Theoret.Appl.Mech. TEOPM7,Vol.40, No.1, pp.17-26, Belgrade 2013*

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*doi:10.2298/TAM1301017D Math. Subj. Class.: 53A35; 53C22; 53C99.

According to: *Tib Journal Abbreviations (C) Mathematical Reviews*, the abbreviation TEOPM7 stands for TEORIJSKA I PRIMENJENA MEHANIKA.

THEORETICAL AND APPLIED MECHANICS
Series: **Special Issue - Address to Mechanics, Vol. 40(S1), pp.17-26, Belgrade 2012**

ON THE DETERMINATION OF SHIFTING OPERATORS ALONG GEODESICS ON A SURFACE

UDC 514.7, 517.9

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Abstract: *A procedure to obtain a closed form of the shifting operators along a known geodesic line on a surface as a solution of a system of linear algebraic equations is proposed. Its correctness is numerically demonstrated in the case of a helicoid surface and a spherical one. The future use of these operators in finite element approximations of tensor fields in non-Euclidean spaces is announced.*

Key words: *surface, geodesic line, parallel transport, shifting operators*

1. INTRODUCTION

It is well known that the system of differential equations for determining the components of a vector **v** parallelly propagated along a curve $u^\alpha = u^\alpha(s)$ on a surface reads[1]

$$\frac{Dv^\alpha}{Ds} = \frac{dv^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma}\, v^\beta \frac{du^\gamma}{ds} = 0 \quad , \tag{1}$$

where u^α are so-called surface coordinates, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the second kind determined for this surface, and s is the arc length of this curve. The fundamental system of solutions $K^{\cdot\alpha}_{\beta}$ of this homogeneous system of differential equations represents the operators of parallel transport (the shifting operators) with respect to the surface along this curve, establishing the relation

$$v^\alpha(P) = K^{\cdot\alpha}_{\beta}(P_o, P)\;\; v^\beta(P_o) \tag{2}$$

between the components of the vector **v** before and after its parallel transport from the point P_o to the point P. However, the existence of this fundamental system, i.e. the existence of shifting operators along the given curve, does not necessarily mean it is easy

[1] Einstein's summation convention for diagonally repeated indices is used; Greek indices have the range {1,2}, while Latin indices will have the range {1,2,3}.

to find them. Namely, *“the explicit form of the function … [$K_{\beta}^{\cdot\alpha}$] is not known”* ([5], p. 260) and even in the case of the geodesic lines on a spherical surface (its great circles) the shifting operators are obtained in [6] by using a *heuristic* procedure (and not by solving the corresponding homogeneous system of differential equations).

2. ALGEBRAIC APPROACH IN THE DETERMINATION OF SHIFTING OPERATORS

Nevertheless, it turned out quite unexpectedly that one can obtain a *closed* form of these operators along a *known* geodesic line on a surface as a solution of a system of linear *algebraic* equations using the fact that the tangent vector of a geodesic is a parallel vector field along this line, i.e. the fact that

$$\frac{du^{\alpha}}{ds} = K_{\beta}^{\cdot\alpha}(P_o, P)\left.\frac{du^{\beta}}{ds}\right|_{P_o} \tag{3}$$

and the insufficiencies of these *two* conditions for the determination of the *four* coefficients $K_{\beta}^{\cdot\alpha}$ is surpassed by introducing an additional vector also *parallelly* propagated along the geodesic line

$$\mathbf{w} = \mathbf{n} \times \mathbf{t} \tag{4}$$

or in the component form

$$w_i = \varepsilon_{ijk} n^j t^k \quad . \tag{5}$$

Namely, this vector – permanently orthogonal to the tangent vector $\mathbf{t}$ of a geodesic line on this surface – is always in the tangent plane of the surface; $\mathbf{n}$ is the normal to the surface and hence (s. [1], p. 214)

$$n_i = \frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{ijk} z_{\alpha}^{j} z_{\beta}^{k} \quad (\varepsilon_{\alpha\beta} = \sqrt{a}\, e_{\alpha\beta}\ ,\ \varepsilon^{\alpha\beta} = e^{\alpha\beta}/\sqrt{a}\ ,\ a \equiv \left|a_{\alpha\beta}\right|\ ) \tag{6}$$

where z are the rectangular Cartesian coordinates, $z_{\alpha}^{i} \equiv \partial z^i / \partial u^{\alpha}$ and $\varepsilon_{ijk} = e_{ijk}$; for the tangent vector $\mathbf{t}$ we have

$$t^i = z_{\alpha}^{i}\, du^{\alpha}/ds \quad ; \tag{7}$$

the surface components of the vector $\mathbf{w}$ (lying in the surface tangent plane) are

$$w_{\alpha} = z_{\alpha}^{i} w_i \quad ; \tag{8}$$

in the case of the *orthogonal* coordinates u^{α} we have $w^{\alpha} = a^{\alpha\beta} w_{\beta}$ and finally (9)

$$w^{\alpha} = a^{\alpha\beta} z_{\beta}^{i} w_i = a^{\alpha\beta} z_{\beta}^{i}\varepsilon_{ijk} n^j t^k = a^{\alpha\beta} z_{\beta}^{i}\varepsilon_{ijk}\delta^{j\ell} n_{\ell} z_{\gamma}^{k}\frac{du^{\gamma}}{ds} = \frac{1}{2}a^{\alpha\beta} z_{\beta}^{i}\varepsilon_{ijk}\delta^{j\ell}\varepsilon^{\mu\nu}\varepsilon_{\ell mn} z_{\mu}^{m} z_{\nu}^{n} z_{\gamma}^{k}\frac{du^{\gamma}}{ds}.$$

Due to the parallel transport of the vector $\mathbf{w}$ along the geodesic line we have

$$w_o^{\beta} K_{\beta}^{\cdot\alpha}(P_o, P) = w_P^{\alpha} \tag{10}$$

and the coefficients $K_{\beta}^{\cdot\alpha}$ can now be determined from (3) and (10)

$$
\left.
\begin{aligned}
K_{1}^{\cdot 1}(P_o,P) &= \frac{1}{\det}\begin{vmatrix} \left.\dfrac{du^1}{ds}\right|_{P} & \left.\dfrac{du^2}{ds}\right|_{P_o} \\ w_P^1 & w_o^2 \end{vmatrix} \qquad
K_{2}^{\cdot 1}(P_o,P) = \frac{1}{\det}\begin{vmatrix} \left.\dfrac{du^1}{ds}\right|_{P_o} & \left.\dfrac{du^1}{ds}\right|_{P} \\ w_o^1 & w_P^1 \end{vmatrix} \\
K_{1}^{\cdot 2}(P_o,P) &= \frac{1}{\det}\begin{vmatrix} \left.\dfrac{du^2}{ds}\right|_{P} & \left.\dfrac{du^2}{ds}\right|_{P_o} \\ w_P^2 & w_o^2 \end{vmatrix} \qquad
K_{2}^{\cdot 2}(P_o,P) = \frac{1}{\det}\begin{vmatrix} \left.\dfrac{du^1}{ds}\right|_{P_o} & \left.\dfrac{du^2}{ds}\right|_{P} \\ w_o^1 & w_P^2 \end{vmatrix}
\end{aligned}
\right\} , \qquad (11)
$$

where

$$
\det = \begin{vmatrix} \left.\dfrac{du^1}{ds}\right|_{P_o} & \left.\dfrac{du^2}{ds}\right|_{P_o} \\ w_o^1 & w_o^2 \end{vmatrix} . \qquad (12)
$$

These expressions are implemented in the corresponding software tool in order to compare some numerical results with the formerly checked ones for the shifting operators on the spherical surface and exceptional coincidence is obtained!

However – bearing in mind that the relations (9) are not the promising ones concerning the determination of the explicit expressions for the shifting operators and that the *surface* components of the vector $\mathbf{w}$ are present in (11) – we can proceed directly, considering $\mathbf{w}$ as a vector in the tangent plane of the surface orthogonal to the tangent vector of a geodesic line on this surface; namely

$$
w^1 = -\frac{\sqrt{a_{22}}}{\sqrt{a_{11}}}\frac{du^2}{ds} \quad , \quad w^2 = \frac{\sqrt{a_{11}}}{\sqrt{a_{22}}}\frac{du^1}{ds} \quad ; \qquad (13)
$$

hence for the coefficients $K_{\beta}^{\cdot\alpha}$ we obtain the expressions

$$
\left.
\begin{aligned}
K_{1}^{\cdot 1}(P_o,P) &= \sqrt{a_o}\left(\frac{\sqrt{a_{11}^{o}}}{\sqrt{a_{22}^{o}}} \left.\frac{du^1}{ds}\right|_{P} \left.\frac{du^1}{ds}\right|_{P_o} + \frac{\sqrt{a_{22}^{P}}}{\sqrt{a_{11}^{P}}} \left.\frac{du^2}{ds}\right|_{P} \left.\frac{du^2}{ds}\right|_{P_o} \right) \\
K_{2}^{\cdot 1}(P_o,P) &= -\sqrt{a_o}\left(\frac{\sqrt{a_{22}^{P}}}{\sqrt{a_{11}^{P}}} \left.\frac{du^2}{ds}\right|_{P} \left.\frac{du^1}{ds}\right|_{P_o} - \frac{\sqrt{a_{22}^{o}}}{\sqrt{a_{11}^{o}}} \left.\frac{du^1}{ds}\right|_{P} \left.\frac{du^2}{ds}\right|_{P_o} \right) \\
K_{1}^{\cdot 2}(P_o,P) &= \sqrt{a_o}\left(\frac{\sqrt{a_{11}^{o}}}{\sqrt{a_{22}^{o}}} \left.\frac{du^2}{ds}\right|_{P} \left.\frac{du^1}{ds}\right|_{P_o} - \frac{\sqrt{a_{11}^{P}}}{\sqrt{a_{22}^{P}}} \left.\frac{du^1}{ds}\right|_{P} \left.\frac{du^2}{ds}\right|_{P_o} \right) \\
K_{2}^{\cdot 2}(P_o,P) &= \sqrt{a_o}\left(\frac{\sqrt{a_{11}^{P}}}{\sqrt{a_{22}^{P}}} \left.\frac{du^1}{ds}\right|_{P} \left.\frac{du^1}{ds}\right|_{P_o} + \frac{\sqrt{a_{22}^{o}}}{\sqrt{a_{11}^{o}}} \left.\frac{du^2}{ds}\right|_{P} \left.\frac{du^2}{ds}\right|_{P_o} \right)
\end{aligned}
\right\} \quad \left(a_o \equiv \left|a_{\alpha\beta}\right|\Big|_{P_o} \right) \qquad (14)
$$

and in every single case one can try to find the explicit expressions for the components of the operators of parallel transport along the known geodesic line on the surface under consideration.

3. EXAMPLES

3.1. Operators of parallel transport along geodesics on a spherical surface

In order to obtain the effective expressions for these operators, we shall use the finite equation of the geodesic line (the great circle) on a spherical surface (with the radius $r \neq 0$) in the form (s. for example [4], p. 167)

$$\mathrm{tg}\,\vartheta = A\cos\varphi + B\sin\varphi \quad , \tag{15}$$

where $\{\varphi, \vartheta\}$ are the geographical coordinates ($u^1 \equiv \varphi$, $u^2 \equiv \vartheta$) and the constants A and B can be obtained from the condition of passing through the points $P_o(\varphi_o, \vartheta_o)$ and $P(\varphi_P, \vartheta_P)$

$$\left.\begin{aligned} \mathrm{tg}\,\vartheta_o &= A\cos\varphi_o + B\sin\varphi_o \\ \mathrm{tg}\,\vartheta_P &= A\cos\varphi_P + B\sin\varphi_P \end{aligned}\right\} \quad ; \tag{16}$$

hence it follows

$$\left.\begin{aligned} A = A(\varphi_o, \varphi_P, \vartheta_o, \vartheta_P) &= \frac{\sin\varphi_P \tan\vartheta_o - \tan\vartheta_P \sin\varphi_o}{\sin(\varphi_P - \varphi_o)} \\ B = B(\varphi_o, \varphi_P, \vartheta_o, \vartheta_P) &= \frac{\tan\vartheta_P \cos\varphi_o - \cos\varphi_P \tan\vartheta_o}{\sin(\varphi_P - \varphi_o)} \end{aligned}\right\} \quad . \tag{17}$$

Knowing that the components of the fundamental metric tensor in the system $\{\varphi, \vartheta\}$ are

$$a_{11} = a_{\varphi\varphi} = r^2\cos^2\vartheta \quad , \quad a_{12} = a_{21} = a_{\varphi\vartheta} = a_{\vartheta\varphi} = 0 \quad , \quad a_{22} = a_{\vartheta\vartheta} = r^2 \tag{18}$$

we have

$$ds^2 = a_{\alpha\beta}\,du^\alpha du^\beta = r^2\left(\cos^2\vartheta\, d\varphi^2 + d\vartheta^2\right) \tag{19}$$

and, bearing in mind the relation (15), we obtain

$$\left.\begin{aligned} \frac{d\varphi}{ds} &= \pm\frac{1}{r\cos\vartheta\sqrt{1+\cos^2\vartheta\left(A\sin\varphi - B\cos\varphi\right)^2}} \\ \frac{d\vartheta}{ds} &= \mp\frac{\cos\vartheta\left(A\sin\varphi - B\cos\varphi\right)}{r\sqrt{1+\cos^2\vartheta\left(A\sin\varphi - B\cos\varphi\right)^2}} \end{aligned}\right\} \quad . \tag{20}$$

Using (17), we have for example

$$\left.\begin{aligned} \left.\frac{d\varphi}{ds}\right|_o &= \frac{\sin(\varphi_P - \varphi_o)}{r\cos\vartheta_o\sqrt{\sin^2(\varphi_P - \varphi_o) + \cos^2\vartheta_o\left[\cos(\varphi_P - \varphi_o)\tan\vartheta_o - \tan\vartheta_P\right]^2}} \\ \left.\frac{d\vartheta}{ds}\right|_o &= -\frac{\cos\vartheta_o\left[\cos(\varphi_P - \varphi_o)\tan\vartheta_o - \tan\vartheta_P\right]}{r\sqrt{\sin^2(\varphi_P - \varphi_o) + \cos^2\vartheta_o\left[\cos(\varphi_P - \varphi_o)\tan\vartheta_o - \tan\vartheta_P\right]^2}} \end{aligned}\right\} \tag{21}$$

and similarly

$$\left.\begin{aligned}
\left.\frac{d\varphi}{ds}\right|_P &= \frac{\sin\left(\varphi_P-\varphi_o\right)}{r\cos\vartheta_P\sqrt{\sin^2\left(\varphi_P-\varphi_o\right)+\cos^2\vartheta_P\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_P-\tan\vartheta_o\right]^2}}\\
\left.\frac{d\vartheta}{ds}\right|_P &= \frac{\cos\vartheta_P\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_P-\tan\vartheta_o\right]}{r\sqrt{\sin^2\left(\varphi_P-\varphi_o\right)+\cos^2\vartheta_P\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_P-\tan\vartheta_o\right]^2}}
\end{aligned}\right\} \quad . \tag{22}$$

Finally – substituting (21) and (22) in (14) – we obtain the following explicit expressions, in the geographical coordinates $\{\varphi,\vartheta\}$, for the operators of parallel transport with respect to a spherical surface along the geodesic line connecting $P_o(\varphi_o,\vartheta_o)$ and $P(\varphi_P,\vartheta_P)$

$$\left.\begin{aligned}
K_1^{\cdot 1}(P_o,P) = {} & \frac{1}{S}\left\{\frac{\cos\vartheta_o}{\cos\vartheta_P}\sin^2\left(\varphi_P-\varphi_o\right)-\right.\\
& \left.-\cos^2\vartheta_o\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_P-\tan\vartheta_o\right]\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_o-\tan\vartheta_P\right]\right\}\\
K_2^{\cdot 1}(P_o,P) = {} & -\frac{1}{S}\sin\left(\varphi_P-\varphi_o\right)\left\{\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_P-\tan\vartheta_o\right]+\right.\\
& \left.+\frac{\cos\vartheta_o}{\cos\vartheta_P}\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_o-\tan\vartheta_P\right]\right\}\\
K_1^{\cdot 2}(P_o,P) = {} & \frac{1}{S}\sin\left(\varphi_P-\varphi_o\right)\left\{\cos\vartheta_P\cos\vartheta_o\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_P-\tan\vartheta_o\right]+\right.\\
& \left.+\cos^2\vartheta_o\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_o-\tan\vartheta_P\right]\right\}\\
K_2^{\cdot 2}(P_o,P) = {} & \frac{1}{S}\left\{\sin^2\left(\varphi_P-\varphi_o\right)-\right.\\
& \left.-\cos\vartheta_P\cos\vartheta_o\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_P-\tan\vartheta_o\right]\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_o-\tan\vartheta_P\right]\right\}
\end{aligned}\right\} \tag{23}$$

where

$$\begin{aligned}
S \equiv {} & \sqrt{\sin^2\left(\varphi_P-\varphi_o\right)+\cos^2\vartheta_o\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_o-\tan\vartheta_P\right]^2}\ \times\\
\times & \sqrt{\sin^2\left(\varphi_P-\varphi_o\right)+\cos^2\vartheta_P\left[\cos\left(\varphi_P-\varphi_o\right)\tan\vartheta_P-\tan\vartheta_o\right]^2} \quad .
\end{aligned} \tag{24}$$

These expressions, in comparison with the ones in **Appendix**, have considerably simpler form. Concerning the correctness of (23), as well of the expressions (11) and (14), the accordance of the four groups of results (quoted in **Table 1.**) for an arbitrarily selected pair of points on the spherical surface represents a *numerical* confirmation of the usefulness of the previously obtained expressions for shifting operators.

3.2. Operators of parallel transport along geodesics on a helicoid surface

In the case of the helicoid surface

$$\left.\begin{aligned}
z^1 &= \rho\cos\varphi\\
z^2 &= \rho\sin\varphi\\
z^3 &= b\varphi
\end{aligned}\right\} \quad (b = const) \tag{25}$$

the components of the fundamental metric tensor in the system $\{\rho,\varphi\}$ ($u^1 \equiv \rho$, $u^2 \equiv \varphi$) are

$$a = a = 1 \ , \ a = a = a = a = 0 \ , \ a = a = \rho + b \tag{26}$$

and we have

$$ds^2 = a_{\alpha\beta} du^\alpha du^\beta = d\rho^2 + \left(\rho^2 + b^2\right) d\varphi^2 \quad . \tag{27}$$

On the other side, the equation of the geodesic line on this surface can be found in the form (s. for example [3], p. 45)

$$\varphi = C \pm \int \frac{D\, d\rho}{\sqrt{\left(\rho^2 + b^2\right)\left(\rho^2 + b^2 - D^2\right)}} \tag{28}$$

and, bearing in mind (27), we obtain

$$\left.\begin{aligned} \frac{d\rho}{ds} &= \pm\sqrt{\frac{\rho^2 + b^2 - D^2}{\rho^2 + b^2}} \\ \frac{d\varphi}{ds} &= \pm\frac{D}{\rho^2 + b^2} \end{aligned}\right\} \quad . \tag{29}$$

Rewriting (28) in the form

$$\varphi = C \pm \int_{\rho_o}^{\rho} \frac{D\, dr}{\sqrt{\left(r^2 + b^2\right)\left(r^2 + b^2 - D^2\right)}} \quad , \tag{30}$$

we find $C = \varphi_o$ (from the condition that $\varphi = \varphi_o$ when $\rho = \rho_o$), while the constant D should be determined from the condition that

$$\varphi_P = \varphi_o \pm \int_{\rho_o}^{\rho_P} \frac{D\, dr}{\sqrt{\left(r^2 + b^2\right)\left(r^2 + b^2 - D^2\right)}} \quad . \tag{31}$$

Due to the monotony of the subintegral function in (31), it is relatively simple to obtain a sufficiently exact value of the constant D as a numerical solution of this equation. With such approximative value for D, the evaluation of the components of the operators of parallel transport along the geodesic line connecting the points $P_o(\rho_o, \varphi_o)$ and $P(\rho_P, \varphi_P)$ on a helicoid surface can be performed according to (14), using the expressions (26) and (29).

In this case, in order to examine the correctness of the whole proposed procedure, the numerical comparison is made between two approaches: the above described one using shifting operators and the one without these operators. In the first case the contravariant components of a vector **v** shifted on this surface from P_o to P is calculated according to the formula

$$v^\alpha(P) = K_{\beta}^{\cdot\alpha}(P_o, P)\, v^\beta(P_o) \tag{32}$$

(where $v^1 \equiv v^\rho$, $v^2 \equiv v^\varphi$) and the Cartesian components of this vector at the point P are determined in the usual way

$$v^i(P) = \left.\frac{\partial z^i}{\partial \rho}\right|_P v^\rho(P) \; + \; \left.\frac{\partial z^i}{\partial \varphi}\right|_P v^\varphi(P) \tag{33}$$

(but now $v^1 \equiv v^x \equiv v^{z^1}$, $v^2 \equiv v^y \equiv v^{z^2}$, $v^3 \equiv v^z \equiv v^{z^3}$). In the second case, the Cartesian components of the vector **v** are obtained directly (without introducing the notion of the operator of parallel transport with respect to a surface) from the condition that a vector shifted along a geodesic line must close a constant angle with this curve at each of its points (s. p. 143 in [2]). The results for an arbitrarily selected pair of points on the helicoid surface (with $b = h/2\pi$ and $h = 5$) are quoted in **Table 2**. and the accordance is evident.

4. Concluding remarks and future activities

The relatively simple and numerically efficient way to obtain the values of components of the operators of parallel transport along a *known* geodesic line passing through two arbitrarily selected points on a surface is described. Although this procedure – based on a solution of a system of linear algebraic equations – can be used to obtain the explicit *analytical* expressions for the shifting operators in some cases, the main benefit is a possibility of its use in the future *numerical* testing of an approach in finite element approximations of tensor fields in non-Euclidean spaces proposed in [7]. Namely, instead of the usual approximation of components of tensor fields, the approximation of the whole field (as a kernel) is performed and the operators of parallel transport play the fundamental role in such approach.

Acknowledgement: *This contribution was prepared to be communicated at the symposium ADDRESS TO MECHANICS on the occasion of the 80th birthday of Professor Veljko A. Vujičić (Mathematical Institute of Serbian Academy of Sciences and Arts, Belgrade), academician of the International Academy of Nonlinear Sciences.*

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Table 1. Geographical components of the shifting operators along the geodesic line connecting the points *Po* and *P* on a spherical surface

$P_o(\varphi_o,\vartheta_o)=P_o(7^o,77^o)$ $P(\varphi_P,\vartheta_P)=P(13^o,78^o)$	$\left\{K_{\beta}^{\cdot\alpha}(P_o,P)\right\}=\begin{Bmatrix} K_1^{\cdot 1}(P_o,P) & K_2^{\cdot 1}(P_o,P) \\ K_1^{\cdot 2}(P_o,P) & K_2^{\cdot 2}(P_o,P)\end{Bmatrix}$
heuristic approach	$\begin{Bmatrix} 1.07630424872861 & 0.490918399454177 \\ -2.296023426624826E-2 & 0.994777456654368 \end{Bmatrix}$
algebraic approach	$\begin{Bmatrix} 1.07630424872861 & 0.490918399454179 \\ -2.296023426624818E-2 & 0.994777456654367 \end{Bmatrix}$
algebraic approach	$\begin{Bmatrix} 1.07630424872861 & 0.490918399454177 \\ -2.296023426624829E-2 & 0.994777456654367 \end{Bmatrix}$
algebraic approach	$\begin{Bmatrix} 1.07630424872861 & 0.490918399454182 \\ -2.296023426624850E-2 & 0.994777456654367 \end{Bmatrix}$

Table 2. Cartesian components of a given vector after parallel transport from the point *Po* to the point P along the geodesic line on a helicoid surface

$P_o(\rho_o,\varphi_o)$ $\parallel$ $P_o(1,85^o)$	$P\,(\rho_P,\varphi_P)$ $\parallel$ $P\,(4,135^o)$	***approach without shifters***	***approach with shifters***
$\mathbf{v}_o=\begin{Bmatrix} v_o^{\langle\rho\rangle} \\ v_o^{\langle\varphi\rangle}\end{Bmatrix}=\begin{Bmatrix}1\\0\end{Bmatrix}$	$\mathbf{v}_P=\begin{Bmatrix} v_P^{z^1} \\ v_P^{z^2} \\ v_P^{z^3}\end{Bmatrix}=$	$\begin{Bmatrix} -2.184434556431047E-2 \\ 0.990432849244363 \\ -0.136255626322580 \end{Bmatrix}$	$\begin{Bmatrix} -2.184434556431003E-2 \\ 0.990432849244364 \\ -0.136255626322580 \end{Bmatrix}$

O ODREĐIVANJU OPERATORA PARALELNOG POMERANJA DUŽ GEODEZIJSKIH LINIJA NA POVRŠIMA

Zoran Drašković

Apstrakt: *Predložen je postupak za dobijanje zatvorenog oblika operatora paralelnog pomeranja duž poznate geodezijske linije na nekoj površi kao rešenja sistema linearnih algebarskih jednačina. Njegova korektnost numerički je pokazana na primeru sferne i helikoidalne površi. Nagoveštena je* *budućа upotreba tih operatora u aproksimacijama konačnim elementima tenzorskih polja u neeuklidskim prostorima.*

Ključne reči: *površ, geodezijska linija, paralelno pomeranje, operatori paralelnog pomeranja*

Appendix: Operators of parallel transport along geodesics on a spherical surface (heuristic approach)

The explicit expressions – obtained in [6] by using a heuristic procedure – for the operators of parallel transport with respect to a spherical surface along the geodesic line (the great circle) connecting $P_o(\varphi_o, \vartheta_o)$ and $P(\varphi_P, \vartheta_P)$ read (geographical coordinates are in question!)

$$\begin{aligned}
K_{1}^{\cdot 1}(P_o,P) = {} & \frac{\cos\vartheta_o}{\cos\vartheta_P}\{[\sin\bar{\varphi}_P \sin(\varphi_P-\psi_{Eu})+\cos\bar{\varphi}_P\cos(\varphi_P-\psi_{Eu})\cos\vartheta_{Eu}]\times \\
& \times[\sin\bar{\varphi}_o\sin(\varphi_o-\psi_{Eu})+\cos\bar{\varphi}_o\cos(\varphi_o-\psi_{Eu})\cos\vartheta_{Eu}]+ \\
& +\cos(\varphi_P-\psi_{Eu})\cos(\varphi_o-\psi_{Eu})\sin^2\vartheta_{Eu}\}
\end{aligned}$$

$$\begin{aligned}
K_{2}^{\cdot 1}(P_o,P) = {} & \frac{1}{\cos\vartheta_P}\{[\sin\bar{\varphi}_P\sin(\varphi_P-\psi_{Eu})+\cos\bar{\varphi}_P\cos(\varphi_P-\psi_{Eu})\cos\vartheta_{Eu}]\times \\
& \times\{\sin\vartheta_o[\sin\bar{\varphi}_o\cos(\varphi_o-\psi_{Eu})-\cos\bar{\varphi}_o\sin(\varphi_o-\psi_{Eu})\cos\vartheta_{Eu}]+ \\
& +\cos\vartheta_o\sin\vartheta_{Eu}\cos\bar{\varphi}_o\}- \\
& -\cos(\varphi_P-\psi_{Eu})\sin\vartheta_{Eu}[\sin\vartheta_o\sin(\varphi_o-\psi_{Eu})\sin\vartheta_{Eu}+\cos\vartheta_o\cos\vartheta_{Eu}]\}
\end{aligned}$$

$$\begin{aligned}
K_{1}^{\cdot 2}(P_o,P) = {} & \cos\vartheta_o\{\{\sin\vartheta_P[\sin\bar{\varphi}_P\cos(\varphi_P-\psi_{Eu})-\cos\bar{\varphi}_P\sin(\varphi_P-\psi_{Eu})\cos\vartheta_{Eu}]+ \\
& +\cos\vartheta_P\sin\vartheta_{Eu}\cos\bar{\varphi}_P\}\times \\
& \times[\sin\bar{\varphi}_o\sin(\varphi_o-\psi_{Eu})+\cos\bar{\varphi}_o\cos(\varphi_o-\psi_{Eu})\cos\vartheta_{Eu}]- \\
& -\cos(\varphi_o-\psi_{Eu})\sin\vartheta_{Eu}[\sin\vartheta_P\sin(\varphi_P-\psi_{Eu})\sin\vartheta_{Eu}+\cos\vartheta_P\cos\vartheta_{Eu}]\}
\end{aligned}$$

$$\begin{aligned}
K_{2}^{\cdot 2}(P_o,P) = {} & \{\sin\vartheta_P[\sin\bar{\varphi}_P\cos(\varphi_P-\psi_{Eu})-\cos\bar{\varphi}_P\sin(\varphi_P-\psi_{Eu})\cos\vartheta_{Eu}]+ \\
& +\cos\vartheta_P\sin\vartheta_{Eu}\cos\bar{\varphi}_P\}\times \\
& \times\{\sin\vartheta_o[\sin\bar{\varphi}_o\cos(\varphi_o-\psi_{Eu})-\cos\bar{\varphi}_o\sin(\varphi_o-\psi_{Eu})\cos\vartheta_{Eu}]+ \\
& +\cos\vartheta_o\sin\vartheta_{Eu}\cos\bar{\varphi}_o\}+ \\
& +[\sin\vartheta_P\sin(\varphi_P-\psi_{Eu})\sin\vartheta_{Eu}+\cos\vartheta_P\cos\vartheta_{Eu}]\times \\
& \times[\sin\vartheta_o\sin(\varphi_o-\psi_{Eu})\sin\vartheta_{Eu}+\cos\vartheta_o\cos\vartheta_{Eu}]
\end{aligned}$$

where

$$\cos\bar{\varphi}_o = \cos\vartheta_o\cos(\varphi_o-\psi_{Eu}) \quad , \quad \cos\bar{\varphi}_P = \cos\vartheta_P\cos(\varphi_P-\psi_{Eu})$$

$$\operatorname{tg}\psi_{Eu} = \frac{\sin\varphi_o\cos\vartheta_o\sin\vartheta_P - \sin\vartheta_o\sin\varphi_P\cos\vartheta_P}{\cos\varphi_o\cos\vartheta_o\sin\vartheta_P - \sin\vartheta_o\cos\varphi_P\cos\vartheta_P}$$

$$\cos\vartheta_{Eu} = \frac{\cos\varphi_o\cos\vartheta_o\sin\varphi_P\cos\vartheta_P - \sin\varphi_o\cos\vartheta_o\cos\varphi_P\cos\vartheta_P}{\sqrt{\begin{aligned}&(\sin\varphi_o\cos\vartheta_o\sin\vartheta_P - \sin\vartheta_o\sin\varphi_P\cos\vartheta_P)^2 + \\ &+(\sin\vartheta_o\cos\varphi_P\cos\vartheta_P - \cos\varphi_o\cos\vartheta_o\sin\vartheta_P)^2 + \\ &+(\cos\varphi_o\cos\vartheta_o\sin\varphi_P\cos\vartheta_P - \sin\varphi_o\cos\vartheta_o\cos\varphi_P\cos\vartheta_P)^2\end{aligned}}}$$

and ψ_{Eu} and ϑ_{Eu} are the Euler angles: the precession ψ_{Eu} is the angle of inclination of the line which represents the intersection of the plane OP_oP and the coordinate plane Oz^1z^2, while the nutation ϑ_{Eu} is the angle between the normals to the planes Oz^1z^2 and OP_oP (the angle of proper rotation is $\varphi_{Eu} = 0$).

Submitted on May 2009, accepted on June 2012.

DOI : 10.2298/TAM12S117D Math.Subj.Class.: 53A35; 53C22; 53C99;