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# NINETY YEARS OF DUFFING'S EQUATION

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**4bstract**. *In the paper the origin of the so named 'Duffing's equation' is shown. The author's generalization of the equation, her published papers dealing with Duffing's equation and some of the solution methods are presented. Three characteristic approximate solution procedures based on the exact solution of the strong cubic Duffing's equation are shown. Using the Jacobi elliptic functions the elliptic-Krylov-Bogolubov (EKB), the homotopy perturbation and the elliptic-Galerkin (EG) methods are described. The methods are compared. The advantages and the disadvantages of the methods are discussed.*

**Key words**: *Duffing's equation, elliptic-Krylov-Bogolubov method, homotopy perturbation method, elliptic-Galerkin method*

## 1. INTRODUCTION

In 1918, in the Edition *Vieweg*, No.41/42 the publication entitled "*Erzwungene Schwingungen bei veranderlicher Eigenfrequenz und ihre technische Bedeuting*" by Georg Duffing (Fig.1), Ingenieur, appears. The first sentence in the Preface of the book [1] is: "Die Anregung zu der vorliegenden Studie wurde mir zunachst durch Beobachtungen an Maschinen gegeben". This statement proves the appropriation of Georg Duffing to experimental-applied dynamics. He was a serious experimentalist who studied mechanical devices to discover geometric properties of dynamical systems [2].
The theory of oscillations was his explicit goal. In *Jahrbuch der Mathematik (1916-1918)*, (see [1]), a reviewer G.H. wrote that the aim of the paper [2] was to clarify the resonant oscillations which are evident in the pendulum (Fig.2) whose motion is given with a differential equation

$$\frac{d^2y}{dt^2}+\gamma^2(\sin y-\sin y_0)+\beta^2(y-y_0)=k\sin\omega t \tag{3}$$

where $y$ is the pendulum displacement, $t$ is time, $\beta^2$ and $\gamma^2$ are positive constants, $y_0$ is the initial deflection, $k$ and $\omega$ are the amplitude and the frequency of the excitation force. Duffing simplified the equation into

$$\frac{d^2y}{dt^2} + a^2 y - \beta y^2 - \gamma y^3 = k \sin \omega t \tag{2}$$

and calculated the first term $H \sin \omega t$ of the periodic solution in the first approximation [3]. He obtained a cubic algebraic equation for $H$ which has three solutions: two stable and an unstable one.

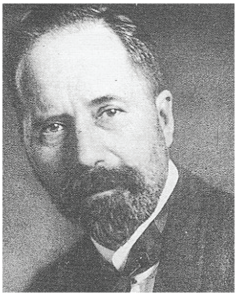

Fig.1. Georg Duffing, 1861-1944.

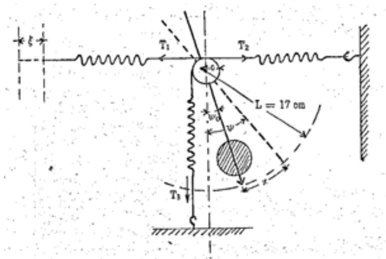


Fig.2. Duffing's oscillator

Duffing gives the problem to mathematicians to give the initial conditions for the unstable motion. Besides, Duffing considered the simplified versions of the Eq. (2) for describing the motion of the symmetrical pendulum

$$\frac{d^2y}{dt^2} + \alpha y - \gamma y^3 = 0 \tag{3}$$

and the unsymmetrical pendulum

$$\frac{d^2y}{dt^2} + \alpha y - \beta y^2 = 0. \tag{4}$$

For the case of small non-linearities ($\gamma \ll \alpha$ and $\beta \ll \alpha$), Duffing gave the approximate solutions in the form of Weierstrass $\wp(t)$ elliptic function [4]. The main disadvantage of the solutions was their complexity and unsuitability for practical use.

During the last years the Eq. (2) is modified and some generalizations are introduced. Usually, the differential equations with polynomial type of non-linearity are called 'Duffing's equation'. The most often investigated type of the Duffing's equation is with the cubic non-linearity

$$\frac{d^2y}{dt^2} + 2\delta \frac{dy}{dt} \pm \alpha y \pm \gamma y^3 = k \sin \omega t \tag{5}$$

where $\delta$ is the damping coefficient.

About 2000 papers are published dealing with qualitative and quantitative analysis of Eq. (5). Two approaches are assumed: one, based on assumption that the non-linearity is small ($\alpha$>0, $\gamma$<<$\alpha$) and the other, the non-linearity is strong ($\gamma\approx\alpha$). Various analytical approximate solving procedures are developed. For the small non-linearity the most widely applied methods are: the method of multiple scales [5], the Bogolubov-Mitropolski [6], the Krylov-Bogolubov method [7], the straightforward expansion [8], Linstedt-Poincare method [9], etc. The author of this review modified the suggested methods for solving a second order differential equation with slow time variable parameters [10], a system of two coupled differential equations with constant coefficients [11]-[15] and slow time variable functions [16]-[21]. For all of these methods it is common that they represent the perturbation to the linear one and the difference between the approximate solution of the non-linear system and the linear one is of small order.

For the case of strong cubic non-linearity, the analytical approximate solution of (5) is based on the exact solution of the differential equation

$$\frac{d^2y}{dt^2} \pm \alpha y \pm \gamma y^3 = 0 \tag{6}$$

The author of this paper developed the approximate analytical solving methods [22] for

$$\frac{d^2y}{dt^2} \pm \alpha y \pm \gamma y^3 = f(y, \frac{dy}{dt}) \tag{7}$$

where $f$ is an additional linear or non-linear function which need not be small, and also for the system of coupled Duffing's equations [23]-[31]. The strong non-linear differential equations with slow time variable parameters are also considered [32]. The chaotic motion in the strong coupled system with constant and changeable parameters is investigated in [33]-[36]. The special cases of differential equations are those with pure non-linear term (see [37]-[40]). In the paper [40] the general form of the pure non-linear differential equation of Duffing's type is introduced.

In spite of the fact that a numerous methods are developed for analytic solving of the strong non-linear differential equations, the asymptotic approaches still need to be considered. Namely, all of the suggested asymptotic solving procedures have beside their advantages also some disadvantages. All the methods can be grouped as: residual methods, perturbation techniques and homotopic methods. In this paper the elliptic-Galerkin method which is the conceptually simplest analytic approximate procedure, the perturbation elliptic-Krylov-Bogolubov method, and the homotopy perturbation method which is adopted for solving of the Duffing's equation, will be shown.

## 2. DIFFERENTIAL EQUATION WITH STRONG CUBIC NON-LINEARITY

The Eq. (6) with initial conditions

$$y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0 \tag{8}$$

has an exact analytic solution in the form

$$y = Yep(\omega t + \theta, k^2), \tag{9}$$

where *ep* denotes a convenient Jacobi elliptic function [41], $\omega(Y)$ is the frequency, $k^2(Y)$ is the modulus of the elliptic function and $Y$ and $\theta$, i.e., the amplitude and phase angle are arbitrary constants dependent on the initial conditions (8).

Dependently on the sign of the coefficients $\alpha$ and $\gamma$ the following type of equations are evident: 1) the hard one: $\alpha>0$ and $\gamma>0$, 2) the hard-soft one: $\alpha>0$ and $\gamma<0$, 3) soft-hard one: $\alpha<0$ and $\gamma>0$.

For the case of strong cubic non-linearity of hardening type, the differential equation

$$\ddot{y}+\alpha y+\gamma y^3=0 \tag{10}$$

has an exact analytical solution in the form of the Jacobi elliptic function

$$y=Ycn(\omega t+\theta,k^2) \tag{11}$$

where *cn* is the Jacobi elliptic function [48], $\omega$ and $k$ are the frequency and modulus of the function

$$\omega^2=\alpha+\gamma Y^2,\quad k^2=\frac{\gamma Y^2}{2(\alpha+\gamma Y^2)} \tag{12}$$

and $Y$ and $\theta$ are arbitrary constants dependent on the initial conditions (8). Substituting (11) and its time derivative into (8), we obtain the amplitude $Y$ and the phase angle $\theta$ according to the relations

$$\frac{\gamma}{2}Y^4+\alpha Y^2-\left[\alpha y_0^2+\frac{\gamma}{2}y_0^4+\dot{y}_0^2\right]=0 \tag{13}$$

and

$$sc(\theta,k^2)dn(\theta,k^2)=-\frac{\dot{y}_0}{y_0\omega} \tag{14}$$

For the special initial conditions

$$y(0)=y_0,\quad \dot{y}(0)=0 \tag{15}$$

the amplitude and phase angle are

$$Y=y_0,\quad \theta=0 \tag{16}$$

and for

$$y(0)=0,\quad \dot{y}(0)=\dot{y}_0 \tag{17}$$

it yields

$$Y=\left[-\frac{\alpha}{\gamma}+\frac{\alpha}{\gamma}\sqrt{1+\frac{2\dot{y}_0^2\gamma}{\alpha^2}}\right]^{1/2},\quad \theta=K(k^2) \tag{18}$$

where $K(k^2)$ is the complete elliptic integral of the first kind [41]. Assuming the series expansion of the square root the approximate amplitude is $Y\approx\dot{y}_0$.

Using the aforementioned procedure Yuste and Bejarano [42] give the solutions for the hard-soft and soft-hard systems.

Table 1 Solutions for the hard-soft and soft-hard systems.

| Type | Solution | Frequency | Modulus |
|---|---|---|---|
| α>0,γ<0 | $y^0 = Ysn(\omega t + \theta, k^2)$ | $\omega^2 = \alpha - \frac{\gamma Y^2}{2} > 0$ | $k^2 = \frac{\gamma Y^2}{2\omega^2}$ |
| α<0,γ>0 | $y^0 = Ycn(\omega t + \theta, k^2)$ | $\omega^2 = \gamma Y^2 - \alpha > 0$ | $k^2 = \frac{\gamma Y^2}{2\omega^2}$ |

**Remarks**:

1. It is obvious that the solution for the hard oscillator exists for all values of parameters $\alpha$ and $\gamma$, but for other oscillators (hard-soft and soft-hard) the motion is oscillatory only for some special relations between the parameters $\alpha$ and $\gamma$ and initial amplitude $Y$.
2. The arbitrary amplitude $Y$ and phase $\theta$ are calculated according to the initial conditions (8), (15) or (17).
3. For the pure cubic equation, when $\alpha$=0, the modulus of the Jacobi elliptic function is constant ($k^2$=1/2) and the frequency is $\omega = Y\sqrt{\gamma}$ . The closed form solution is

$$y = Ycn(Yt\sqrt{\gamma} + \theta, 1/2) \tag{19}$$

For the initial conditions (8), the amplitude of vibration is

$$Y = \left[ y_0^4 + \frac{2}{\gamma}\dot{y}_0^2 \right]$$

## 3. The elliptic homotopy perturbation method

Let us rewrite the differential equation (7) in the form

$$\ddot{y} + \alpha y + \gamma y^3 = -g(y, \dot{y}) \tag{20}$$

and apply the initial conditions (15). For $g$=0 the differential equation has the exact solution (11) with (16). According to this result, we assume the initial approximate solution of (20) in the form

$$Y_0(t) \equiv Y_0 = y_0 cn(\omega_1 t, k_1^2) = y_0 cn_1 \tag{21}$$

where $\omega_1$ and $k_1^2$ transform into $\omega$ and $k^2$ when $g$=0. Due to definition of homotopy $X$:$\Omega$x[0,1]→$R$, that the two continuous functions from one topological space can be "continuously deformed" into the other, and introducing the embedding artificial parameter $p$ with the values in the interval [0,1], as it was suggested by He [43], a homotopy transformation of the differential equation (20) yields

$$(1-p)[(\ddot{X}+\alpha X+\gamma X^3)-(\ddot{Y}_0+\alpha Y_0+\gamma Y_0^3)]+p[\ddot{X}+\alpha X+\gamma X^3+g(X,\dot{X})]=0 \tag{22}$$

with initial conditions

$$X(0,p)=y_0, \quad \dot{X}(0,p)=0 \tag{23}$$

Namely, for $p$=0 the Eq. (22) simplifies into

$$\ddot{X}+\alpha X+\gamma X^3=0 \tag{24}$$

with the exact solution

$$X(t,0)=y^0(t)=Ycn(\omega t+\theta,k^2) \tag{25}$$

When $p$=1 the equation has the same form as the original equation

$$\ddot{X}+\alpha X+\gamma X^3=-g(X,\dot{X}) \tag{26}$$

and the solution is

$$X(t,1)=y(t). \tag{27}$$

It can be concluded that for the change of $p$ from zero to unity the solution is continually changing from (25) to (27).

For $X(t,p)$, which is the solution of (22) in the whole domain $p\in[0,1]$ and is smooth enough to have $k$th order partial derivatives respect to $p$ at $p$=0, the Maclaurin series is as follows

$$X(t,p)=Y_0(t)+\sum_{k=1}^{\infty}\left(\frac{y_k(t)}{k!}\right)p^k \tag{28}$$

Substituting (28) into (22) and separating the terms with the same order of the parameter $p$ a system of linear differential equations is obtained. For $p^1$ the first-order deformation equation is

$$\ddot{y}_1+\alpha y_1+3\gamma Y_0^2 y_1=-[\ddot{Y}_0+\alpha Y_0+\gamma Y_0^3+g(Y_0,\dot{Y}_0)] \tag{29}$$

with initial conditions

$$y_1(0)=0, \quad \dot{y}_1(0)=0 \tag{30}$$

Introducing (21) into (29), yields

$$\begin{aligned}\ddot{y}_1+\alpha y_1+3\gamma y_0^2 cn_1^2 y_1 &=-[-y_0\omega_1^2 cn_1(1-2k_1^2+2k_1^2 cn_1^2)+\alpha y_0 cn_1+\gamma y_0^3 cn_1^3\\ &+g(y_0 cn_1,-y_0\omega_1 sn_1 dn_1)],\end{aligned} \tag{31}$$

where $sn_1 \equiv sn(\omega_1 t,k_1)$ and $dn_1 \equiv dn(\omega_1 t,k_1)$ are Jacobi elliptic functions [48].

For

$$\alpha=0, \quad g(y,\dot{y})=-\beta y^2, \quad \gamma>\beta \tag{32}$$

the Eq. (31) is specified as

$$\ddot{y}_1 + 3\gamma y_0^2 cn_1^2 y_1 = -[-y_0\omega_1^2 cn_1(1-2k_1^2 + 2k_1^2 cn_1^2) + \gamma y_0^3 cn_1^3 + \beta y_0^2 cn_1^2)] \tag{33}$$

Solution of (33) is assumed as a sum of a constant and a linear term of elliptic function $cn_1$

$$y_1 = K_1 + K_2 cn_1 \tag{34}$$

Substituting (34) into (33) and separating the terms with the same order of elliptic function $cn_1$ the following system of algebraic equations is obtained

$$\begin{aligned} &(K_1 + y_0)\omega_1^2(1-2k_1^2) = 0, \\ &3\gamma y_0^2 K_0 + \beta K_0^2 = 0, \\ &3\beta y_0^2 K_1 + \gamma y_0^3 - 2(K_1 + y_0)k_1^2\omega_1^2 = 0. \end{aligned} \tag{35}$$

Due to initial conditions (30) the relation $K_0$ and $K_1$ is

$$K_0 + K_1 = 0 \tag{36}$$

Solving equations (35) and (36) it follows:

$$\omega_1^2 = 3\gamma y_0^2 \frac{\gamma y_0 + \beta}{3\gamma y_0 + \beta}, \quad k_1^2 = \frac{1}{2}, \quad K_0 = -K_1 = -\frac{\beta}{3\gamma} \tag{37}$$

Using the relations (21) and (34) with (37) and according to (27) and (28) the solution in the first approximation is

$$y(t) = -\frac{\beta}{3\gamma} + (y_0 + \frac{\beta}{3\gamma})cn(y_0 t\sqrt{3\gamma\frac{\gamma y_0 + \beta}{3\gamma y_0 + \beta}}, \frac{1}{2}) \tag{38}$$

Analyzing (37) it is obvious that the coefficient $\beta$ has no influence on modulus of Jacobi function. Frequency and argument of Jacobi function and also the accuracy of the approximate solution (38) depend on coefficient ratio $\beta/\gamma$. For smaller ratio $(\beta/\gamma)<<1$ the difference between exact solution and approximate solution is negligible. For higher values of the ratio $\beta/\gamma$ the difference is significant and the solution in the first approximation is not acceptable.

## 4. Elliptic-Krylov-Bogolubov (EKB) method

Let us modify the differential equation (7) by introducing the small parameter $\varepsilon<<1$

$$\frac{d^2y}{dt^2} + \alpha y + \gamma y^3 = \varepsilon f(y, \frac{dy}{dt}) \tag{39}$$

Due to idea of Krylov and Bogolubov [44], Eq. (39) can be transformed into a system of two coupled first order differential equations. Namely, the solution of (39) is assumed in the form of the solution (19) for $\varepsilon$=0, i.e.,

$$y(t) = Y(t) cn(\psi(t), k^2) \equiv Ycn \tag{40}$$

where the amplitude and the phase are time dependent

$$\psi(t) = \int_0^t \omega(s) ds + \theta(t) \tag{41}$$

and, also, the frequency and the modulus of Jacobi elliptic function (12)

$$\omega(t) = \sqrt{\alpha + \gamma [Y(t)]^2}, \quad k^2(t) = \frac{\gamma [Y(t)]^2}{2(\alpha + \gamma [Y(t)]^2)} \tag{42}$$

The first time derivative of (40) is

$$\frac{dy}{dt} = Y\omega \frac{\partial (cn)}{\partial \psi} \tag{43}$$

with the constraint

$$\frac{dY}{dt} cn + Y \frac{d\theta}{dt} \frac{\partial (cn)}{\partial \psi} + Y \frac{\partial (cn)}{\partial (k^2)} \frac{\partial (k^2)}{\partial Y} \frac{dY}{dt} = 0 \tag{44}$$

Substituting (40), (43) and the time derivative of (43) into (39), we obtain

$$\frac{dY}{dt} [(\omega + Y \frac{\partial \omega}{\partial Y}) \frac{\partial (cn)}{\partial \psi} + Y\omega \frac{\partial^2 (cn)}{\partial \psi \partial (k^2)} \frac{\partial (k^2)}{\partial Y}] + Y\omega \frac{d\theta}{dt} \frac{\partial^2 (cn)}{\partial \psi^2} = \varepsilon f (Ycn, Y\omega \frac{\partial (cn)}{\partial \psi}) \tag{45}$$

After some transformation of (44) and (45), the two coupled first order differential equations follow

$$\begin{aligned}
&\frac{dY}{dt} [(\omega + Y \frac{\partial \omega}{\partial Y})(cn_\psi)^2 + Y\omega cn_{\psi k} cn_\psi \frac{\partial (k^2)}{\partial Y} - \omega cn_{\psi\psi} (cn + Ycn_k \frac{\partial (k^2)}{\partial Y})] \\
&= \varepsilon [f(Ycn, Y\omega cn_\psi)] cn_\psi, \\
&Y \frac{d\theta}{dt} [(\omega + Y \frac{\partial \omega}{\partial Y})(cn_\psi)^2 + Y\omega cn_{\psi k} cn_\psi \frac{\partial (k^2)}{\partial Y} - \omega cn_{\psi\psi} (cn + Ycn_k \frac{\partial (k^2)}{\partial Y}) \\
&= -\varepsilon [f(Ycn, Y\omega cn_\psi)](cn + Ycn_k \frac{\partial (k^2)}{\partial Y}),
\end{aligned} \tag{46}$$

where $cn_\psi \equiv \frac{\partial (cn)}{\partial \psi}, cn_{\psi\psi} \equiv \frac{\partial^2 (cn)}{\partial \psi^2}, cn_{\psi k} \equiv \frac{\partial^2 (cn)}{\partial \psi \partial (k^2)}$.

The aim is to solve the system of differential equations.

For the pure cubic Duffing's equation, where $\alpha$=0 and the modulus of the Jacobi elliptic function is constant (see (19)), the system (46) simplifies to

$$\begin{aligned} &\frac{dY}{dt}[(\omega+Y\frac{\partial\omega}{\partial Y})(cn_\psi)^2-\omega cn_{\psi\psi}cn]=\varepsilon[f(Ycn,Y\omega cn_\psi)]cn_\psi,\\ &Y\frac{d\theta}{dt}[(\omega+Y\frac{\partial\omega}{\partial Y})(cn_\psi)^2-\omega cn_{\psi\psi}cn]=-\varepsilon[f(Ycn,Y\omega cn_\psi)]cn, \end{aligned} \tag{47}$$

where $cn_\psi=-sndn, cn_{\psi\psi}=-cn(1-2k^2+2k^2cn^2)$. Using the value for the modulus ($k^2$=1/2) and the frequency ($\omega=Y\sqrt{\gamma}$) of the Jacobi elliptic function, and also the relations between the elliptic functions, we have

$$\begin{aligned} &Y^2\sqrt{\gamma}\frac{d\theta}{dt}=-\varepsilon[f(Ycn,-Y^2\sqrt{\gamma}sndn]cn,\\ &Y\sqrt{\gamma}\frac{dY}{dt}=-\varepsilon[f(Ycn,-Y^2\sqrt{\gamma}sndn]sndn. \end{aligned} \tag{48}$$

At this point the averaging procedure is introduced. The averaging is over the period of the elliptic function $4K(k^2)$, where $K(k^2)$ is the complete elliptic integral of the first kind. The averaged first order differential equations (48) are

$$\frac{dY}{dt}=-\frac{\varepsilon}{\omega}\frac{1}{4K}\int_0^{4K}f(Ycn,-Y\omega\, sn\, dn)sn\, dn\, d\psi \tag{49}$$

$$\frac{d\theta}{dt}=-\frac{\varepsilon}{Y\omega}\frac{1}{4K}\int_0^{4K}f(Ycn,-Y\omega\, sn\, dn)cn\, d\psi \tag{50}$$

where for the modulus $k^2$=1/2 the elliptic integral is $K=K(1/2)=1.85407$ and $cn=cn(\psi,1/2)$, $sn=sn(\psi,1/2)$, $dn=dn(\psi,1/2)$.

a) For the special case when the small non-linear function depends only on the deflection, i.e., $f\equiv f(y)$, the first order differential equations (49) and (50) simplify to

$$\frac{dY}{dt}=0,\quad \frac{d\theta}{dt}=\frac{\varepsilon}{Y\omega}\frac{1}{4K}\int_0^{4K}f(Ycn)cn\, d\psi \tag{51}$$

i.e., $Y$=const. and $\theta=\varepsilon\theta(Y)$,
where

$$\theta=\frac{1}{4KY\omega}\int_0^{4K}f(Ycn)cn\, d\psi \tag{52}$$

Then, the EKB approximate solution is

$$y=Ycn[(\omega+\varepsilon\theta)t+\theta_0,1/2] \tag{53}$$

where from the initial conditions of the oscillations $Y$ and $\theta_0$ are obtained.
b) For $f \equiv f(dy/dt)$, the Eqs. (49) and (50) have the form

$$\frac{dY}{dt} = -\frac{\varepsilon}{\omega}\frac{1}{4K}\int_0^{4K} f(-Y\omega\, sn\, dn) sn\, dn\, d\psi \tag{54}$$

$$\frac{d\theta}{dt} = 0 \tag{55}$$

The EKB solution yields

$$y = Y(t)cn[(\sqrt{\gamma}\int_0^t Y(t)dt) + \theta_0, 1/2] \tag{56}$$

where $Y(t)$ is the solution of (54).
**Examples**
1) For the differential equation with strong non-linear cubic term and the weak linear part

$$\ddot{y} + \gamma y^3 + \varepsilon y = 0 \tag{57}$$

the phase angle is due to (52)

$$\theta = \frac{1}{4K\omega}\int_0^{4K} cn^2\, d\psi = \frac{1}{\omega}(\frac{2E}{K} - 1) = \frac{0.4569}{\omega} \tag{58}$$

and the approximate solution of (57) is according to (53) and (58)

$$y = Ycn[(\omega + \frac{0.4569\varepsilon}{\omega})t + \theta_0, 1/2] \tag{59}$$

where $E$=$E$(1/2)=1.35064 is the complete elliptic integral of the second kind for modulus $k^2$=1/2 and $\omega = Y\sqrt{\gamma}$ .
The exact solution of (57) is according to (11), (12) and (15)

$$y = Ycn[t\sqrt{\varepsilon + \gamma Y^2} + \theta_0, \gamma Y^2 / 2(\varepsilon + \gamma Y^2)] \tag{60}$$

For ε<<1 using the series expansion of the functions in (60) the approximate solution is obtained

$$y = Ycn[t(\omega + \frac{\varepsilon}{2\omega}) + \theta_0, 1/2] \tag{61}$$

Comparing (59) and (61) it is evident that the difference is negligible.
2) For the differential equation with small linear damping term

$$\ddot{y} + \gamma y^3 + 2\zeta\dot{y} = 0 \tag{62}$$

the amplitude of vibration is according to (54)

$$\frac{dY}{dt} = -\frac{\varepsilon(2\zeta)Y}{4K}\int_0^{4K} sn^2\, dn^2\, d\psi = -\frac{\varepsilon(2\zeta)Y}{4K}Q \tag{63}$$

i.e.,

$$Y = Y_0 \exp(-\frac{\varepsilon(2\zeta)}{4K}Q)t \tag{64}$$

where $Q = \int_0^{4K} sn^2\, dn^2\, d\psi$ . Using (56) and (64) the solution of (62) follows

$$y = Y_0 \exp(-\frac{\varepsilon(2\zeta)t}{4K}Q)cn[Y_0\sqrt{\gamma}\,\frac{4K}{\varepsilon(2\zeta)Q}(1-\exp(-\frac{\varepsilon t(2\zeta)Q}{4K})) + \theta_0, 1/2]$$

After some simplification the approximate solution is

$$y = Y_0 \exp(-\frac{\varepsilon(2\zeta)t}{4K}Q)cn[Y_0 t\sqrt{\gamma} + \theta_0, 1/2] \tag{65}$$

The amplitude of vibration decreases exponentially. The period of vibration increases, but very slow. It allows the assumption of the constant frequency of vibration.

**Remark**:

The EKB method is usual known as that with time variable amplitude and phase, as it is assumed that the perturbed amplitude and phase of the solution differs for a small value to trial solution.

## 5. Elliptic-Galerkin (EG) method

Let us consider the differential equation

$$\ddot{y} + \alpha y + \beta y^2 + \gamma Y^3 = 0 \tag{66}$$

The approximate solution will be obtained by applying the Galerkin method which represents one of the weighted residual methods. In the previous papers, usually, the trial solution of (66) is assumed as a linear combination of the circular functions and the arbitrary weight function also belongs to that group. Our intention is to extend the method by applying of the Jacobi elliptic function.

We introduce a trail solution to (66) as a linear combination of independent *cn* Jacobi elliptic functions

$$y = K_1 cn(\omega t, k^2) + K_2 cn^2(\omega t, k^2) = K_1 cn + K_2 cn^2 \tag{67}$$

where $K_1$ and $K_2$ are constants, $\omega$ and $k$ are the frequency and modulus of the *cn* elliptic function which have to be calculated.

Substituting (67) into (66), the residual function is obtained

$$r(\psi)\equiv K_1\omega^2(cn)''+2K_2\omega^2[(cn)']^2+2K_2\omega^2(cn)(cn)''+\alpha[K_1cn+K_2(cn)^2]$$
$$+\beta[K_1^2(cn)^2+2K_1K_2(cn)^3+K_2^2(cn)^4]+\gamma[K_1^3(cn)^3+3K_1^2K_2(cn)^4 \qquad (68)$$
$$+3K_1K_2^2(cn)^5+K_2^3(cn)^6],$$

where $(\cdot)'=d(\cdot)/d\psi$, $(\cdot)''=d^2(\cdot)/d\psi^2$ and $\psi=\omega t$. If (67) is the closed form solution of Eq. (66), the residual function $r(\psi)$ is zero. The goal is to construct $y(\psi)$ so that the integral of the residual will be zero for some choices of weight functions $w(\psi)$. As the weight function is arbitrary one, let us choose it as the derivatives of the constant $K_1$ and $K_2$, respectively, of the assumed solution (67), i.e., $(\partial y/\partial K_1)$ and $(\partial y/\partial K_2)$. Multiplying (68) with the weight function and integrating over the interval $[0,4K(k)]$, where $K$ is the total elliptic integral of the first kind and $4K$ is the period of $cn$ function, one obtains

$$\int_0^{4K(k)} r(\psi)cn\,d\psi=0,\quad \int_0^{4K(k)} r(\psi)cn^2d\psi=0 \qquad (69)$$

i.e.

$$\int_0^{4K(k)} -K_1\omega^2cn^2(1-k^2+k^2cn^2)+\alpha K_1cn^2+2\beta K_1K_2cn^4+\gamma[K_1^3cn^4+3K_1K_2^2cn^6]d\psi=0$$

$$\int_0^{4K(k)} -2K_2cn^4\omega^2(1-k^2+k^2cn^2)+2K_2\omega^2sn^2cn^2dn^2+\alpha K_2cn^4$$
$$+\beta[K_1^2cn^4+K_2^2cn^6]+\gamma[3K_1^2K_2cn^6+K_2^3cn^8]d\psi=0. \qquad (70)$$

Eqs. (66) and (69), i.e. (70), are equivalent, because $w(\psi)$ is any arbitrary function. To apply the method, all we need to do is to solve the two algebraic equations for the coefficients $\omega=\omega(K_1,K_2)$ and $k^2=k^2(K_1,K_2)$

$$\omega^2[(1-k^2)C_2+k^2C_4]=\alpha C_2+(2\beta K_2+\gamma K_1^2)C_4+3K_2^2\gamma C_6 \qquad (71)$$

$$K_2\omega^2[2(2-3k^2)C_4+4k^2C_6-2(1-k^2)C_2]$$
$$=(\beta K_1^2+\alpha K_2)C_4+(\beta K_2+3\gamma K_1^2)K_2C_6+\gamma K_2^3C_8, \qquad (72)$$

where (see [48]): $k'^2=1-k^2$, $C_0=4K(k)$, $C_2=\frac{4}{k^2}(E-k'^2K)$ and

$$C_{2m+2}=\frac{2m(2k^2-1)C_{2m}+(2m-1)k'^2C_{2m-2}}{(2m+1)k^2} \qquad \text{for} \quad m=1,2,3.$$

Eliminating $\omega$ from (71) and (72) we obtain the algebraic equation

$$\frac{K_2}{(1-k^2)C_2+k^2C_4}[\alpha C_2+(2\beta K_2+\gamma K_1^2)C_4+3K_2^2\gamma C_6] \\ =\frac{(\beta K_1^2+\alpha K_2)C_4+(\beta K_2+3\gamma K_1^2)K_2C_6+\gamma K_2^3C_8}{2(2-3k^2)C_4+4k^2C_6-2(1-k^2)C_2}, \tag{73}$$

which is not easy to be solved. Analyzing the relation (71) and (73) it is seen that the both strong non-linear terms have a significant influence on the modulus and the frequency of the Jacobi elliptic function. Only, for the case when $\alpha=\beta=0$ the modulus of the Jacobi elliptic function is independent on the initial amplitude, but the frequency depends, as it is previously stated in Eq. (19).
**Remark**: The accuracy of the solution depends on our ability to find the most convenient combination of functions.

## 6. CONCLUSION

In the paper three procedures for solving of the so called Duffing’s equation are shown: the elliptic-Galerkin (EG) method, the elliptic-Krylov-Bogolubov (EKB) method and the homotopy perturbation method. For all of the methods is common that the solutions are based on the exact solution of the strong cubic differential equation given with Jacobi elliptic function. The mentioned methods have some advantages, but also disadvantages. The EG method is one of conceptually simplest analytical approximate procedure which leads to algebraic equations; however the results may be with small accuracy as it depends on the investigator to chose the most adequate weight function. The elliptic-Krylov-Bogolubov (EKB) method is of perturbation type. The perturbation method is based on the assumption that a small parameter ($\varepsilon$<<1) must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation technique. Many non-linear problems described with Duffing’s equation have no small parameter at all. Then, an appropriate choice of a small parameter leads to accurate results, but an unsuitable choice to a bad result. The homotopy method does not require a small parameter in the equation and eliminates the previous limitations. The main disadvantage is the question of convergence of the solution. Farther investigations are necessary.

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# DEVEDESET GODINA DUFINGOVE JEDNAČINE

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*Abstrakt. U radu su dati izvorni podaci za tzv. Dufingovu jednačinu. Pirkazana je generalizacija koju je izvršio autor ovog rada, prikazani su njeni radovi koji se bave Dufingovom jednačinom kao i neke od metoda za rešavanje ove jednačine. Date su tri najznačajnije procedure nalaženja približnog rešenja na bazi tačnog rešenja Dufingove jednačine sa jakom kubnom nelinearnosti. Koristeći Jakobijeve eliptičke funkcije opisane su sledeće metode: eliptički-Krilov-Boboljubov (EKB) metod, homotopijski perturbacioni i eliptičke-Galerkinov (EG) metod. Metode su poredjene. Prikazane su prednosti i nedostaci pojedinih procedura.*

**Ključne reči**: *Dufingova jednačina, eliptički-Krilov-Boboljubov (EKB) metod, homotopijski perturbacioni metod, eliptičke-Galerkinov (EG) metod*

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