**Key words:** *linear system, dissipation, residual motion*

### 1. INTRODUCTION

Some of the simplest and most fundamental vibrating systems can be described by a differential equation of the form

$$A\ddot{q} + B\dot{q} + Cq = 0, \quad q \in \mathfrak{R}^n \quad (1)$$

where  $A$ ,  $B$ , and  $C$  are  $n \times n$  constant real symmetric matrices,  $q$  is the  $n$ -dimensional vector of generalized coordinates and dots denote derivatives with respect to  $t$  (the time). The inertia matrix  $A$  and stiffness matrix  $C$  are positive definite ( $> 0$ ), and the damping matrix  $B$  may be positive definite or positive semi-definite ( $\geq 0$ ). In the case  $B > 0$  dissipation is complete, and the case  $B \geq 0$  corresponds to incomplete dissipation. In the latter case the system is called partially dissipative (damped).

It is convenient, although not necessary, to rewrite equation (1) in the form

$$\ddot{x} + D\dot{x} + Kx = 0, \quad (2)$$

using the congruent transformation  $x = A^{1/2}q$ , where  $A^{1/2}$  denotes the unique positive definite square root of the matrix  $A$ , and  $D = A^{-1/2}BA^{-1/2}$ , and  $K = A^{-1/2}CA^{-1/2}$ .

All solutions  $x(t)$  of the equation (2) (or  $q(t)$  of (1)) can be characterized algebraically using properties of the quadratic matrix polynomial

$$L(\lambda) = \lambda^2 I + \lambda D + K, \quad (3)$$

where  $I$  is the identity matrix. The eigenvalues of the system are zeros of the characteristic polynomial

$$\Delta(\lambda) = \det(L(\lambda)) \quad (4)$$

Since (4) is a polynomial of degree  $2n$  with respect to  $\lambda$ , there are  $2n$  eigenvalues, counting multiplicities. If  $\lambda$  is an eigenvalue, the nonzero vectors  $X$  in the nullspace of  $L(\lambda)$  are the eigenvectors associated with  $\lambda$ , i. e.,

$$L(\lambda)X = 0 \quad (5)$$

In general, eigenvalues and corresponding eigenvectors may be real or may appear in complex conjugate pairs.

If the dissipation is complete, it is well-known that the system (2) (or (1)) is asymptotically stable ( $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all solutions  $x(t)$ ), see [1]. On the other hand, the partially damped system (2) may or may not be asymptotically stable, although it is obviously stable in the Lyapunov sense (any solution of equation (2) remains bounded). Consequently, all eigenvalues of this system lie in the *closed* left-half of the complex plane ( $\text{Re } \lambda \leq 0$ ). Notice that if the system is asymptotically stable, then  $\text{Re } \lambda < 0$ .

Recently some attention has been paid to the question whether or not a damped system has pure imaginary eigenvalues, i. e., in the terminology of the mechanical vibrations, whether or not undamped modes are possible in such system (see [2] and quoted references). From the above discussion it is clear that nonexistence of undamped motions (also called “residual motions”) is equivalent to the asymptotic stability of the system, and consequently, any test for asymptotic stability gives the answer of the question. A survey of the stability criteria for linear second order systems is given in [3]. Also, it should be mentioned that the paper [2] rediscovered an old criterion for asymptotic stability of the system [4], as was recently stressed in [5].

In this paper we are interested in the determination of the number of pure imaginary eigenvalues of the system without computing the zeros of the characteristic polynomial (4). The main result given in section 3 (Theorem 2) recently derived in our paper [6]. This result is based on the well-known condition of asymptotic stability [7], which coincides with the rank condition of controllability of a linear system (see [8]), and a transformation converting the system (2) into two uncoupled subsystems; one of them is  $r$ -dimensional undamped subsystem, where  $r$  is the number of conjugate pairs of purely imaginary eigenvalues of the system *including multiplicity*, the second is  $(n-r)$ -dimensional damped asymptotically stable subsystem. When the matrix  $K$  has *all distinct* eigenvalues, and  $r$  its eigenvectors lie in the nullspace of the damping matrix, the decomposability of the system in *modal* coordinates was observed in [4]. In sections 4 and 5, when one of two matrices  $D$  and  $K$  is transformed on diagonal form, two useful results are stated. Finally, in section 6 the results of section 3 are generalized to asymmetric and symmetrizable systems.

## 2. THE DECOMPOSABILITY OF THE SYSTEM

**Theorem 1.** Let  $\pm i\omega_1, \dots, \pm i\omega_r$  be eigenvalues of  $L(\lambda)$ . Then there exists an orthogonal matrix  $Q$  such that

$$Q^T D Q = \hat{D} = \begin{pmatrix} 0_r & | & 0 \\ \hline 0 & | & \hat{D}_{n-r} \end{pmatrix}, \quad (6)$$

and

$$Q^T K Q = \hat{K} = \begin{pmatrix} \Omega_r & | & 0 \\ \hline 0 & | & \hat{K}_{n-r} \end{pmatrix}, \quad (7)$$

where  $0_r$  is the zero square matrix of order  $r$ , and  $\Omega_r = \text{diag}(\omega_1^2, \dots, \omega_r^2)$ .

To prove Theorem 1 we need the following lemmas.

**Lemma 1.** Let  $(i\omega, X)$ ,  $\omega \in \Re$ ,  $i = \sqrt{-1}$ , be an eigenpair of  $L(\lambda)$ . Then  $(\omega^2, X)$  and  $(0, X)$  are eigenpairs of the matrices  $K$  and  $D$ , respectively.

*Proof.* From

$$L(i\omega)X = (-\omega^2 I + i\omega D + K)X = 0, \quad (8)$$

we obtain

$$\langle X, (K - \omega^2 I)X \rangle + i\omega \langle X, DX \rangle = 0, \quad (9)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product, and  $\langle X, (K - \omega^2 I)X \rangle$ , and  $\langle X, DX \rangle$  are real quantities, since  $K$  and  $D$  are real symmetric matrices. Then  $\langle X, DX \rangle = 0$ , which implies  $DX = 0$ , since  $D \geq 0$ . This together with  $L(i\omega)X = 0$  gives  $KX = \omega^2 X$ .  $\square$

It is clear that the eigenvector  $X$  in Lemma 1 can be taken to be *unit* ( $\langle X, X \rangle = 1$ ) and *real*.

**Lemma 2.** a) If  $(i\omega_1, X^{(1)})$  and  $(i\omega_2, X^{(2)})$  are eigenpairs of  $L(\lambda)$  with  $\omega_1^2 \neq \omega_2^2$ , then  $\langle X^{(1)}, X^{(2)} \rangle = 0$ .

b) If the eigenvalue  $i\omega$  of  $L(\lambda)$  has multiplicity  $k$ , it possesses  $k$  eigenvectors which are mutually orthogonal.

*Proof.* a) The result follows from Lemma 1 and the additional fact that eigenvectors associated with distinct eigenvalues of a symmetric matrix are orthogonal.

b) Since the system (2) is stable, the multiple eigenvalue  $i\omega$  must be semi-simple, which means that the eigenvalue has  $k$  linearly independent eigenvectors. Since a linear combination of these  $k$  vectors is also an eigenvector of  $L(\lambda)$  associated with  $i\omega$ , the Gram-Schmidt process (see [9]) can be used to obtain  $k$  mutually orthogonal eigenvectors.  $\square$

It follows from Lemma 1 and 2 that the number of independent undamped modes is equal to the number of conjugate pairs of purely imaginary eigenvalues (natural frequencies), including multiplicity.

*Proof of Theorem 1.* By lemmas 1 and 2, there exists an orthonormal set of  $r$  vectors  $X^{(1)}, \dots, X^{(r)}$ , such that

$$DX^{(j)} = 0, KX^{(j)} = \omega_j^2 X^{(j)}, j=1, \dots, r \quad (10)$$

Now, consider an orthogonal matrix  $Q$  having the vectors  $X^{(1)}, \dots, X^{(r)}$  as its first  $r$  columns,

$$Q = (X^{(1)}, \dots, X^{(n)}) \quad (11)$$

The matrices  $D$  and  $K$  are then orthogonally congruent to matrices  $\hat{D}$  and  $\hat{K}$ , respectively, described by

$$\hat{D} = Q^T D Q = (\langle X^{(i)}, DX^{(j)} \rangle) \quad (12)$$

and

$$\hat{K} = Q^T K Q = (\langle X^{(i)}, KX^{(j)} \rangle), \quad (13)$$

where  $i, j = 1, \dots, n$ . Using (10) and  $\langle X^{(i)}, X^{(j)} \rangle = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker delta and  $i, j = 1, \dots, n$ , we compute

$$\langle X^{(i)}, DX^{(j)} \rangle = 0 \quad (14)$$

and

$$\langle X^{(i)}, KX^{(j)} \rangle = \omega_j^2 \delta_{ij}, \quad (15)$$

where  $i = 1, \dots, n$  and  $j = 1, \dots, r$ . The relations (14) and (15) show that  $\hat{D}$  and  $\hat{K}$  have the partitioned forms (6) and (7).  $\square$

### 3. THE MAIN RESULTS

Introduce the  $n \times n^2$  matrix

$$\Phi = \begin{pmatrix} D & \vdots & KD & \vdots & \vdots & \vdots & \vdots & \vdots & K^{n-1}D \end{pmatrix} \quad (16)$$

which plays key role in a test for asymptotic stability of the system [7].

**Theorem 2.** The system (2) has  $r = n - \text{rank} \Phi$  conjugate pairs of purely imaginary eigenvalues, including multiplicity.

**Corollary 1.** If  $\text{rank} D = m$ , then  $0 \leq r \leq n - m$ .

This follows immediately from  $\text{rank} D \leq \text{rank} \Phi \leq n$ .

*Proof of Theorem 2.* Suppose that  $\Delta(\pm i\omega_j) = 0$ ,  $\omega_j \in \mathfrak{R}$ ,  $j = 1, \dots, r$  and that remaining zeros of  $\Delta(\lambda)$  take places on the open left-half of the complex plane. Then from Theorem 1 it follows that there exists an orthogonal coordinate transformation

$$x = Q \begin{pmatrix} y \\ z \end{pmatrix}, y \in \mathfrak{R}^r, z \in \mathfrak{R}^{n-r}, \quad (17)$$

which transforms equation (2) to the form

$$\begin{pmatrix} \ddot{y} \\ \ddot{z} \end{pmatrix} + \hat{D} \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} + \hat{K} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (18)$$

where  $\hat{D}$  and  $\hat{K}$  have the partitioned forms (6) and (7). Under the above assumptions it is clear that the  $(n-r)$  dimensional subsystem of (18)

$$\ddot{z} + \hat{D}_{n-r} \dot{z} + \hat{K}_{n-r} z = 0, z \in \mathfrak{R}^{n-r} \quad (19)$$

is asymptotically stable and, according to well-known result [7], we have

$$\text{rank}\left(\hat{D}_{n-r} \mid \hat{K}_{n-r} \hat{D}_{n-r} \mid \dots \mid \hat{K}_{n-r}^{n-r-1} \hat{D}_{n-r}\right) = n-r \quad (20)$$

On the other hand, the matrix  $\Phi$  coincides with the matrix

$$Q\left(\hat{D} \mid \hat{K}\hat{D} \mid \dots \mid \hat{K}^{n-1}\hat{D}\right)P, \quad (21)$$

where  $P = \text{diag}(Q^T, \dots, Q^T)$ . Then

$$\text{rank}\Phi = \text{rank}\left(\hat{D}_{n-r} \mid \hat{K}_{n-r} \hat{D}_{n-r} \mid \dots \mid \hat{K}_{n-r}^{n-1} \hat{D}_{n-r}\right), \quad (22)$$

since  $Q$  and  $P$  are nonsingular, and  $\hat{D} = \text{diag}(0_r, \hat{D}_{n-r})$ , and  $\hat{K}^j \hat{D} = \text{diag}(0_r, \hat{K}_{n-r}^j \hat{D}_{n-r})$ . Now, according to the Cayley-Hamilton theorem (see [9]), every matrix  $\hat{K}_{n-r}^j \hat{D}_{n-r}$  with integer  $j \geq n-r$  can be represented by a linear combination of the matrices  $\hat{D}_{n-r}, \hat{K}_{n-r} \hat{D}_{n-r}, \dots, \hat{K}_{n-r}^{n-r-1} \hat{D}_{n-r}$ , and, consequently

$$\text{rank}\left(\hat{D}_{n-r} \mid \dots \mid \hat{K}_{n-r}^{n-1} \hat{D}_{n-r}\right) = \text{rank}\left(\hat{D}_{n-r} \mid \dots \mid \hat{K}_{n-r}^{n-r-1} \hat{D}_{n-r}\right) \quad (23)$$

The result then follows from (20), (22) and (23).  $\square$

**Remark 1.** The matrix (16) can be expressed in terms of the original matrices as

$$\Phi = A^{-1/2} \tilde{\Phi} \text{diag}(A^{-1/2}, \dots, A^{-1/2}), \quad (24)$$

where

$$\tilde{\Phi} = \left( B \mid (CA^{-1})B \mid \dots \mid (CA^{-1})^{n-1}B \right) \quad (25)$$

Consequently,  $\text{rank}\Phi = \text{rank}\tilde{\Phi}$ , since  $A$  is nonsingular.

In the case of “classical damping” in which  $D$  and  $K$  commute the following result as a consequence of Theorem 2 can be obtained.

**Theorem 3.** If  $DK = KD$ , then the system has  $r = n - \text{rank}D$  conjugate pairs of purely imaginary eigenvalues.

*Proof.* Since  $D$  and  $K$  commute there exists an orthogonal matrix such that both  $D$  and  $K$  are orthogonally congruent to diagonal matrices [9]. Then, evidently,  $\text{rank}\Phi = \text{rank}D$ , and Theorem 3 follows from Theorem 2.  $\square$

In the next, two examples are given to illustrate the application of the above results.

*Example 1.* Consider the two-degree-of-freedom system shown in Fig. 1, where  $c_1 > 0$  and  $\beta > 0$  stand for the spring constants and coefficient of viscous damping, respectively, and  $q_1$  and  $q_2$  are the displacements from equilibrium positions of masses  $m_1$  and  $m_2$ .



Fig. 1 The system of example

The inertia, damping and stiffness matrices of this system are as follows

$$A = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, B = \beta \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad (26)$$

It is clear that  $\text{rank}B = 1$ , and consequently, the system is partially damped. The matrix (25) takes the form

$$\tilde{\Phi} = \beta \begin{pmatrix} 1 & -1 & \frac{c_1}{m_1} & -\frac{c_1}{m_1} \\ -1 & 1 & -\frac{c_2}{m_2} & \frac{c_2}{m_2} \end{pmatrix} \quad (27)$$

Thus, by Theorem 2, we have

$$r = 2 - \text{rank}\tilde{\Phi} = \begin{cases} 0, c_1m_2 \neq c_2m_1 \\ 1, c_1m_2 = c_2m_1 \end{cases} \quad (28)$$

In the case  $c_1m_2 = c_2m_1$ , the system can oscillate such that relative motion between the masses is absent, so that the damper dissipates no energy. If  $c_1m_2 \neq c_2m_1$ , the system does not have pure imaginary eigenvalues, and all motions lead up to dissipation of energy.

*Example 2.* Consider the three-degree-of-freedom system (2) with

$$D = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \text{ and } K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad (29)$$

previously studied in [10].

It can be easily verified that  $\text{rank}D = 1$ , and that  $DK = KD$ . Thus, by Theorem 3, system of this example has two conjugate pairs of purely imaginary eigenvalues.

#### 4. THE CASE WHEN $K$ IS DIAGONAL (PRINCIPAL COORDINATES)

It is well known that there exists an orthogonal matrix  $Q$  such that

$$Q^T K Q = \Omega = \text{diag}(\omega_1^2 I_{n_1}, \dots, \omega_k^2 I_{n_k}), \quad (30)$$

where  $\omega_j$  and  $I_{n_j}$  denote the distinct natural frequencies of the undamped system ( $D=0$  in (2)) with multiplicity  $n_j \geq 1$  and the identity matrices of order  $n_j$ ,  $n_1 + \dots + n_k = n$ . Multiple natural frequencies are typical in vibrating systems with symmetry or as a result of optimization.

On transforming to principal (modal) coordinates defined by  $p = Q^T x$  and using (30), (2) reduces to

$$\ddot{p} + R\dot{p} + \Omega p = 0, \quad (31)$$

where the matrix  $R = Q^T D Q$  is known as the modal damping matrix. Form a consistent partition of  $R$  with  $\Omega$ :

$$R = \begin{pmatrix} R_{11} & R_{12} & \cdot & R_{1k} \\ R_{21} & R_{22} & \cdot & R_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ R_{k1} & R_{k2} & \cdot & R_{kk} \end{pmatrix} \quad (32)$$

We need the following statement.

**Lemma 3.** The system (31) is asymptotically stable if and only if  $n_j = \text{rank}R_{jj}$ ,  $j = 1, \dots, k$ .

*Proof.* See [11].  $\square$

**Theorem 4.** The system (31) has  $r = n - \sum_{j=1}^k \text{rank}R_{jj}$  conjugate pairs of purely imaginary eigenvalues, including multiplicity. If  $\omega_j i$  is an eigenvalue of the system, then its multiplicity is equal to  $n_j - \text{rank}R_{jj}$ .

*Proof.* It follows from Theorem 1 and Lemma 3.  $\square$

Since  $\text{rank}R_{jj} \leq m = \text{rank}R$ , the next result recently formulated in [5] follows directly from Theorem 4.

**Corollary 2.** If  $\max_j(n_j) > \text{rank}R$ , then system (31) has residual motion.

Next, we apply Theorem 4 to the example 1. For this example, the matrices  $R$  and  $\Omega$  take the forms

$$R = \frac{\beta}{\sqrt{m_1 m_2}} \begin{pmatrix} \sqrt{m_2/m_1} & -1 \\ -1 & \sqrt{m_1/m_2} \end{pmatrix}, \Omega = \begin{pmatrix} c_1/m_1 & 0 \\ 0 & c_2/m_2 \end{pmatrix} \quad (33)$$

Thus, by Theorem 4,  $r = 0$  if  $c_1 m_2 \neq c_2 m_1$ , i. e., the system does not have residual motion (the system is asymptotically stable), and  $r = 1$  if  $c_1 m_2 = c_2 m_1$ .

## 5. THE CASE WHEN D IS DIAGONAL

Since  $D = D^T \geq 0$  and  $\text{rank}D = m$ , there exists an orthogonal matrix  $Q$  such that

$$S = Q^T D Q = \begin{pmatrix} S_{11} & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 0 \end{pmatrix} \quad (34)$$

where  $S_{11} = \text{diag}(s_1, \dots, s_m, 0, \dots, 0)$ ,  $s_j > 0$  for all  $j = 1, \dots, m$ . By the coordinate transformation

$$x = Q \begin{pmatrix} u \\ v \end{pmatrix}, u \in \mathfrak{R}^m, v \in \mathfrak{R}^{n-m} \quad (35)$$

the system (2) reduces to the form

$$\begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} + \begin{pmatrix} S_{11} & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 0 \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \begin{pmatrix} P_{11} & \cdot & P_{12} \\ \cdot & \cdot & \cdot \\ P_{21} & \cdot & P_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (36)$$

where



$$P = Q^T K Q = \left( \begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array} \right) \quad (37)$$

is the transformed stiffness matrix written in the consistent partitioned form with (34).

Introduce the  $(n-m) \times m(n-m)$  matrix

$$F = \left( P_{21} \mid P_{22} P_{21} \mid \dots \mid P_{22}^{n-m-1} P_{21} \right) \quad (38)$$

**Lemma 4.**  $\text{rank}\Phi = m + \text{rank}F$ .

*Proof.* Substituting (34) and (37) into (16), after some rank persevering manipulations, we obtain

$$\text{rank}\Phi = \text{rank} \left( \begin{array}{c|c|c|c|c|c|c} S_{11} & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & P_{21} & P_{22} P_{21} & P_{22}^2 P_{21} & \dots & P_{22}^{n-1} P_{21} & 0 \end{array} \right) \quad (39)$$

Since every matrix  $P_{22}^j P_{21}$  with  $j \geq n-m$ , according to the Cayley-Hamilton theorem, can be represented by a linear combination of the matrices  $P_{21}, P_{22} P_{21}, \dots, P_{22}^{n-m-1} P_{21}$  the lemma is proved.  $\square$

**Theorem 5.** The system (36) has  $r = n - m - \text{rank}F$  conjugate pairs of purely imaginary eigenvalues, including multiplicity.

*Proof.* It directly follows from Theorem 2 and Lemma 4.  $\square$

Let us give an example illustrating Theorem 5.

*Example 3.* (taken from [2]). Consider the tree-degree-of-freedom system (2) with

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } K = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{pmatrix} \quad (40)$$

The damping matrix is positive semi-definite with  $m = \text{rank}D = 1$ , whereas the stiffness matrix is positive definite. By the orthogonal matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (41)$$

we obtain

$$S = Q^T D Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } P = Q^T K Q = \begin{pmatrix} 5 & -2 & -3 \\ -2 & 3 & 0 \\ -3 & 0 & 3 \end{pmatrix} \quad (42)$$

The matrix F takes the form

$$F = \begin{pmatrix} -2 & -6 \\ -3 & -9 \end{pmatrix} \quad (43)$$

Thus, by Theorem 5, we have  $r = (3-1) - \text{rank}F = 1$ . This fact can be corroborated by computing the eigenvalues of the system; the eigenvalues are

$$\pm 1.7321i, -0.4203 \pm 0.3473i, -0.5797 \pm 2.5283i.$$

## 6. SOME GENERALIZATIONS TO ASYMMETRIC SYSTEMS

In this section, it is shown that theorems 2 and 3 can be generalized for a class of asymmetric systems (i. e., the symmetry restriction are not met by inertia, damping and stiffness matrices) commonly known as symmetrizable systems. Asymmetric coefficient matrices appear in problems involving follower forces, gyroscopy, aero-/hydro-elasticity and control effects, etc.

Assuming that the inertia matrix  $A$  is nonsingular, the equations of motion can be written as

$$\ddot{q} + A^{-1}B\dot{q} + A^{-1}Cq = 0 \quad (44)$$

The symmetrizable systems are defined in [12] as systems that have symmetrizable matrices  $A^{-1}B$  and  $A^{-1}C$ , i. e., such that factorizations  $A^{-1}B = S_1S_2$  and  $A^{-1}C = S_1S_3$  are permissible, where  $S_1$  is symmetric and positive definite, while  $S_2$  and  $S_3$  need only be symmetric. Additionally, it is supposed that  $A^{-1}B$  has nonnegative real eigenvalues and  $A^{-1}C$  has positive real eigenvalues. Then,  $S_2$  and  $S_3$  are positive semi-definite and positive definite, respectively, and the system described by (44) is stable [12]. Consequently, all eigenvalues of this system lie in the closed left-half of the complex plane.

Using the transformation  $q = S_1^{1/2}x$ , Eq. (44) is reduced to

$$\ddot{x} + D\dot{x} + Kx = 0, \quad (45)$$

where  $D = D^T = S_1^{1/2}S_2S_1^{1/2}$  and  $K = K^T = S_1^{1/2}S_3S_1^{1/2}$ . Since  $D = D^T \geq 0$  and  $K = K^T > 0$ , the results developed in the section 3 can be applied to Eq. (45). From the factorizations of  $A^{-1}B$  and  $A^{-1}C$ , we have

$$D = S_1^{-1/2}A^{-1}BS^{1/2} \quad (46)$$

and

$$K = S_1^{-1/2}A^{-1}CS^{1/2} \quad (47)$$

Substituting (46) and (47) into (16) results in

$$\Phi = S_1^{-1/2}\bar{\Phi}diag(S_1^{1/2}, \dots, S_1^{1/2}), \quad (48)$$

where

$$\bar{\Phi} = \left( A^{-1}B \mid A^{-1}CA^{-1}B \mid \dots \mid (A^{-1}C)^{n-1}A^{-1}B \right) \quad (49)$$

It is clear that  $rank\Phi = rank\bar{\Phi}$ . Thus, the following proposition is proved.

**Theorem 6.** The symmetrizable system described by Eq. (44), where  $A^{-1}B$  and  $A^{-1}C$  have non-negative and positive eigenvalues, respectively, has  $r = n - rank\bar{\Phi}$  conjugate pairs of purely imaginary eigenvalues.

**Remark 2.** It is clear that  $rank\bar{\Phi}$  is the same as rank of the matrix (25), because  $\tilde{\Phi} = A\bar{\Phi}$ .

Also, the following result can be easily established.

**Theorem 7.** Suppose that  $A^{-1}B$  and  $A^{-1}C$  have non-negative and positive eigenvalues, respectively. If  $A^{-1}B$  and  $A^{-1}C$  commute in multiplication, then the system (44) has  $r = n - \text{rank}(A^{-1}B)$  conjugate pairs of purely imaginary eigenvalues.

*Example 4.* Consider the asymmetric system described by

$$\ddot{q} + \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \dot{q} + \begin{pmatrix} 4 & 4 \\ 1 & 4 \end{pmatrix} q = 0 \quad (50)$$

Here note that

$$A^{-1}B = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1.2461 & -0.2769 \\ -0.2769 & 0.3115 \end{pmatrix} \begin{pmatrix} 2.8889 & 5.7778 \\ 5.7778 & 11.5556 \end{pmatrix}, \quad (51)$$

and

$$A^{-1}C = \begin{pmatrix} 4 & 4 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1.2461 & -0.2769 \\ -0.2769 & 0.3115 \end{pmatrix} \begin{pmatrix} 4.8897 & 7.5573 \\ 7.5573 & 19.5599 \end{pmatrix}, \quad (52)$$

so that the coefficient matrices have a common positive definite factor. On the other hand, the eigenvalues of  $A^{-1}B$  in this example are 0 and 4, and those of  $A^{-1}C$  are 2 and 6. Hence,  $A^{-1}B$  and  $A^{-1}C$  have nonnegative and positive real eigenvalues, respectively. Thus, the Theorem can be applied. The matrix (49) takes the form

$$\overline{\Phi} = \begin{pmatrix} 2 & 4 & 12 & 24 \\ 1 & 2 & 6 & 12 \end{pmatrix} \quad (53)$$

and, consequently,  $r = 2 - \text{rank}\overline{\Phi} = 1$ . This is in agreement with the eigenvalue calculation for the system, which yields  $\lambda_{1,2} = \pm i\sqrt{2}$  and  $\lambda_{3,4} = -2 \pm i\sqrt{2}$ .

Finally, observe that  $A^{-1}B$  and  $A^{-1}C$  commute for this example,  $\text{rank}(A^{-1}B) = 1$ , and, according to the Theorem 7,  $r = 1$ .

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## O REZIDUALNOM KRETANJU PRIGUŠENO OSCILUJUĆIH SISTEMA

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*U radu se razmatraju linearni oscilujućii sistemi čije su matrice inercije i krutosti simetrične i pozitivno definitne a matrica prigušenja pozitivno semidefinitna. Izvedeno je nekoliko formula za određivanje broja nezavisnih neprigušenih modova koji odgovaraju imaginarnim sopstvenim vrijednostima razmatranih sistema. Ovi rezultati su uopšteni na klasu simetrizabilnih asimetričnih sistema. Ispravnost i pogodnost dobijenih rezultata je ilustrovana na nekoliko primjera..*

Ključne riječi: *linearni system, prigušenje, rezidualno kretanje*