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Miloš M. Jovanović

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VORTICITY EVOLUTION IN PERTURBED POISEUILLE FLOW

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Miloš M. Jovanović

University of Niš, Faculty of Mechanical Engineering, Aleksandra Medvedeva 14, 18000 Niš, Serbia, e-mail: jmilos@masfak.ni.ac.rs

Abstract. *We consider numerical simulation of temporal hydrodynamic instability with finite amplitude perturbations in plane incompressible Poiseuille flow. Two dimensional Navier Stokes equations have been used and reduced to vorticity-stream function form. Trigonometric polynomials have been used in homogeneous direction and Chebyshev polynomials in inhomogeneous direction. The problem of boundary conditions for vorticity has been solved by using the method of influence matrices. The Orr-Sommerfeld equation has been solved by Chebyshev polynomials, and linear combination of the obtained eigenfunctions has been optimized with regard to the corresponding eigenvalue. We present here the results of simulation for the perturbations optimized in regard to the least stable eigenvalue for the Reynolds number $Re = 1000$.*

Key words: *direct numerical simulation, perturbed Poiseuille flow, subcritical instability, optimized perturbation, pseudospectral method*

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1. INTRODUCTION

It is well known that classical Hydrodynamic stability theory is not capable of describing the initial transient growth mechanism that has been observed by experimentators for viscous channel flows. The reason for such behavior has been ascribed to the asymptotic behavior of the unstable eigenvalue, since the perturbation is formed only by the unstable eigenvalue and the corresponding eigenfunction. So, for Poiseuille flow the critical Reynolds number is 5772, and for this value of Reynolds number the eigenvalue has the positive imaginary part, and so the flow is asymptotically unstable, i.e. for large values of time, when $t \to \infty$

We have simulated the streamfunction-vorticity form of the 2D Navier-Stokes equations, and carried out the perturbation of laminar Poiseuille flow, by forming the optimized linear combination of the all eigenfunctions normalized on the corresponding eigenvalue, in this case the least stable eigenvalue. The simulation has been carried out for the subcritical Reynolds number Re=1000, defined on channel half height H, middle channel maximal fluid velocity U_{max} and fluid kinematic viscosity-ν

2. PROBLEM STATEMENT AND GOVERNING EQUATIONS

We consider the problem for plane Poiseuille flow, where H is channel half height, L-is its lenght. Incompressible fluid flows through the channel from left to right, whereby the pressure at the inlet is p_i and at the outlet cross section is p_o. Momentum equation by means of which we describe this isothermal incompressible flow can be written in nondimensional form [1]

$$St\frac{\partial \vec{V}}{\partial t}+\left(\vec{V}\cdot\nabla\right)\vec{V}=\frac{1}{Fr}\vec{F}-Eu\nabla p+\frac{1}{Re}\Delta\vec{V}, \tag{1}$$

where is ∇-Hamilton's differential operator. In the above expression Δ-designates nondimensional Laplace's differential operator. The continuity equation reads

$$\nabla\cdot\vec{V}=div\vec{V}=0. \tag{2}$$

In the above expressions $\vec{V}$ is nondimensional velocity vector of 2D flow in Cartesian coordinates, $\vec{F}$ –nondimensional force field, p–nondimensional pressure, t–nondimensional time, St–Strouhal number, Fr–Froude number, Eu–Euler number, Re–Reynolds number. Nondimensional form of the momentum equation has been obtained by using the following caracteristic scales for various independent and dependent variables: $L_0=H$ – for all lengths, $V_0=U_{max}$ –maximal velocity at the middle of the channel, for all velocities, p_o–pressure at the outlet of the channel, for pressure, and g–gravity acceleration for body force. Four dimensionless parameters are thus occuring, namely,

$$St=\frac{L_0}{t_0V_0},\quad Fr=\frac{gL_0}{V_0^2},\quad Eu=\frac{p_0}{\rho V_0^2},\quad Re=\frac{L_0V_0\rho}{\mu}, \tag{3}$$

where are μ - dynamic viscosity, ρ - fluid density. We take that St =1, Eu =1 and Fr =1, and we introduce the ν-dimensionless kinematic viscosity, which is the inverse of Reynolds number. So we have now

$$\frac{\partial \vec{V}}{\partial t}+\left(\vec{V}\cdot\nabla\right)\vec{V}=\vec{F}-\nabla p+\nu\Delta\vec{V}, \tag{4}$$

If we take the curl of this equation, and having in mind the definition of vorticity

$$\vec{\omega}=\nabla\times\vec{V}=curl\vec{V}, \tag{5}$$

then we obtain the following transport expression for vorticity

$$\frac{\partial \vec{\omega}}{\partial t} + \left(\vec{V} \cdot \nabla\right) \vec{\omega} = \nabla \times \vec{F} + \nu \Delta \vec{\omega} . \tag{6}$$

We have taken into account that the curl of arbitrary scalar function gradient, in this case pressure function, is by definition equal to zero. The velocity vector can be expressed as curl of stream function

$$\vec{V} = \nabla \times \vec{\psi} = curl \vec{\psi}, \tag{7}$$

and after substitution of this expression (7) into (6), and after projection to z-axe, we obtain the following equation for transport of vorticity

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} + \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right). \tag{8}$$

If we substitute the expression (7) in (2) the continuity equation is identically satisfied, and can not be used for closure the system of equations. For closure the system of equations we use the definition of vorticity given by the expression (5) and velocity vector given through the streamfunction vector by (7). So the second equation for closure the system of equations reads

$$\omega + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{9}$$

The system of the equations (8) and (9) with appropriate initial and boundary conditions should be solved in space and time. Boundary conditions can be formulated in the following manner

$$\psi(x,1,t) = g_+(x,t), \quad \frac{\partial \psi}{\partial y}(x,1,t) = h_+(x,t) \;\; on \;\; \Gamma_u \times \mathrm{T}, \tag{10}$$

$$\psi(x,-1,t) = g_-(x,t), \quad \frac{\partial \psi}{\partial y}(x,-1,t) = h_-(x,t) \;\; on \;\; \Gamma_l \times \mathrm{T}, \tag{11}$$

$$\psi(x,y,0) = \psi_0(x,y) \;\; on \;\; \Omega, \tag{12}$$

here domain Ω is defined as $\Omega = \{ (x,y) \in \hat{\mathrm{h}}^2 \mid 0 \le x \le 2\pi \wedge -1 < y < 1 \}$. We have designated the upper domain boundary $\Gamma_u = \{(x,y) \in \hat{\mathrm{h}}^2 \mid 0 \le x \le 2\pi \wedge y = 1 \}$ and the lower domain boundary $\Gamma_l = \{ (x,y) \in \hat{\mathrm{h}}^2 \mid 0 \le x \le 2\pi \wedge y = -1\}$. The time domain is defined as $\mathrm{T} = \{ t \in \hat{\mathrm{h}} \mid 0 \le t \le T_e \}$, where T_e is the end of the simulation. We have anticipated the periodic boundary conditions in streamwise direction (x-axe), which are in accordance with the periodic perturbations obtained by the solution of Orr-Sommerfeld equation of hydrodynamic stability.

3. NUMERICAL PROCEDURE FOR THE SOLUTION OF PROBLEM

For the problem stated in the previous section, for the basis function in x-direction we have taken trigonometric polynomials, and for y-direction we have taken Chebyshev polynomials. The domain in x-direction is equally discretised $\Delta x = 2\pi/N$, and domain in y-direction is discretised by Gauss-Lobatto-Chebyshev points defined as $y_j = \cos(\pi j/N)$ for $0 \le j \le N$, where is N-number of discretization points in x- and y-direction. For streamwise direction we have used Fourier-Galerkin method, and for stream normal direction Chebyshev-collocation method. The truncated Fourier series for streamfunction and vorticity read

$$\omega_N\left(x,y,t\right) = \sum_{k=-N/2}^{k=N/2} \hat{\omega}_k\left(y,t\right)\ e^{Ikx}, \tag{13}$$

$$\psi_N\left(x,y,t\right) = \sum_{k=-N/2}^{k=N/2} \hat{\psi}_k\left(y,t\right)\ e^{Ikx}, \tag{14}$$

In the above expressions $I = \sqrt{-1}$ is imaginary unit, k-wave number, $\hat{\omega}_k\left(y_j,t\right)$ and $\hat{\psi}_k\left(y_j,t\right)$ are Fourier coefficients for vorticity and streamfunction respectively. In order to have 2π-periodicity in the flow domain, we have chosen that wave number must be from the set of integers, $k\in$Ħ. In order to implement Fourier-Galerkin method to the system of equation (8) and (9), we firstly approximate nonlinear convective terms on left hand side, in the following manner

$$N_1 = \left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right)_N \left(x,y,t\right) = \sum_{k=-N/2}^{N/2} \left[\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right]_k \left(y,t\right) e^{Ikx} \tag{15}$$

$$N_2 = \left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right)_N \left(x,y,t\right) = \sum_{k=-N/2}^{N/2} \left[\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right]_k \left(y,t\right)\ e^{Ikx} \tag{16}$$

Substituting the (13), (14), (15) and (16) in the (8) and (9), we obtain the following residual equations

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{k=-N/2}^{N/2}\hat{\omega}_k\left(y,t\right)e^{Ikx} + \sum_{k=-N/2}^{N/2}\left[\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right]_k\left(y,t\right)e^{Ikx} - \sum_{k=-N/2}^{N/2}\left[\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right]_k\left(y,t\right)e^{Ikx} - \\ - \sum_{k=-N/2}^{N/2}\hat{F}_k\left(y,t\right)e^{Ikx} - \nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sum_{k=-N/2}^{N/2}\hat{\omega}_k\left(y,t\right)e^{Ikx} \neq 0,\end{aligned} \tag{17}$$

$$\sum_{k=-N/2}^{N/2}\hat{\omega}_k\left(y,t\right)e^{Ikx} + \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sum_{k=-N/2}^{N/2}\hat{\psi}_k\left(y,t\right)e^{Ikx} \neq 0. \tag{18}$$

We have introduced the following expression for the curl of body force,

$$\mathrm{F} = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} = \sum_{k=-N/2}^{N/2} \hat{\mathrm{F}}_k\left(x,y,t\right)e^{Ikx} \tag{19}$$

If we apply Galerkin method to the equations (17) and (18), i.e. we take for the weight functions the same as basis functions , we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_{k=-N/2}^{N/2}\hat{\omega}_k\left(y,t\right)\left\langle e^{Ikx},e^{Ilx}\right\rangle + \sum_{k=-N/2}^{N/2}\left[\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right]_k\left(y,t\right)\left\langle e^{Ikx},e^{Ilx}\right\rangle - \\ &-\sum_{k=-N/2}^{N/2}\left[\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right]_k\left(y,t\right)\left\langle e^{Ikx},e^{Ilx}\right\rangle - \sum_{k=-N/2}^{N/2}\hat{\mathrm{F}}_k\left(y,t\right)\left\langle e^{Ikx},e^{Ilx}\right\rangle - \\ &-\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sum_{k=-N/2}^{N/2}\hat{\omega}_k\left(y,t\right)\left\langle e^{Ikx},e^{Ilx}\right\rangle = 0, \quad l = 0,1,\ldots,N/2.\end{aligned} \tag{20}$$

$$\begin{aligned}&\sum_{k=-N/2}^{N/2}\hat{\omega}_k\left(x,y,t\right)\left\langle e^{Ikx},e^{Ilx}\right\rangle + \left(-k^2+\frac{\partial^2}{\partial y^2}\right)\sum_{k=-N/2}^{N/2}\hat{\psi}_k\left(x,y,t\right)\left\langle e^{Ikx},e^{Ilx}\right\rangle = 0, \\ &l = 0,1,\ldots,N/2.\end{aligned} \tag{21}$$

where we have denoted with $\langle\ ,\ \rangle$ the inner product. Recalling that $\hat{\omega}_{-k} = \hat{\omega}_k^*$, i.e. Fourier coefficient of inverse wave number is complex conjugate of the corresponding wave number, it is not necessary to take $l = -N/2,...,N/2$, but $l = 0,1,...,N/2$. Having in mind the orthogonality relation

$$\left\langle e^{Ikx},e^{Ilx}\right\rangle = \int_0^{2\pi} e^{Ikx}e^{-Ilx}dx = \begin{cases}2\pi, & l = k,\\ 0, & l \neq k,\end{cases} \tag{22}$$

the system of equations (20) and (21) takes the following form

$$\begin{aligned}&\frac{\partial\hat{\omega}_k(y,t)}{\partial t} + \left[\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right]_k\left(y,t\right) - \left[\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right]_k\left(y,t\right) = \hat{\mathrm{F}}_k\left(y,t\right) + \nu\left(\frac{\partial^2}{\partial y^2}-k^2\right)\hat{\omega}_k\left(y,t\right) \\ &l = k = 0,1,\ldots,N/2.\end{aligned} \tag{23}$$

$$\hat{\omega}_k\left(y,t\right) + \left(-k^2+\frac{\partial^2}{\partial y^2}\right)\hat{\psi}_k\left(y,t\right) = 0, \quad l = k = 0,1,\ldots,N/2. \tag{24}$$

Applying now the Chebyshev-collocation method in inhomogenuous direction (*y*-axe) to the above system of equations, we obtain the following system

$$\frac{\partial\hat{\omega}_{kN}\left(y_j,t\right)}{\partial t}+\left[\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right]_{kN}\left(y_j\right)-\left[\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right]_{kN}\left(y_j\right)=\hat{F}_{kN}\left(y_j,t\right)+$$
$$+\nu\left(-k^2+\sum_{l=0}^{N}d_{j,l}^{(2)}\right)\hat{\omega}_{kN}\left(y_j,t\right),\quad k=0,1,\ldots,N/2,\quad j=1,\ldots,N-1, \tag{25}$$

$$\hat{\omega}_{kN}\left(y_j,t\right)+\left(-k^2+\sum_{l=0}^{N}d_{j,l}^{(2)}\right)\hat{\psi}_{kN}\left(y_j,t\right)=0,$$
$$k=0,1,\ldots,N/2,\quad j=1,\ldots,N-1. \tag{26}$$

Here the subscript index N denotes the number of discretization points in y-direction, and it is worth reminding that we have taken $N_y=N_x=N$. Differentiation with regard to y-variable has been substituted with Chebyshev differentiation expressions [2] given by

$$\frac{\partial^2\hat{\omega}_{kN}}{\partial y^2}\left(y_j\right)=\hat{\omega}_{kN}^{(2)}\left(y_j\right)=\sum_{l=0}^{N}d_{j,l}^{(2)}\hat{\omega}_{kN}\left(y_l\right),\quad j=0,\ldots,N, \tag{27}$$

$$\frac{\partial^2\hat{\psi}_{kN}}{\partial y^2}\left(y_j\right)=\hat{\psi}_{kN}^{(2)}\left(y_j\right)=\sum_{l=0}^{N}d_{j,l}^{(2)}\hat{\psi}_{kN}\left(y_l\right),\quad j=0,\ldots,N. \tag{28}$$

The next step to be carried out is the temporal discretization of the governig equations. For this purpose we have used the following two-step generalized method defined in the following way

$$\frac{\left(1+\varepsilon\right)\hat{\omega}_{kN}^{n+1}\left(y_j\right)-2\varepsilon\hat{\omega}_{kN}^{n}\left(y_j\right)-\left(1-\varepsilon\right)\hat{\omega}_{kN}^{n-1}\left(y_j\right)}{2\Delta t}+$$
$$+\left[\gamma_1\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right)_{kN}^{n+1}\left(y_j\right)+\gamma_2\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right)_{kN}^{n}\left(y_j\right)+\left(1-\gamma_1-\gamma_2\right)\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right)_{kN}^{n-1}\left(y_j\right)\right]- \tag{29}$$
$$-\left[\gamma_1\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right)_{kN}^{n+1}\left(y_j\right)+\gamma_2\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right)_{kN}^{n}\left(y_j\right)+\left(1-\gamma_1-\gamma_2\right)\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right)_{kN}^{n-1}\left(y_j\right)\right]+$$
$$+\nu k^2\left[\theta_1\hat{\omega}_{kN}^{n+1}\left(y_j\right)+\theta_2\hat{\omega}_{kN}^{n}\left(y_j\right)+\left(1-\theta_1-\theta_2\right)\hat{\omega}_{kN}^{n-1}\left(y_j\right)\right]-$$
$$-\nu\sum_{l=0}^{N}d_{j,l}^{(2)}\left[\theta_1\hat{\omega}_{kN}^{n+1}\left(y_l\right)+\theta_2\hat{\omega}_{kN}^{n}\left(y_l\right)+\left(1-\theta_1-\theta_2\right)\hat{\omega}_{kN}^{n-1}\left(y_l\right)\right]=$$
$$=\theta_1\hat{F}_{kN}^{n+1}\left(y_j\right)+\theta_2\hat{F}_{kN}^{n}\left(y_j\right)+\left(1-\theta_1-\theta_2\right)\hat{F}_{kN}^{n-1}\left(y_j\right),$$
$$k=0,\ldots,N/2,\ \ j=1,\ldots N-1,\ \ n=1,\ldots,M.$$

for the spatially discretised momentum equation (25) and

$$\hat{\omega}_j^{n+1} - k^2 \hat{\psi}_j^{n+1} + \sum_{l=0}^{N} d_{j,l}^{(2)} \hat{\psi}_l^{n+1} = 0,$$
$$k = 0,\ldots,N/2, \quad j = 1,\ldots,N-1, \quad n = 1,\ldots,M. \tag{30}$$

for the spatially discretised the definition of vorticity equation (26). Here is denoted $t=n\Delta t$, where Δt-time step, n-the number of time step.

The same procedure must be carried out to boundary and initial conditions.

$$g_{+,N/2}(x,t) = \sum_{k=-N/2}^{N/2} \hat{g}_{+,k}(t) \ e^{I\,k\,x} \approx g_+(x,t) \tag{31}$$

$$g_{-,N/2}(x,t) = \sum_{k=-N/2}^{N/2} \hat{g}_{-,k}(t) \ e^{I\,k\,x} \approx g_-(x,t) \tag{32}$$

$$h_{+,N/2}(x,t) = \sum_{k=-N/2}^{N/2} \hat{h}_{+,k}(t) \ e^{I\,k\,x} \approx h_+(x,t) \tag{33}$$

$$h_{+,N/2}(x,t) = \sum_{k=-N/2}^{N/2} \hat{h}_{+,k}(t) \ e^{I\,k\,x} \approx h_+(x,t) \tag{34}$$

$$\psi_{0,N/2}(x,y) = \sum_{k=-N/2}^{N/2} \hat{\psi}_{0,k}(y) \ e^{I\,k\,x} \approx \psi_0(x,y) \tag{35}$$

Having in mind the expression (14) for streamfunction, boundary conditions (10) and (11), as well as their trigonometric polynomial approximation given by (31)-(35), after implementing Galerkin method and applying the orthogonality relation (22), we obtain the following boundary conditions in space of Fourier coefficients

$$\hat{\psi}_k(1,t) = \hat{g}_{+,k}(t), \quad k = 0,\ldots,N/2. \tag{36}$$

$$\hat{\psi}_k(-1,t) = \hat{g}_{-,k}(t), \quad k = 0,\ldots,N/2, \tag{37}$$

$$\frac{\partial \hat{\psi}_k}{\partial y}(1,t) = \hat{h}_{+,k}(t), \quad k = 0,\ldots,N/2, \tag{38}$$

$$\frac{\partial \hat{\psi}_k}{\partial y}(-1,t) = \hat{h}_{-,k}(t), \quad k = 0,\ldots,N/2. \tag{39}$$

After time discretization and application of Chebyshev collocation method for boundary conditions, the above boundary conditions read as follow

$$\hat{\psi}_{k,0}^{n+1} = \hat{g}_{+,k}^{n+1}, \quad y = 1, \quad k = 0,\ldots,N/2, \tag{40}$$

$$\hat{\psi}_{k,N}^{n+1} = \hat{g}_{-,k}^{n+1}, \quad y = -1, \quad k = 0,\ldots,N/2, \tag{41}$$

$$\sum_{l=0}^{N} d_{0,l}^{(1)} \ \hat{\psi}_{k,l}^{n+1} = \hat{h}_{+,k}^{n+1}, \ \ y = 1, \ \ k = 0,\ldots,N/2, \tag{42}$$

$$\sum_{l=0}^{N} d_{N,l}^{(1)}\,\hat{\Psi}_{k,l}^{n+1} = \hat{h}_{-,k}^{n+1},\ \ y=-1,\ \ k=0,\dots,N/2. \tag{43}$$

This system of equations (29) and (30) together with boundary conditions (40) to (43) should be solved numerically. The system is represented by 2(N+1)×2(N+1) three time levels matrix equation. The nonlinear advective terms have been computed by pseudospectral technique [3], so that full Navier-Stokes equation in vorticity-stream-function formulation can be simulated for the case of 2D viscous channel flow. The problem of two boundary conditions for stream function and none for vorticity has been successfully resolved by applying the influence matrix method [4].

4. TEMPORAL HYDRODYNAMIC STABILTY

In order to simulate the process of instability of viscous fluid flow between two parallel horizontal plates, we solved Orr-Sommerfeld equation of hydrodynamic linear stability for the given velocity profile, in this case for plane Poiseuille flow, and formed the streamfunction perturbation as the linear combination of the eigenfunctions obtained as the solution of this equation. It is well known fact that there is no analytical solution of this equation. The first numerical solution of this equation is given in the paper [5], and the critical Reynolds number of this type of flow has been found to be 5772. In order to show how we perturbed the Poiseuille flow, we start from the equation (8) in which we substitute

$$\omega = \Omega + \omega', \quad \psi = \Psi + \psi', \quad \mathrm{F} = F + f' \tag{44}$$

where we have taken in account the expression (19), so that we have

$$\begin{aligned}&\frac{\partial}{\partial t}(\Omega+\omega') + \frac{\partial}{\partial y}(\Psi+\psi')\frac{\partial}{\partial x}(\Omega+\omega') - \frac{\partial}{\partial x}(\Psi+\psi')\frac{\partial}{\partial y}(\Omega+\omega') = \\ &= F + f' + \nu\left(\frac{\partial^2}{\partial x^2}(\Omega+\omega') + \frac{\partial^2}{\partial y^2}(\Omega+\omega')\right)\end{aligned} \tag{45}$$

The capital letters designates the basic flow values, and the prime denotes perturbations of the corresponding physical values. If we subtract (8) from (45) the above equation is reduced to following form

$$\begin{aligned}&\frac{\partial\omega'}{\partial t} + \frac{\partial\Psi}{\partial y}\frac{\partial\omega'}{\partial x} + \frac{\partial\psi'}{\partial y}\frac{\partial\Omega}{\partial x} + \frac{\partial\psi'}{\partial y}\frac{\partial\omega'}{\partial x} - \frac{\partial\psi'}{\partial x}\frac{\partial\Omega}{\partial y} + \frac{\partial\Psi}{\partial x}\frac{\partial\omega'}{\partial y} - \\ &-\frac{\partial\psi'}{\partial x}\frac{\partial\omega'}{\partial y} = f' + \nu\left(\frac{\partial^2\omega'}{\partial x^2} + \frac{\partial^2\omega'}{\partial y^2}\right)\end{aligned} \tag{46}$$

Neglecting the nonlinear terms of perturbed values as small values of higher order than we have

$$\frac{\partial\omega'}{\partial t} + \frac{\partial\Psi}{\partial y}\frac{\partial\omega'}{\partial x} + \frac{\partial\psi'}{\partial y}\frac{\partial\Omega}{\partial x} - \frac{\partial\psi'}{\partial x}\frac{\partial\Omega}{\partial y} - \frac{\partial\Psi}{\partial x}\frac{\partial\omega'}{\partial y} = f' + \nu\left(\frac{\partial^2\omega'}{\partial x^2} + \frac{\partial^2\omega'}{\partial y^2}\right) \tag{47}$$

In plane channel flow, better to say, in Poiseuille flow we have $U=\partial\Psi/\partial y=1-y^2$, $V=-\partial\Psi/\partial x=0$, $\Omega=-dU/dy$, and also $\partial\Omega/\partial x=0$, so finally we have

$$\frac{\partial\omega'}{\partial t}+\frac{\partial\Psi}{\partial y}\frac{\partial\omega'}{\partial x}-\frac{\partial\psi'}{\partial x}\frac{\partial\Omega}{\partial y}=f'+\nu\left(\frac{\partial^2\omega'}{\partial x^2}+\frac{\partial^2\omega'}{\partial y^2}\right) \tag{48}$$

We anticipate that $f'=0$, and $\omega'=-\Delta\psi'$, so that we have now

$$\frac{\partial}{\partial t}\Delta\psi'+U\frac{\partial}{\partial x}\Delta\psi'-\frac{\partial\psi'}{\partial x}\frac{d^2U}{dy^2}=\nu\,\Delta^2\psi' \tag{49}$$

with boundary conditions

$$\psi'(y=\pm1)=0 \qquad \frac{\partial\psi'}{\partial y}(y=\pm1)=0 \tag{50}$$

We have used modal approach for solving this problem, anticipating the that the perturbations are 2π periodic in x-direction, and are represented in the following way

$$\psi'(x,y,t)=\frac{1}{2}\left(\psi_1+\psi_1^*\right)=\frac{1}{2}\left(\hat\psi(y)e^{i\alpha(x-ct)}+\hat\psi(y)^*e^{-i\alpha^*(x-c^*t)}\right) \tag{51}$$

Here ψ_1 is the normal mode form of stream function perturbation, and the values denoted by $*$ designate their complex-conjugate value. Thus the sum of normal mode and its complex-conjugate gives the real valued function ψ'. Since the complex-conjugate value ψ_1^* can be easily obtained from the complex-valued function ψ_1 itself, it is only necessarily to substitute ψ_1 in the (49), so that it reads

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Delta\hat\psi(y)e^{i\alpha(x-ct)}-\frac{d^2U}{dy^2}\frac{\partial}{\partial x}\hat\psi(y)e^{i\alpha(x-ct)}=\nu\,\Delta^2\hat\psi(y)e^{i\alpha(x-ct)}, \tag{52}$$

which after some differentiation and rearrangements gives

$$(U-c)\left(\frac{d^2\hat\psi}{dy^2}-\alpha^2\hat\psi\right)-\hat\psi\frac{d^2U}{dy^2}=\frac{\nu}{i\alpha}\left(\alpha^4\hat\psi-2\alpha^2\frac{d^2\hat\psi}{dy^2}+\frac{d^4\hat\psi}{dy^4}\right). \tag{53}$$

This equation known as Orr-Sommerfeld equation, together with appropriate boundary conditions

$$\hat\psi(1)=0,\quad \hat\psi(-1)=0,\quad \frac{d\hat\psi}{dy}(1)=0,\quad \frac{d\hat\psi}{dy}(-1)=0, \tag{54}$$

should be solved numerically. Numerical procedures are given for examples in [5], [6], [7] and [8]. This equation can be reduced to operator form

$$(U-c)\left(\mathrm{D}^2\hat{\psi}-\alpha^2\hat{\psi}\right)-\hat{\psi}\frac{d^2U}{dy^2}=\frac{\nu}{i\alpha}\left(\alpha^4\hat{\psi}-2\alpha^2\mathrm{D}^2\hat{\psi}+\mathrm{D}^4\hat{\psi}\right) \tag{55}$$

where is $\Delta = d/dy$ differential operator, which can be solved as generalized eigenvalue problem, where c's are eigenvalues and $\hat{\psi}$'s are eigenfunctions.

We can fix now two parameters α and ν, and compute the eigenvalue c, what correspods to the case of temporal hydrodynamic stability. If we choose to fix c and ν, and to compute α, that would be the case of spatial hydrodynamic stability. In the first case we have $\alpha\in\hat{\hbar}$ and $c\in\hbar$, and in the second case $c\in\hat{\hbar}$ and $\alpha \in\hbar$. Here c is the velocity of traveling wave, and α is its wave number. In general case $c=c_{Re}+ i\, c_{Im}$ and $\alpha=\alpha_{Re}+ i\, \alpha_{Im}$, but since we have only one equation and two unknowns, we have to make some assumptions concerning these unknowns. In our calculations we have anticipated $\alpha=1$.

The results obtained for $Re=1000$ ($\nu=1/1000$) are presented in fig.1 and are obtained for the N=128 Gauss-Lobatto-Chebyshev point in y-directions. For creating the perturbation that can exhibit the transieth growth mechanism, we have used the optimized linear combination of all eigenvectors which is normalized with regard to the least stable eigenvalue. This optimized perturbation was superposed to the initial unperturbed velocity profile, and the flow was driven by the force term determined from the perturbed Navier Stokes equation. This transient growth is possible due to non-normality of Orr-Sommerfeld operator, but the all eigenvalues and eigenvectors have to be used for creating linear combination, not only the least stable eigenvalue and correspoding eigenvector, see [9],[10],[11]. The results of simulation Fig.1and Fig.2 are given in the next section for the dimensionless time $t=n\pi$, $n=1,...,10$.

5. THE RESULTS OF TRANSIENT FLOW SIMULATION

The initial condition for our simulation is the solution of the problem for laminar Poiseuille 2D-flow is

$$U(x,y,0)=1-y^2,\quad V(x,y,0)=0,\quad on\quad \Omega. \tag{56}$$

Our goal is to simulate the transition process from laminar to perturbed state for the value of Reynolds number $Re=1000$ which is beneath the critical value $Re_c=5772$, to simulate the transient growth of kinetic energy and enstrophy. We have carried out this simulation by imposing the perturbations obtained by solution of Orr-Sommerfeld equation on laminar velocity profile. The simulations are driven by forcing term which is determined by the perturbed Navier-Stokes equation,

$$\begin{aligned} F_{pert} &= \frac{\partial}{\partial t}(\Omega+\omega')+\frac{\partial}{\partial y}(\Psi+\psi')\frac{\partial}{\partial x}(\Omega+\omega')- \\ &-\frac{\partial}{\partial x}(\Psi+\psi')\frac{\partial}{\partial y}(\Omega+\omega')-\nu\left[\frac{\partial^2}{\partial x^2}(\Omega+\omega')+\frac{\partial^2}{\partial y^2}(\Omega+\omega')\right]. \end{aligned} \tag{57}$$

Here Ω and Ψ are the values determined from (56) at initial time, and later they are results obtained from our numerical procedure and our MATLAB code, and the ψ' and

ω' are obtained as solution of Orr-Sommerfeld equation, and as optimized linear combination of all eigenvectors,

$$\psi'(x,y,t)=\sum_{n=1}^{N'}\beta_n\hat{\psi}_n(y)e^{i\alpha(x-c_n t)}=\sum_{n=1}^{N'}\beta_n\hat{\psi}_n(y)e^{i\alpha\left(x-(c_{\mathcal{Re}}+ic_{\mathcal{Im}})_n t\right)} \tag{58}$$

and,

$$\omega'(x,y,t)=-\Delta\psi'. \tag{59}$$

Here $\hat{\psi}_k$'s are eigenvectors and c_n's are eigenvalues of generalized eigenvalue problem of Orr-Sommerfeld equation for the case of plane Poiseuille flow, and β_n are coefficient which should be determined by appropriate optimization procedure. The procedure we used in this paper is according to [7, p.121,fig.4.7] and is based on method first developed in [11].

Here β is perturbation spectar obtained by using the matrix Ψ, whose columns are eigenvectors $\hat{\psi}_n(y)$, in the following way

$$\beta=\Psi^{-1}\psi'(y,0). \tag{60}$$

Functional to be minimized is

$$f=\psi'^{*}\cdot\psi'=(\Psi\beta)^{*}\cdot(\Psi\beta)=\beta^{*}\cdot\Psi^{*}\,\Psi\beta=\beta^{*}\cdot A\beta\,. \tag{61}$$

In other words, the functional is the dot product of perturbation vector of stream function and its complex conjugate. If we put the condition that the *i*-th mode is of unit magnitude, then the variational problem can be reduced to the following function

$$f=\beta^{*}\cdot A\beta+\lambda(\beta\cdot\boldsymbol{e}_i-1). \tag{62}$$

Here we have designated with e_i – the unit vector, i.e. the column vector whose the only the *i-th* element is different from null. Let find the derivative with respect to β, e.i. let find the first variation of the above function f and equal it with zero, so that we have

$$\frac{d f}{d\beta}=\frac{d}{d\beta}\left[\beta^{*}\cdot A\beta+\lambda(\beta\cdot\boldsymbol{e}_i-1)\right]=A\beta+\lambda\,\boldsymbol{e}_i=\mathbf{0}. \tag{63}$$

And after rearrangements

$$A\beta=-\lambda\,\boldsymbol{e}_i\,, \tag{64}$$

so that after multiplication both side with inverse matrice $A^{-1}$from the left we have

$$\beta=-\lambda\,A^{-1}\boldsymbol{e}_i\,. \tag{65}$$

The optimized spectar can be normalized by appropriate calculation of coefficient λ, so that the value β_i=1 can be obtained. Having this in mind we have

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_N \end{bmatrix} = -\lambda \begin{bmatrix} a_{11}^{-1} & a_{12}^{-1} & \dots & a_{1i}^{-1} & \dots & a_{1N}^{-1} \\ a_{21}^{-1} & a_{22}^{-1} & \dots & a_{2i}^{-1} & \dots & a_{2N}^{-1} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1}^{-1} & a_{i2}^{-1} & \dots & a_{ii}^{-1} & \dots & a_{iN}^{-1} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{N1}^{-1} & a_{N2}^{-1} & \dots & a_{Ni}^{-1} & \dots & a_{NN}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \tag{66}$$

and the value of λ is determined by this expression

$$\lambda = \frac{-\beta_i}{a_{ii}^{-1}} = \frac{-1}{a_{ii}^{-1}}. \tag{67}$$

Fig.1 shows the vorticity fields for six different times, for dimensionless time $t=n\pi$, $n=1,...,6$. Since the least stable eigenvalue for $Re=1000$ equals $c=0.3462-i0.0421$, we can see that the velocity of traveling wave perturbation optimized with regard to this mode has 34.62% of the fluid velocity in the middle of the channel. In the time $t=\pi$ we can see the vorticity distribution similar to that given in reference [6, p.121, fig.4.7], and as we can see the middle of the channel is almost unperturbed. In the next time for $t=2\pi$ we can see that vorticity has advected and diffused in the direction of the middle of channel, and that the maximal amplitudes has been decreased, what is in our opinion the consequence of the relaxation from the initial perturbation to the exact solution to Navier-Stokes given by (56). This can be seen on colorbars in the first two figures, where on the first one we have maximal magnitude 6.2 and on the second one 5.5 in dimensionless vorticity units. In the next instant of time for $t=3\pi$, it can be observed the merging of perviously separated vortexes with the same sign, which are deformed due to wall normal velocity perturbation, which is not shown here due to space limitation.

In $t=4\pi$, two circled vortexes have been formed with the centers located at $y=0.2$ positive one and $y=-0.2$ the negative one. The positive one with counter-clockwise rotation (red color) and negative vortex with clockwise rotation (blue color), in the middle of the channel, where this vortex pair is being deformed by the velocity gradient of the flud flow. The vortex pairs on upper and lower wall are in the form of romboid since the velocity of perturbation is greater than the velocity of surronding fluid near the walls, because the velocity of perturbation is $c_{Re}=0.3462$ and velocity of the fluid is given by (56). This velocity difference decreases with going away from the walls according with this expression till the normal coordinate reaches the value where these two velocities are equal, better to say, till to the value of critical fluid layer $y\approx0.8$, since $U=1-0.64=0.36$. So we can notice that the velocity of perturbation traveling wave (phase velocity) is much higher in the wall region than the streamwise fluid velocity, but opposite is valid for the middle of the channel, where fluid velocity U ($-0.8<y<0.8$) is greater than phase velocity.

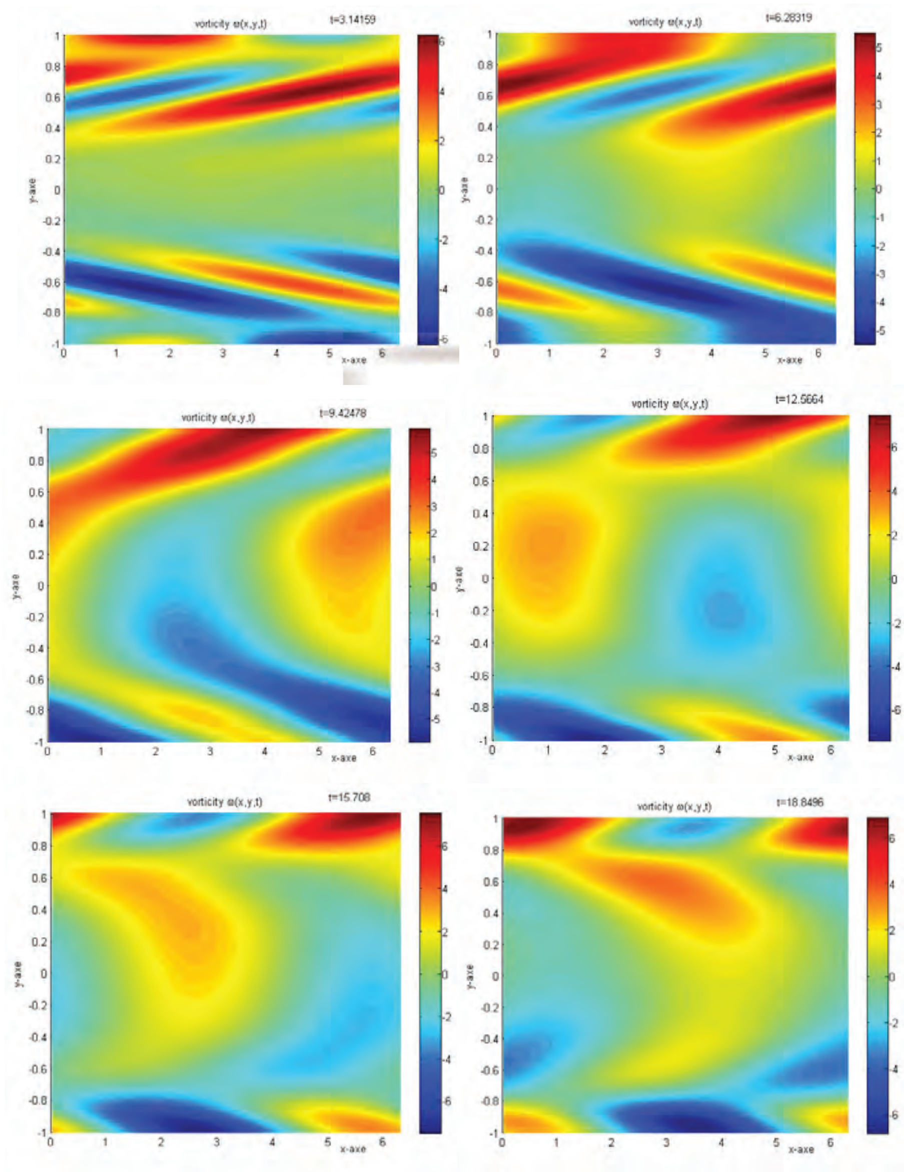

Fig.1. vorticity distribution $\omega(x,y,t)$ in 2D channel viscous fluid flow for $t=\pi,\ldots,6\pi$ for streamfunction perturbation optimized to least stable eigenmode.

In the next instant of time for $t=5\pi$ it can be noticed that these vortex have been deformed in streamwise direction, by the mean velocity gradient., and this process is continued in the next instant of time $t=6\pi$.

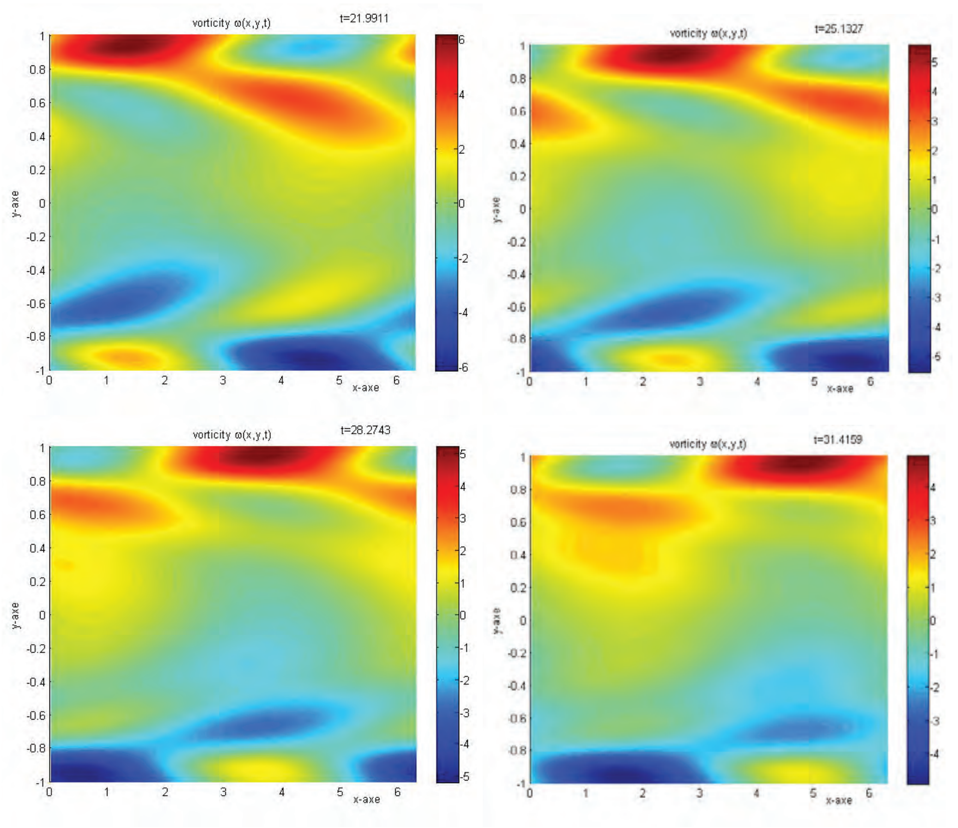

Fig.2: vorticity distribution $\omega(x,y,t)$ in 2D channel viscous fluid flow for t=7π,…,10π for streamfunction perturbation optimized to least stable eigenmode.

In the next vorticity fields for t=7,...,10π, the process of vortex distorsion is continued, and advected in downstream direction. The maximal and minimal values of vorticity are on the upper and lower wall respectively , and their values decrease with increase of time. Vortex pair on lower plate consist of two vortexes, the negative one which atains the value $\omega_{min}=-4.8$ and positive one with the value $\omega_{max}=1$ at instant of time $t=10\pi$. The opposite is true on the upper wall; the negative vortex attains the value $\omega_{min}=-1$ and vortex with counter-clockwise rotation (red color) reaches the value $\omega_{max}=4.8$. These vortex pairs moves in downstream direction with phase velocity $c_{Re}=0.3462$, which can be seen on the figures above, since the displacement of the center of the vortex between two instant of time can be determined in the following way: $s=c_{Re}\,\Delta t=0.3462\cdot 3.1416=1.0876$, and this is what we can see on this fig.2 between four different instant of time.

6. CONCLUSION

The most important results can be seen on the colorbars for ten different instant of times. It can be noticed that the maximal vorticity displayed on colorbars attains its maximal value at time t=4π (ω_{max}=6.85) and t=5π (ω_{max}=6.75), and afterwards the intensity of vorticity monotonically decreases, so that for *t*=10π we have the value ω_{max} =4.94. In this way we have two time periods, the first one when the maximal values of vorticity increases with time until it reaches t=4π, and second one when the extreme values of vortex intensity decline and the kinetic energy and enstrophy are monotonicaly decreasing functions of time.

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EVOLUCIJA VRTLOŽNOSTI U POREMEĆENOM POASEJEVOM STRUJANJU

Miloš M. Jovanović

Apstrakt: U radu se razmatra direktna numerička simulacija vrtložnosti viskoznog nestišljivog fluida za slučaj Poasejevog strujanja (strujanja između horizontalnih paralelnih ploča pod dejstvom gradienta pritiska u horizontalnom pravcu), kod koga je polju strujne funkcije fluida pridodato poremećajno polje konačne amplitude. Ovo poremećajno polje je dobijeno optimizacijom linearne kombinacije svih sopstvenih vektora dobijenih kao rešenje Orr-Sommerfeld-ove jednačine za granične uslove koji odgovaraju opisanom primeru.

Navije-Stoksova jednačina u obliku strujna funkcija–vrtložnost je numerički simulirana korišćenjem pseudospekttralnog metoda. Za aproksimaciju u pravcu x-ose korišćen je Furije-Galerkinov metod, dok u nehomogenom pravucu, u pravcu y-ose korišćen je Čebišeljev kolokacioni metod. Za diskretizaciju po vremenu korišćen je poluimplicitni metod Adams-Bašvorta koji je drugog reda tačnosti. U radu su prikazana poremećajna polja vrtložnosti za deset različitih trenutaka vremena, u periodu tzv. prelaznog rasta energije do trenutka kada ona počinje da opada, odnosno do početka procesa relaminarizacije.

Ključne reči: *Direktna numerička simualacija, poremećajno Poasejevo strujanje, podkritična nesabilnost strujanja. optimizovani poremećaji, pseudospektralni metod.*

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