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DELAY DEPENDENT STABILITY OF LINEAR TIME-DELAY SYSTEMS

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Abstract. *This paper deals with the problem of delay dependent stability for both ordinary and large-scale time-delay systems. Some necessary and sufficient conditions for delay-dependent asymptotic stability of continuous and discrete linear time-delay systems are derived. These results have been extended to the large-scale time-delay systems covering the cases of two and multiple existing subsystems. The delay-dependent criteria are derived by Lyapunov's direct method and are exclusively based on the solvents of particular matrix equation and Lyapunov equation for non-delay systems. Obtained stability conditions do not possess conservatism. Numerical examples have been worked out to show the applicability of results derived.*

Key words (bold): *continuous time-delay systems, discrete time-delay systems, large-scale time-delay systems, delay-dependent stability, Lyapunov stability, necessary and sufficient conditions*

1. INTRODUCTION

The problem of investigation of time-delay systems has been exploited over many years. Time-delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability.

During the last three decades, the problem of stability analysis of time-delay systems has received considerable attention and many papers dealing with this problem have appeared. In the literature, various stability analysis techniques have been utilized to derive stability criteria for asymptotic stability of the time-delay systems by many researchers.

The developed stability criteria are classified often into two categories according to their dependence on the size of the delay: delay-dependent and delay-independent stability criteria. It has been shown that delay-dependent stability conditions that take into account the size of delays, are generally less conservative than delay-independent ones which do not include any information on the size of delays.

Further, the delay-dependent stability conditions can be classified into two classes: frequency-domain (which are suitable for systems with a small number of heterogeneous delays) and time-domain approaches (for systems with a many heterogeneous delays).

In the first approach, we can include the two or several variable polynomials [1], [2] or the small gain theorem based approach.

In the second approach, we have the comparison principle based techniques for functional differential equations [3], [4] and respectively the Lyapunov stability approach with the Krasovskii and Razumikhin based methods [5], [6]. The stability problem is thus reduced to one of finding solutions to Lyapunov [7] or Riccati equations [8], solving linear matrix inequalities (LMIs) [9], [10], [11] [12] or analyzing eigenvalue distribution of appropriate finite-dimensional matrices [13] or matrix pencils [14]. For further remarks on the methods see also the guided tours proposed by [15], [16], [17], [18], [19], [20].

It is well-known [21] that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving stability conditions. The general form of this functional leads to a complicated system of partial differential equations [22]. Special forms of Lyapunov–Krasovskii functionals lead to simpler delay-independent (Boyd et al., 1994; Verriest & Niculescu, 1998; Kolmanovskii & Richard, 1999) [9], [23], [21] and (less conservative) delay-dependent conditions [24], [25], [21], [26], [27], [28]. Note that the latter simpler conditions are appropriate in the case of unknown delay, either unbounded (delay-independent conditions) or bounded by a known upper bound (delay-dependent conditions).

In the delay-dependent stability case, special attention has been focused on the first delay interval guaranteeing the stability property, under some appropriate assumptions on the system free of delay. Thus, algorithms for computing optimal (or suboptimal) bounds on the delay size are proposed in [14] (frequency-based approach), in [29] (integral quadratic constraints interpretations), in [10], [11], [7] (Lyapunov-Razumikhin function approach) or in [12] (discretization schemes for some Lyapunov- Krasovskii functionals). For computing general delay intervals, see, for instance, the frequency based approaches proposed in [30].

In the past few years, there have been various approaches to reduce the conservatism of delay-dependent conditions by using new bounding for cross terms or choosing new Lyapunov–Krasovskii functional and model transformation. The delay-dependent stability criterion of [31], [26] is based on a so-called Park's inequality for bounding cross terms. However, major drawback in using the bounding of [31] and [26] is that some matrix variables should be limited to a certain structure to obtain controller synthesis conditions in terms of LMIs. This limitation introduces some conservatism. In [32] a new inequality, which is more general than the Park's inequality, was introduced for bounding cross terms and controller synthesis conditions were presented in terms of nonlinear matrix inequalities in order to reduce the conservatism. It has been shown that the bounding technique in [32] is less conservative than earlier ones. An iterative algorithm was developed to solve the nonlinear matrix inequalities [32].

Further, in order to reduce the conservatism of these stability conditions, various model transformations have been proposed. However, the model transformation may introduce additional dynamics. In [33] the sources for the conservatism of the delay-dependent methods under four model transformations, which transform a system with discrete delays into one with distributed delays are analyzed. It has been demonstrated that descriptor transformation, that has been proposed in [34], leads to a system which is equivalent to the original one, does not depend on additional assumptions for stability of the transformed system and requires bounding of fewer cross-terms. In order to reduce the conservatism [35], [36] proposed some new methods to avoid using model transformation and bounding technique for cross terms.

In [37] both the descriptor system approach and the bounding technique using by [32] are utilized and the delay-dependent stability results are performed. The derived stability criteria have been demonstrated to be less conservative than existing ones in the literature.

Delay-dependent stability conditions in terms of linear matrix inequalities (LMIs) have been obtained for retarded and neutral type systems. These conditions are based on four main model transformations of the original system and application mentioned inequalities.

The majority of stability conditions in the literature available, of both continual and discrete time-delay systems, are sufficient conditions. Only a small number of works provide both necessary and sufficient conditions [38], [39], [47], [49], [50], [53] which are in their nature mainly dependent of time-delay. These conditions do not possess conservatism but often require more complex numerical computations. In our paper we represent some necessary and sufficient stability conditions.

Less attention has been drawn to the corresponding results for discrete-time delay systems [40], [41], [42], [43], [44], [45], [54]. This is mainly due to the fact that such systems can be transformed into augmented high dimensional systems (equivalent systems) without delay [22], [46]. This augmentation of the systems is, however, inappropriate for systems with unknown delays or systems with time varying delays. Moreover, for systems with large known delay amounts, this augmentation leads to large-dimensional systems. Therefore, in these cases the stability analysis of discrete time-delay systems can not be to reduce on stability of discrete systems without delay.

In our paper we present delay-dependent stability criteria for particular classes of time-delay systems: continuous and discrete time-delay systems and continuous and discrete time-delay large-scale systems. Thereat, these stability criteria are express in form necessary and sufficient conditions.

2. STABILITY OF TIME-DELAY SYSTEMS

Throughout this paper we use the following notation. R and C denote real (complex) vector space or the set of real (complex) numbers, T^+ denotes the set of all non-negative integers, λ^* means conjugate of $\lambda \in C$ and F^* conjugate transpose of matrix $F \in C^{n\times n}$.

Re(s) is the real part of $s \in C$. The superscript T denotes transposition. For real matrix F the notation $F > 0$ means that the matrix F is positive definite. $\lambda_i(F)$ is the eigenvalue of matrix F. Spectrum of matrix F is denoted with $\sigma(F)$ and spectral radius with $\rho(F)$.

2.1. Continuous time-delay systems

Considers class of continuous time-delay systems described by

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau), \quad \mathbf{x}(t) = \varphi(t), \quad -\tau \le t < 0 \tag{1}$$

Theorem 1. [38] Let the system be described by (1). If for any given matrix $Q = Q^* > 0$ there exist matrix $P = P^* > 0$, such that

$$P\left(A_0 + T(0)\right) + \left(A_0 + T(0)\right)^T P = -Q \tag{2}$$

where $T(t)$ is continuous and differentiable matrix function which satisfies

$$\dot{T}(t) = \begin{cases} \left(A_0 + T(0)\right)T(t), & 0 \le t \le \tau, \quad T(\tau) = A_1 \\ 0, & t > \tau \end{cases} \tag{3}$$

then the system (1) is asymptotically stable.

In paper [38] it is emphasized that the key to the success in the construction of a Lyapunov function corresponding to the system (1) is the existence of at least one solution $T(t)$ of (3) with boundary condition $T(\tau) = A_1$. In other words, it is required that the nonlinear algebraic matrix equation

$$e^{\left(A_0 + T(0)\right)\tau}\, T(0) = A_1 \tag{4}$$

has at least one solution for $T(0)$. It is asserted, there, that asymptotic stability of the system (*Theorem 1*) can be determined based on the knowledge of only one or any solution of the particular nonlinear matrix equation. However, [47] gives counterexample which denies this maintenance.

2.1.1 Main results

If we introduce a new matrix,

$$R \triangleq A_1 + T(0) \tag{5}$$

then condition (2) reads

$$PR + R^*P = -Q \tag{6}$$

which presents a well-known Lyapunov's equation for the system without time-delay. This condition will be fulfilled if and only if R is a stable matrix:

$$\mathrm{Re}\,\lambda_i(R) < 0 \tag{7}$$

Let Ω_T and Ω_R denote sets of all solutions of eq. (4) per T(0) and (6) per R, respectively.

Equation (4) can be written in a different form as follows,

$$R - A_0 - e^{-R\tau} A_1 = 0 \tag{8}$$

and there follows

$$\det\left(R - A_0 - e^{-R\tau} A_1\right) = 0 \tag{9}$$

Substituting a matrix variable R by scalar variable s in (7), the characteristic equation of the system (1) is obtained as

$$f(s) = \det\left(sI - A_0 - e^{-s\tau} A_1\right) = 0 \tag{10}$$

Let us denote

$$\Sigma \triangleq \{s \mid f(s) = 0\} \tag{11}$$

a set of all characteristic roots of the system (1). The necessity for the correctness of desired results, forced us to propose new formulations of *Theorem 1*.

Theorem 2. [47] Suppose that there exist(s) the solution(s) $T(0) \in \Omega_T$ of (4). Then, the system (1) is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (2) holds for all solutions $T(0) \in \Omega_T$ of (4).

Conclusion 1. Statement *Theorem 2* require that condition (2) is fulfilled for all solutions $T(0) \in \Omega_T$ of (4). In other words, it is requested that condition (7) holds for all solution R of (8), especially for $R = R_{\max}$, where the matrix $R_m \in \Omega_R$ is maximal solvent of (8) that contains eigenvalue with a maximal real part $\lambda_m \in \Sigma$: $\mathrm{Re}\,\lambda_m = \max_{s \in \Sigma} \mathrm{Re}\, s$. Therefore, from (7) follows condition $\mathrm{Re}\,\lambda_i(R_m) < 0$.

On the basis of *Conclusion 1*, it is possible to reformulate *Theorem 2* in the following way.

Theorem 3. [47] Suppose that there exists maximal solvent R_{m} of (8). Then, the system (1) is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (6) holds for the solution $R = R_{\mathrm{m}}$ of (8).

2.2 Continuous large scale time-delay systems

Consider a linear continuous large scale time-delay autonomous systems composed of N interconnected subsystems. Each subsystem is described as:

$$\dot{\mathbf{x}}_i(t) = A_i\mathbf{x}_i(t) + \sum_{j=1}^{N} A_{ij}\mathbf{x}_j(t-\tau_{ij}),\ 1 \le i \le N \tag{12}$$

with an associated function of initial state $\mathbf{x}_i(\theta) = \varphi_i(\theta)$, $\theta \in [-\tau_{m_i},\ 0]$, $1 \le i \le N$. $\mathbf{x}_i(t) \in R^{n_i}$ is state vector, $A_i \in R^{n_i \times n_i}$ denote the system matrix, $A_{ij} \in \mathbf{R}^{n_i \times n_j}$ represents the interconnection matrix between the i-th and the j-th subsystems, and τ_{ij} is constant delay.

For the sake of brevity, we first observe system (12) made up of two subsystems ($N = 2$). For this system, we derive new necessary and sufficient delay-dependent conditions for stability, by Lyapunov's direct method. The derived results are then extended to the linear continuous large scale time-delay systems with multiple subsystems.

2.2.1. Main results

Theorem 4. [49] Given the following system of matrix equations (SME)

$$R_1 - A_1 - e^{-R_1\tau_{11}}A_{11} - e^{-R_1\tau_{21}}S_2A_{21} = 0 \tag{13}$$

$$R_1S_2 - S_2A_2 - e^{-R_1\tau_{12}}A_{12} - e^{-R_1\tau_{22}}S_2A_{22} = 0 \tag{14}$$

where A_1, A_2, A_{12}, A_{21} and A_{22} are matrices of system (12) for $N = 2$, n_i subsystem orders and τ_{ij} time-delays of the system. If there exists solution of SME (13)-(14) upon unknown matrices $R_1 \in R^{n_1 \times n_1}$ and $S_2 \in C^{n_1 \times n_2}$, then the eigenvalues of matrix R_1 belong to a set of roots of the characteristic equation of system (12) for $N = 2$.

Proof. By introducing the time-delay operator $e^{-\tau s}$, the system (12) can be expressed in the form

$$\begin{gathered}\dot{\mathbf{x}}(t) = \begin{bmatrix} A_1 + A_{11}e^{-\tau_{11}s} & A_{12}e^{-\tau_{12}s} \\ A_{21}e^{-\tau_{21}s} & A_2 + A_{22}e^{-\tau_{22}s} \end{bmatrix}\mathbf{x}(t) = A_e(s)\mathbf{x}(t), \\ \mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1^T(t) & \mathbf{x}_2^T(t) \end{bmatrix}^T \end{gathered} \tag{15}$$

Let us form the following matrix

$$F(s) = [F_{ij}(s)] = sI_{n_1+n_2} - A_e(s) = \begin{bmatrix} sI_{n_1} - A_1 - A_{11}e^{-\tau_{11}s} & -A_{12}e^{-\tau_{12}s} \\ -A_{21}e^{-\tau_{21}s} & sI_{n_2} - A_2 - A_{22}s^{-\tau_{22}s} \end{bmatrix} \tag{16}$$

Its determinant is

$$\det F(s) = \det \begin{bmatrix} F_{11}(s) + S_2 F_{21}(s) & F_{12}(s) + S_2 F_{22}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} \\ = \det \begin{bmatrix} G_{11}(s, S_2) & G_{12}(s, S_2) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \tag{17}$$

$$G_{11}(s, S_2) = sI_{n1} - A_1 - A_{11}e^{-\tau_{11}s} - S_2 A_{21} e^{-\tau_{21}s} \tag{18}$$

$$G_{12}(s, S_2) = sS_2 - S_2 A_2 - A_{12}e^{-\tau_{12}s} - S_2 A_{22} e^{-\tau_{22}s} \tag{19}$$

The characteristic polynomial of system (12) for $N = 2$, defined by

$$f(s) \hat{=} \det\left(sI_N - A_e(s)\right) = \det G(s, S_2) \tag{20}$$

is independent of the choice of matrix S_2, because the determinant of matrix $G(s, S_2)$ is invariant with respect to elementary row operation of type 3. Let us designate a set of roots of the characteristic equation of system (12) by $\Sigma \hat{=} \{ s \mid f(s) = 0 \}$. Substituting scalar variable s by matrix X in $G(s, S_2)$ we obtain

$$G(X, S_2) = \begin{bmatrix} G_{11}(X, S_2) & G_{12}(X, S_2) \\ G_{21}(X) & G_{22}(X) \end{bmatrix} \tag{21}$$

If there exist transformational matrix S_2 and matrix $R_1 \in \mathbf{C}^{n_1 \times n_1}$ such that $G_{11}(R_1, S_2) = 0$ and $G_{12}(R_1, S_2) = 0$ is satisfied, i.e. if (13)-(14) hold, then

$$f(R_1) = \det G_{11}(R_1, S_2) \cdot \det G_{22}(R_1) = 0 \tag{22}$$

So, the characteristic polynomial (20) of system (12) is annihilating polynomial [48] for the square matrix R_1, defined by (13)-(14). In other words, $\sigma(R_1) \subset \Sigma$.

Theorem 5. [49] Given the following SME

$$R_2 - A_2 - e^{-R_2\tau_{12}} S_1 A_{12} - e^{-R_2\tau_{22}} A_{22} = 0 \tag{23}$$

$$R_2 S_1 - S_1 A_1 - e^{-R_2\tau_{11}} S_1 A_{11} - e^{-R_2\tau_{21}} A_{21} = 0 \tag{24}$$

where A_1, A_2, A_{12}, A_{21} and A_{22} are matrices of system (12) for $N = 2$, n_i subsystem orders and τ_{ij} time-delays of the system. If there exists solution of SME (23)-(24) upon unknown matrices $R_2 \in \mathbf{C}^{n_2 \times n_2}$ and $S_1 \in \mathbf{C}^{n_2 \times n_1}$, then the eigenvalues of matrix R_2 belong to a set of roots of the characteristic equation of system (12) for $N = 2$.

Proof. Proof is similarly with the proof of *Theorem 4*.

Definition 1. The matrix R_1 (R_2) is referred to as *solvent* of SME (13)-(14) ((23)-(24)).

Definition 2. Each root λ_m of the characteristic equation (20) of the system (12) which satisfies the following condition: $\operatorname{Re}\lambda_m = \max \operatorname{Re} s,\ \ s \in \Sigma$ will be referred to as *maximal eigenvalue* of system (12).

Definition 3. Each solvent R_{1m} (R_{2m}) of SME (13)-(14) ((23)-(24)), whose spectrum contains maximal eigenvalue λ_m of system (12), is referred to as *maximal solvent* of SME (13)-(14) ((23)-(24)).

Theorem 6. [49] Suppose that there exists maximal solvent of SME (23)-(24) and let R_{1m} denote one of them. Then, system (12), for $N = 2$, is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$R_{1m}^* \ P + P \ R_{1m} = -Q \tag{25}$$

Proof. *Sufficient condition.* Define the following vector continuous functions

$$\begin{aligned} \mathbf{x}_{ti} &= \mathbf{x}_i\left(t+\theta\right), \quad \theta \in \left[-\tau_{m_i},\ 0\right], \\ \mathbf{z}\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) &= \sum_{i=1}^{2} S_i \left(\mathbf{x}_i\left(t\right) + \sum_{j=1}^{2} \int_0^{\tau_{ji}} T_{ji}\left(\eta\right) \mathbf{x}_i\left(t-\eta\right) d\eta \right) \end{aligned} \tag{26}$$

where $T_{ji}\left(t\right) \in \mathbf{C}^{n_i \times n_i}$, $j = 1, 2$ are varying continuous matrix functions and $S_1 = I_{n_1}$, $S_2 \in \mathbf{C}^{n_1 \times n_2}$.

The proof of the theorem follows immediately by defining Lyapunov functional for system (12) as

$$V\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) = \mathbf{z}^*\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) P \, \mathbf{z}\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right), \quad P = P^* > 0 \tag{27}$$

Derivative of (27), along the solutions of system (12) is

$$\dot{V}\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) = \dot{\mathbf{z}}^*\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) P \ \mathbf{z}\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) + \mathbf{z}^*\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) P \ \dot{\mathbf{z}}\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) \tag{28}$$

$$\begin{aligned} \dot{\mathbf{z}}\left(\mathbf{x}_{t1}, \mathbf{x}_{t2}\right) = \sum_{i=1}^{2} \Bigg\{ & S_i \left(A_i + \sum_{j=1}^{2} T_{ji}\left(0\right) \right) \mathbf{x}_i\left(t\right) \\ & + \sum_{j=1}^{2} \left(S_j A_{ji} - S_i T_{ji}\left(\tau_{ji}\right) \right) \mathbf{x}_i\left(t - \tau_{ji}\right) + \sum_{j=1}^{2} \int_0^{\tau_{ji}} S_i T_{ji}'\left(\eta\right) \mathbf{x}_i\left(t-\eta\right) d\eta \Bigg\} \end{aligned} \tag{29}$$

If we define new matrices

$$R_i = A_i + \sum_{j=1}^{2} T_{ji}\left(0\right), \ \ i = 1, 2 \tag{30}$$

and if one adopts

$$S_i T_{ji}\left(\tau_{ji}\right) = S_j A_{ji}\,, \ \ i, j = 1, 2 \tag{31}$$

$$S_i T_{ji}^{'}(\eta) = R_1 S_i T_{ji}(\eta), \quad S_i R_i = R_1 S_i, \quad i, j = 1, 2 \tag{32}$$

then

$$\dot{\mathbf{z}}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) = R_1 \mathbf{z}(\mathbf{x}_{t1}, \mathbf{x}_{t2}), \quad \dot{V}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) = \mathbf{z}^*(\mathbf{x}_{t1}, \mathbf{x}_{t2})(R_1^* P + P R_1)\mathbf{z}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) \tag{33}$$

It is obvious that if the following equation is satisfied

$$R_1^* P + P R_1 = -Q < 0\,, \tag{34}$$

then $\dot{V}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) < 0$, $\forall \mathbf{x}_{ti} \neq \mathbf{0}$.

In the Lyapunov matrix equation (25), of all possible solvents R_1 only one of maximal solvents R_{1m} is of importance, because it is containing maximal eigenvalue $\lambda_m \in \Sigma$, which has dominant influence on the stability of the system.

Necessary condition. Let us assume that system (12) for $N = 2$ is asymptotically stable, i.e. $\forall s \in \Sigma$, $\operatorname{Re} s < 0$ hold. Since $\sigma(R_{1m}) \subset \Sigma$ follows $\operatorname{Re}\lambda(R_{1m}) < 0$ and the positive definite solution of Lyapunov matrix equation (25) exists.

From (31)-(32) follows

$$S_j A_{ji} = e^{R_1 \tau_{ji}} S_i T_{ji}(0), \;\; S_1 = I_{n_1}, \;\; i = 1, 2, \; j = 1, 2 \tag{35}$$

Using (30) and (35), for $i = 1$, we obtain (13). Multiplying (30) (for $i = 2$) from the left by matrix S_2 and using (32) and (35) we obtain (14). Taking a solvent with eigenvalue $\lambda_m \in \Sigma$ (if it exists) as a solution of the system of equations (13)-(14), we arrive at a maximal solvent R_{1m}.

Theorem 7. [49] Suppose that there exists maximal solvent of SME (23)-(24) and let R_{2m} denote one of them. Then, system (12), for $N = 2$, is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$R_{2m}^* P + P R_{2m} = -Q \tag{36}$$

Proof. Proof is almost identical to that exposed for *Theorem 6*.

Theorem 8. [49] Given the following system of matrix equations

$$R_k S_i - S_i A_i - \sum_{j=1}^{N} e^{-R_k \tau_{ji}} S_j A_{ji} = 0, \quad S_i \in \mathbf{C}^{n_k \times n_i}, \quad S_k = I_{n_k}, \quad 1 \leq i \leq N \tag{37}$$

for a given k, $1 \leq k \leq N$, where A_i and A_{ji}, $1 \leq i \leq N$, $1 \leq j \leq N$ are matrices of system (12) and τ_{ji} is time-delay in the system. If there is a solvent of (37) upon unknown matrices $R_k \in \mathbf{C}^{n_k \times n_k}$ and S_i, $1 \leq i \leq N$, $i \neq k$, then the eigenvalues of matrix R_k belong to a set of roots of the characteristic equation of system (12).

Proof. Proof of this theorem is a generalization of proof of *Theorem* 4 or *Theorem* 5.

Theorem 9 [49] Suppose that there exists maximal solvent of (37) for given k,

$1 \le k \le N$ and let R_{km} denote one of them. Then, linear discrete large scale time-delay system (12) is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$R_{km}^{*}\ P + P\ R_{km} = -Q \tag{38}$$

Proof. Proof is based on generalization of proof for *Theorem* 6 or *Theorem* 7. It is sufficient to take arbitrary N instead of $N = 2$.

Example 1. Consider following continuous large scale time-delay system with delay interconnections

$$\dot{x}_1(t) = A_1 x_1(t) + A_{12} x_2(t-\tau_{12}), \quad \dot{x}_2(t) = A_2 x_2(t) + A_{21} x_1(t-\tau_{21}) + A_{23} x_3(t-\tau_{23})$$
$$\dot{x}_3(t) = A_3 x_3(t) + A_{31} x_1(t-\tau_{31}) + A_{32} x_2(t-\tau_{32}) \tag{39}$$

$$A_1 = \begin{bmatrix} -6 & 2 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -10.9 \end{bmatrix}, \ A_{12} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 0 & 3 \\ -2 & 1 & 2 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1.87 & 4.91 & 10.30 \\ -2.23 & -16.51 & -24.11 \\ 1.87 & -3.91 & -10.30 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -1 & 0 & -2 \\ 3 & 0 & 5 \\ 1 & 0 & 2 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -18.5 & -17.5 \\ -13.5 & -18.5 \end{bmatrix}, \quad A_{31} = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad A_{32} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \end{bmatrix},$$

Applying *Theorem* 8 to a given system, for $k = 1$, the following SME is obtained

$$\mathrm{R}_1 - A_1 - e^{-R_1\tau_{21}} S_2 A_{21} - e^{-R_1\tau_{31}} S_3 A_{31} = 0, \quad R_1 S_2 - S_2 A_2 - e^{-R_1\tau_{12}} A_{12} - e^{-R_1\tau_{32}} S_3 A_{32} = 0$$
$$\mathrm{R}_1 S_3 - S_3 A_3 - e^{-R_1\tau_{23}} S_2 A_{23} = 0$$

If for pure time-delays we adopt the following values: $\tau_{12} = 5$, $\tau_{21} = 2$, $\tau_{23} = 4$, $\tau_{31} = 5$ and $\tau_{32} = 3$, by applying the nonlinear least squares algorithms, we obtain a great number of solutions upon R_1 . Among those solutions is a maximal solution:

$$R_{1m} = \begin{bmatrix} -0.0484 & -0.0996 & 0.0934 \\ 0.2789 & -0.3123 & 0.2104 \\ 1.1798 & -1.1970 & -0.3798 \end{bmatrix}$$

The eigenvalues of matrix R_{1m} amount to: $\lambda_1 = -0.2517$, $\lambda_{2,3} = -0.2444 \pm j\ 0.3726$.

Therefore, for a maximal eigenvalue λ_m one of the values from the set $\{\lambda_2, \lambda_3\}$ can be adopted. Based on *Theorem 9*, it follows that the large scale time-delay system is asymptotically stable.

2.3 Discrete time-delay systems

A linear, discrete time-delay system can be represented by the difference equation

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-h) \tag{40}$$

with an associated function of initial state

$$\mathbf{x}(\theta) = \boldsymbol{\psi}(\theta), \qquad \theta \in \{-h, -h+1, \dots, 0\} \tag{41}$$

The equation (40) is referred to as homogenous or the unforced state equation. Vector $\mathbf{x}(k) \in R^n$ is a state vector and $A_0, A_1 \in R^{n\times n}$ are constant matrices of appropriate dimensions, and pure time-delay is expressed by integers $h \in T^+$.

System (40) can be expressed with the following representation without delay [22], [46].

$$\begin{gathered} \mathbf{x}_a(k) = \begin{bmatrix} \mathbf{x}^T(k-h) & \mathbf{x}^T(k-h+1) & \cdots & \mathbf{x}^T(k) \end{bmatrix} \in R^N, \qquad N \mathrel{\hat{=}} n(h+1) \\ \mathbf{x}_a(k+1) = A_a\mathbf{x}_a(k), \quad A_a = \begin{bmatrix} 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \\ A_1 & 0 & \cdots & A_0 \end{bmatrix} \in R^{N\times N} \end{gathered} \tag{42}$$

The system defined by (42) is called the augmented system, while matrix A_a, the matrix of augmented system. Characteristic polynomial of system (40) is given with:

$$f(\lambda) \mathrel{\hat{=}} \det M(\lambda) = \sum_{j=0}^{n(h+1)} a_j\lambda^j, \quad a_j \in R, \quad M(\lambda) = I_n\lambda^{h+1} - A_0\lambda^h - A_1 \tag{43}$$

Denote with

$$\Omega \mathrel{\hat{=}} \{\, \lambda \mid f(\lambda) = 0 \,\} = \lambda(A_{eq}) \tag{44}$$

the set of all characteristic roots of system (40). The number of these roots amounts to $n(h+1)$. A root λ_m of Ω with maximal module:

$$\lambda_m \in \Omega : \ |\lambda_m| = \max|\lambda(A_a)| \tag{45}$$

let us call maximal eigenvalue.

2.3.1. Main results

If scalar variable λ in the characteristic polynomial is replaced by matrix $X \in C^{n\times n}$ the following monic matrix polynomial is obtained

$$M(X) = X^{h+1} - X^hA_0 - A_1 \tag{46}$$

For the needs stability of system (40) only the maximal solvents of (46) are usable, whose spectrums contain maximal eigenvalue λ_m. A special case of maximal solvent is the so called dominant solvent [51], [52] which can be computed in a simple way by Bernoulii or Traub algorithm.

Definition 4. Every solvent R_m of (46) whose spectrum $\sigma(R_m)$ contains maximal eigenvalue λ_m of Ω is a *maximal solvent*.

Definition 5. [51], [52] Matrix A dominates matrix B if all the eigenvalues of A are greater, in modulus, then those of B. In particular, if the solvent R_1 of (46) dominates the solvents $R_2, \ldots, R_l$ we say it is a *dominant solvent*.

Theorem 10. [50] Suppose that there exists maximal solvent of (46) and let R_m denote one of them. Then, linear discrete time-delay system (40) is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$R_m^* P R_m - P = -Q \tag{47}$$

Proof. S*ufficient condition.* Define the following vector discrete functions

$$\mathbf{x}_k = \mathbf{x}(k+\theta), \quad \theta \in \{-h, -h+1, \ldots, 0\}, \quad \mathbf{z}(\mathbf{x}_k) = \mathbf{x}(k) + \sum_{j=1}^{h} T(j)\mathbf{x}(k-j) \tag{48}$$

where, $T(k) \in C^{n\times n}$ is, in general, some time varying discrete matrix function. The conclusion of the theorem follows immediately by defining Lyapunov functional for the system (40) as

$$V(\mathbf{x}_k) = \mathbf{z}^*(\mathbf{x}_k) P\, \mathbf{z}(\mathbf{x}_k), \quad P = P^* > 0 \tag{49}$$

It is obvious that $\mathbf{z}(\mathbf{x}_k) = \mathbf{0}$ if and only if $\mathbf{x}_k = \mathbf{0}$, so it follows that $V(\mathbf{x}_k) > 0$ for $\forall \mathbf{x}_k \neq 0$.

The forward difference of (49), along the solutions of system (40) is

$$\Delta V(\mathbf{x}_k) = \Delta\mathbf{z}^*(\mathbf{x}_k) P\mathbf{z}(k) + \mathbf{z}^*(\mathbf{x}_k) P\Delta\mathbf{z}(\mathbf{x}_k) + \Delta\mathbf{z}^*(\mathbf{x}_k) P\Delta\mathbf{z}(\mathbf{x}_k) \tag{50}$$

A difference of $\Delta\mathbf{z}(\mathbf{x}_k)$ can be determined in the following manner

$$\begin{gathered}\Delta\mathbf{z}(\mathbf{x}_k) = \Delta\mathbf{x}(k) + \sum_{j=1}^{h} T(j)\Delta\mathbf{x}(k-j), \quad \Delta\mathbf{x}(k) = (A_0 - I_n)\mathbf{x}(k) + A_1\mathbf{x}(k-h) \\ \sum_{j=1}^{h} T(j)\Delta\mathbf{x}(k-j) = T(1)\mathbf{x}(k) - T(h)\mathbf{x}(k-h) + \big(T(2) - T(1)\big)\mathbf{x}(k-1) + \cdots \\ + \big(T(h) - T(h-1)\big)\mathbf{x}(k-h+1)\end{gathered} \tag{51}$$

Define a new matrix R by

$$R \triangleq A_0 + T(1) \tag{52}$$

If

$$\Delta T(h) = A_1 - T(h) \tag{53}$$

then $\Delta\mathbf{z}(\mathbf{x}_k)$ has a form

$$\Delta\mathbf{z}(\mathbf{x}_k) = (R - I_n)\mathbf{x}(k) + \sum_{j=1}^{h}\left[\Delta T(j)\cdot\mathbf{x}(k-j)\right] \tag{54}$$

If one adopts

$$\Delta T(j) = (R - I_n)T(j), \quad j = 1, 2, \dots, h \tag{55}$$

then (50) becomes

$$\Delta V(\mathbf{x}_k) = \mathbf{z}^*(\mathbf{x}_k)\left(R^*PR - P\right)\mathbf{z}(\mathbf{x}_k) \tag{56}$$

It is obvious that if the following equation is satisfied

$$R^*PR - P = -Q, \quad Q = Q^* > 0 \tag{57}$$

then $\Delta V(\mathbf{x}_k) < 0, \ \ \mathbf{x}_k \neq \mathbf{0}$.

In the Lyapunov matrix equation (57), of all possible solvents R of (46), only one of maximal solvents R_m is of importance, because it is containing maximal eigenvalue $\lambda_m \in \Omega$, which has dominant influence on the stability of the system. So, (47) represent stability sufficient condition for system given by (40).

Necessary condition. If the system (40) is asymptotically stable then all roots $\lambda_i \in \Omega$ are located within unit circle. Since $\sigma(R_m) \subset \Omega$, follows $\rho(R_m) < 1$, so the positive definite solution of Lyapunov matrix equation (47) exists.

Matrix $T(1)$ can be determined in the following way. From (55), follows

$$T(h+1) = R^h\, T(1) \tag{58}$$

and using (52)-(53) one can get (46).

Corollary 1. [50] Suppose that there exists maximal solvent of (46) and let R_m denote one of them. Then, system (40) is asymptotically stable if and only if $\rho(R_m) < 1$.

Proof. Follows directly from *Theorem 10*.

Corollary 2. [50] Suppose that there exists dominant solvent R_1 of (46). Then, system (40) is asymptotically stable if and only if $\rho(R_1) < 1$.

Proof. Follows directly from *Corollary* 1, since dominant solution is, at the same time, maximal solvent.

Example 2. Let us consider linear discrete systems with delayed state (40) with

$$A_0 = \begin{bmatrix} 7/10 & -1/2 \\ 1/2 & 17/10 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1/75 & 1/3 \\ -1/3 & -49/75 \end{bmatrix} \tag{59}$$

A. For $h = 1$ there are two solvents of matrix polynomial equation (46) ($R^2 - RA_0 - A_1 = 0$):

$$R_1 = \begin{bmatrix} 19/30 & -1/6 \\ 1/6 & 29/30 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1/15 & -1/3 \\ 1/3 & 11/15 \end{bmatrix},$$

Since $\lambda(R_1) = \{4/5, 4/5\}$, $\lambda(R_2) = \{2/5, 2/5\}$, dominant solvent is R_1. For getting the dominant solvent Bernoulli or Traub's algorithm may be used. After $(4+3)$ iterations for Traub's algorithm [52] and 17 iterations for Bernoulli algorithm [52], dominant solvent can be found with accuracy of 10^{-4}. Since $\rho(R_1) = 4/5 < 1$, based on *Corollary* 2, it follows that the system under consideration is asymptotically stable.

B. For $h = 20$ applying Bernoulli or Traub's algorithm for computation the dominant solvent R_1 of matrix polynomial equation (46) ($R^{21} - R^{20}A_0 - A_1 = 0$), we obtain

$$R_1 = \begin{bmatrix} 0.6034 & -0.5868 \\ 0.5868 & 1.7769 \end{bmatrix}$$

Based on *Corollary* 2, the system is not asymptotically stable because $\rho(R_1) = 1.1902 > 1$.

2.4 Discrete large scale time-delay systems

Consider a large-scale linear discrete time-delay systems composed of N interconnected S_i. Each subsystem S_i, $1 \le i \le N$ is described as

$$S_i: \quad \mathbf{x}_i(k+1) = A_i \mathbf{x}_i(k) + \sum_{j=1}^{N} A_{ij} \mathbf{x}_j(k - h_{ij}) \tag{60}$$

with an associated function of initial state

$$\mathbf{x}_i(\theta) = \boldsymbol{\psi}_i(\theta), \quad \theta \in \{-h_{m_i}, -h_{m_i}+1, \dots, 0\} \tag{61}$$

where $\mathbf{x}_i(k) \in R^{n_i}$ is state vector, $A_i \in R^{n_i \times n_i}$ denotes the system matrix, $A_{ij} \in R^{n_i \times n_j}$ represents the interconnection matrix between the i-th and the j-th subsystems and the constant delay $h_{ij} \in T^+$.

Lemma 1. System (60) will be asymptotically stable if and only if

$$|\lambda_{\max}(A_a)| < 1 \tag{62}$$

holds, where matrix

$$A_a = [A_{aij}] \in R^{N_a \times N_a}, \quad N_a = \sum_{i=1}^{N} N_i, \quad N_i = n_i(h_{m_i} + 1), \quad h_{m_i} = \max_j h_{ji} \tag{63}$$

is defined in the following way

$$A_{aii} = \begin{bmatrix} \overset{1}{\downarrow} & & \cdots & \overset{h_{ii}+1}{\downarrow} & & & \\ A_i & 0 & \cdots & A_{ii} & \cdots & 0 & 0 \\ \hline I_{n_i} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \vdots & 0 & \vdots & I_{n_i} & 0 \end{bmatrix} \in R^{N_i \times N_i},$$

$$A_{aij} = \begin{bmatrix} \overset{1}{\downarrow} & \cdots & \overset{h_{ij}+1}{\downarrow} & & \\ 0 & \cdots & A_{ij} & \cdots & 0 \\ \hline 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \in R^{N_i \times N_i} \tag{64}$$

where A_i and A_{ij}, $1 \le i \le N$, $1 \le j \le N$, are matrices of system (60).

2.4.1. Main results

Theorem 11. [53] Given the following system of monic matrix polynomial equations

$$R_l^{h_{m_i}+1} S_i - R_l^{h_{m_i}} S_i A_i - \sum_{j=1}^{N} R_l^{h_{m_i}-h_{ji}} S_j A_{ji} = 0, \quad S_i \in C^{n_l \times n_i}, \quad S_l = I_{n_l} \tag{65}$$

for a given l, $1 \le l \le N$, where A_i and A_{ji}, $1 \le i \le N$, $1 \le j \le N$ are matrices of system (60) and h_{ji} is time-delay in the system, $h_{m_i} = \max_j h_{ji}, \quad 1 \le i \le N$.

If there is a solution of (65) upon unknown matrices $R_l \in C^{n_l \times n_l}$ and S_i, $1 \le i \le N$, $i \ne l$, then $\lambda(R_l) \subset \lambda(A_a)$ holds, where matrix A_a is defined by (63)-(64).

Proof. By introducing time-delay operator z^{-h}, system (60) can be expressed in the following form

$$\mathbf{x}(k+1) = A_e(z)\mathbf{x}(k), \quad \mathbf{x}(k) = \begin{bmatrix} \mathbf{x}_1^T(k) & \mathbf{x}_2^T(k) & \cdots & \mathbf{x}_N^T(k) \end{bmatrix}^T$$

$$A_e(z) = \begin{bmatrix} A_1 + A_{11} z^{-h_{11}} & \cdots & A_{1N} z^{-h_{1N}} \\ \vdots & \ddots & \vdots \\ A_{N1} z^{-h_{N1}} & \cdots & A_N + A_{NN} z^{-h_{NN}} \end{bmatrix} \tag{66}$$

Let us form the following matrix.

$$F(z) = zI_{N_e} - A_e(z) = \left[F_{ij}(z) \right] \tag{67}$$

If we add to the arbitrarily chosen l-th block row of this matrix the rest of its block rows previously multiplied from the left by the matrices $S_j \ne 0$, $1 \le j \le N$, $j \ne l$ respectively and after multiplying i-th of the block column, $1 \le i \le N$, of the preceding matrix by $z^{h_{m_i}}$ and after integrating the matrix $S_l = I_{n_l}$, we obtain

$$z^{\sum_{i=1}^{N} n_i h_{m_i}} \cdot \det F(z) = \det \begin{bmatrix} z^{h_{m_1}} F_{11}(z) & \cdots & z^{h_{m_N}} F_{1N}(z) \\ \vdots & \vdots & \vdots \\ z^{h_{m_1}} \sum_{j=1}^{N} S_j F_{j1}(z) & \cdots & z^{h_{m_N}} \sum_{j=1}^{N} S_j F_{jN}(z) \\ \vdots & \ddots & \vdots \\ z^{h_{m_1}} F_{N1}(z) & \cdots & z^{h_{m_N}} F_{NN}(z) \end{bmatrix}$$

$$= \det \begin{bmatrix} G_{11}(z) & \cdots & G_{1N}(z) \\ \vdots & \vdots & \vdots \\ G_{l1}(z,S) & \cdots & G_{lN}(z,S) \\ \vdots & \vdots & \vdots \\ G_{N1}(z) & \cdots & G_{NN}(z) \end{bmatrix} \tag{68}$$

$$= \det G(z,S), \quad S = \{S_1, \cdots, S_N\}$$

The l -th block row of the $N \times N$ block matrix $G(z,S)$ is defined by

$$G_{li}(z,S) = z^{h_{m\,i}+1} S_i - z^{h_{m\,i}} S_i A_i - \sum_{j=1}^{N} z^{h_{m\,i}-h_{ji}} S_j A_{ji}, \quad 1 \le i \le N, \qquad S_l = I_{n_l} \tag{69}$$

The characteristic polynomial of system (60) [46]

$$g(z) \hat{=} \det G(z,S) = \sum_{j=0}^{N_e} a_j z^j, \quad N_e = \sum_{i=1}^{N} n_i \left(h_{m_i} + 1\right), a_j \in R, 0 \le j \le N_e \tag{70}$$

does not depend on the choice of transformation matrices $S_1, \cdots, S_N$) [48].

Let us denote

$$\sum \hat{=} \{ z \mid g(z) = 0 \} \tag{71}$$

a set of all characteristic roots of system (60). This set of roots equals the set $\lambda(A_a)$. Substituting a scalar variable z by matrix $X \in C^{n_l \times n_l}$ in $G(z,S)$, a new block matrix is obtained $G(X,S)$. If there exist the transformation matrices S_i, $1 \le i \le N$, $i \ne l$ and solvent $R_l \in C^{n_l \times n_l}$ such that for the l -th block row of $G(X,S)$ holds $G_{li}(R_l, S) = 0, \quad 1 \le i \le N$ i.e. holds (65), then

$$g(R_l) = 0 \tag{72}$$

Therefore, the characteristic polynomial of system (60) is annihilating polynomial for the square matrix R_l and $\lambda(R_l) \subset \sum$ holds. The mentioned assertion holds $\forall l, \quad 1 \le l \le N$.

Definition 6. Each solvent R_{lm} of (65), for the given l, $1 \le l \le N$, whose spectrum contains maximal eigenvalue λ_m of system (60), is referred to as *maximal solvent* of (65) .

Theorem 12 [53] Suppose that there exist at least one l, $1 \le l \le N$, that there exists maximal solvent of (65) and let R_{lm} denote one of them. Then, linear discrete large-scale time-delay system (60) is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists matrix $P = P^* > 0$ such that

$$R_{lm}^{\ *}\ P\ R_{lm} - P = -Q\ . \tag{73}$$

Proof. S*ufficient condition.* Define the following vector discrete functions

$$\begin{gathered}\mathbf{v}\left(\mathbf{x}_{k1},\cdots,\mathbf{x}_{kN}\right) = \sum_{i=1}^{N} S_i \left[\mathbf{x}_i\left(k\right) + \sum_{j=1}^{N}\sum_{l=1}^{h_{ji}} T_{ji}\left(l\right)\mathbf{x}_i\left(k-l\right)\right], \\ \mathbf{x}_{ki} = \mathbf{x}_i\left(k+\theta\right),\ \ \theta \in \left\{-h_{m_i},\ \ldots\ ,0\right\}\end{gathered} \tag{74}$$

where $T_{ji}\left(k\right) \in C^{n_i \times n_i}$, $1 \le j \le N$, $1 \le i \le N$ are, in general, some time-varying discrete matrix functions and $S_l = I_{n_l}$, $S_i \in C^{n_\ell \times n_i}$, $1 \le i \le N$, $i \ne l$. The conclusion of the theorem follows immediately by defining Lyapunov functional for system (60) as

$$V\left(\mathbf{x}_{k1},\cdots,\mathbf{x}_{kN}\right) = \mathbf{v}^*\left(\cdot,\cdots,\cdot\right) P\, \mathbf{v}\left(\cdot,\cdots,\cdot\right), \quad P = P^* > 0 \tag{75}$$

It is obvious that $V\left(\cdot,\cdots,\cdot\right) > 0$ for $\forall \mathbf{x}_{ki} \ne \mathbf{0}$, $1 \le i \le N$.

The forward difference of (75), along the solutions of system (60) is

$$\begin{gathered}\mathrm{D}V\left(\cdot,\cdots,\cdot\right) = \mathrm{D}\,\mathbf{v}^*\left(\cdot,\cdots,\cdot\right) P\ \mathbf{v}\left(\cdot,\cdots,\cdot\right) + \mathbf{v}^*\left(\cdot,\cdots,\cdot\right) P\ \mathrm{D}\,\mathbf{v}\left(\cdot,\cdots,\cdot\right) \\ +\mathrm{D}\,\mathbf{v}^*\left(\cdot,\cdots,\cdot\right) P\ \mathrm{D}\,\mathbf{v}\left(\cdot,\cdots,\cdot\right)\end{gathered} \tag{76}$$

A difference of $\mathbf{v}\left(\cdot,\cdots,\cdot\right)$ can be determined in the following manner

$$\begin{aligned}\mathrm{D}\,\mathbf{v}\left(\cdot,\cdots,\cdot\right) = \sum_{i=1}^{N} S_i \Bigg[\Bigg(A_i - I_{n_i} + \sum_{j=1}^{N} T_{ji}\left(1\right)\Bigg)\mathbf{x}_i\left(k\right) + \sum_{j=1}^{N} T_{ji}\left(h_{ji}\right)\mathbf{x}_i\left(k-h_{ji}\right) \\ + \sum_{j=1}^{N}\sum_{l=1}^{h_{ji}-1} \mathrm{D}\,T_{ji}\left(l\right)\mathbf{x}_i\left(k-l\right) + \sum_{j=1}^{N} A_{ij}\mathbf{x}_j\left(k-h_{ij}\right)\Bigg]\end{aligned} \tag{77}$$

If we define new matrices

$$R_i = A_i + \sum_{j=1}^{N} T_{ji}\left(1\right),\ 1 \le i \le N \tag{78}$$

then $\mathrm{D}\,\mathbf{v}\left(\cdot,\cdots,\cdot\right)$ has a form

$$\begin{aligned}\mathrm{D}\,\mathbf{v}(\cdot,\cdots,\cdot) = \sum_{i=1}^{N}\Bigg[& S_i\left(R_i - I_{n_i}\right)\mathbf{x}_i(k) + \sum_{j=1}^{N}\left(S_j A_{ji} - S_i T_{ji}\left(h_{ji}\right)\right)\mathbf{x}_i\left(k-h_{ji}\right) \\ & + \sum_{j=1}^{N}\sum_{l=1}^{h_{ji}-1} S_i \Delta T_{ji}(l)\,\mathbf{x}_i(k-l)\Bigg]\end{aligned} \tag{79}$$

If

$$S_j A_{ji} - S_i T_{ji}\left(h_{ji}\right) = S_i \mathrm{D}\, T_{ji}\left(h_{ji}\right), \quad 1 \le i \le N, \quad 1 \le j \le N \tag{80}$$

$$S_i\left(R_i - I_{n_i}\right) = \left(R_l - I_{n_l}\right)S_i, \quad 1 \le i \le N \tag{81}$$

$$S_i \mathrm{D}\, T_{ji}(l) = \left(R_l - I_{n_l}\right)S_i\, T_{ji}(l), \quad 1 \le i \le N, \quad 1 \le j \le N \tag{82}$$

then

$$\begin{aligned}\mathrm{D}\,\mathbf{v}(\cdot,\cdots,\cdot) &= \left(R_l - I_{n_\ell}\right)\mathbf{v}(\cdot,\cdots,\cdot), \\ \mathrm{D}V(\cdot,\cdots,\cdot) &= \mathbf{v}^*(\cdot,\cdots,\cdot)\left(R_l^* P\, R_l - P\right)\mathbf{v}(\cdot,\cdots,\cdot)\end{aligned} \tag{83}$$

It is obvious that if the following equation is satisfied

$$R_l^* P\, R_l - P = -Q, \qquad Q = Q^* > 0 \tag{84}$$

then $\mathrm{D}V(\cdot,\cdots,\cdot) < 0$, $\forall \mathbf{x}_{ki} \ne \mathbf{0}$, $1 \le i \le N$.

In the Lyapunov matrix equation (73), of all possible solvents R_l of (65), only one of maximal solvents R_{lm} is of importance, for it is the only one that contains maximal eigenvalue $\lambda_m \in \Sigma$, which has dominant influence on the stability of the system.

Necessary condition. If system (60) is asymptotically stable, then $\forall \lambda_i \in \Sigma$, $|\lambda_i| < 1$. Since $\lambda\left(R_{lm}\right) \subset \Sigma$,it follows that $\rho\left(R_{lm}\right) < 1$, therefore the positive definite solution of Lyapunov matrix equation (60) exists.

If it exists, maximal solvent R_{lm} can be determined in the following way. From (80) and (82) we obtain

$$S_j A_{ji} = R_l^{h_{ji}} S_i T_{ji}(1), \quad S_l = I_{n_l}, \quad 1 \le i \le N, \quad 1 \le j \le N \tag{85}$$

Multiplying i-th equation of the system of matrix equations (78) from the left by matrix $R_l^{h_{mi}} S_i$ and using (81) and (85), we obtain equation (65). Taking solvent with eigenvalue $\lambda_m \in \Sigma$ (if it exists) as a solution of the system of equations (65), we arrive at maximal solvent R_{lm}.

Corollary 3. Suppose that for the given l, $1 \le l \le N$, there exists matrix R_l being solution of (65). If system (60) is asymptotically stable, then matrix R_l is discrete stable.

Proof. If system (60) is asymptotically stable, then $\forall\, z \in \Sigma \quad |z| < 1$. Since $\lambda(R_l) \subset \Sigma$, it follows that $\forall\, \lambda \in \lambda(R_l), \quad |\lambda| < 1$, i.e. matrix R_l is discrete stable.

Example 3. Consider a large-scale linear discrete time-delay systems, consisting of three subsystems

$$\begin{aligned} S_1:\ & x_1(k+1) = A_1 x_1(k) + B_1 u_1(k) + A_{12} x_2(k - h_{12}), \\ S_2:\ & x_2(k+1) = A_2 x_2(k) + B_2 u_2(k) + A_{21} x_1(k - h_{21}) + A_{23} x_3(k - h_{23}), \\ S_3:\ & x_3(k+1) = A_3 x_3(k) + B_3 u_3(k) + A_{31} x_1(k - h_{31}) \end{aligned} \tag{86}$$

$$A_1 = \begin{bmatrix} 0.8 & 0.6 \\ 0.4 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0.7 & 0 & -0.5 \\ -0.1 & 6 & -0.1 \\ -0.6 & 1 & 0.8 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & -0.1 \\ 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} -0.1 & -0.2 \\ 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.2 \\ 0.1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 0.8 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, A_{31} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}$$

The overall system is stabilized by employing a local memory-less state feedback control for each subsystem

$$\mathbf{u}_i(k) = K_i \mathbf{x}_i(k), \ K_1 = \begin{bmatrix} -6 & -7 \end{bmatrix}, K_2 = \begin{bmatrix} -7 & -45 & 10 \\ 4 & -4 & -4 \end{bmatrix}, K_3 = \begin{bmatrix} -5 & -1 \\ 1 & -4 \end{bmatrix}$$

Substituting the inputs into this system, we obtain the equivalent closed loop system representations

$$S_i:\ \mathbf{x}_i(k+1) = \hat{A}_i\, \mathbf{x}_i(k) + \sum_{j=1}^{3} A_{ij} \mathbf{x}_j(k - h_{ij}), \quad 1 \le i \le 3, \quad \hat{A}_i = A_i + B_i K_i$$

For time-delay in the system, let us adopt: $h_{12} = 5$, $h_{21} = 2$, $h_{23} = 4$ and $h_{31} = 5$. Applying *Theorem 11* to a given closed loop system, for $l = 1$ we obtain

$$R_1^{\,6} - R_1^{\,5} \hat{A}_1 - R_1^{\,3} S_2 A_{21} - S_3 A_{31} = 0,$$

$$R_1^{\,6} S_2 - R_1^{\,5} S_2 \hat{A}_2 - A_{12} = 0,$$

$$R_1^{\,5} S_3 - R_1^{\,4} S_3 \hat{A}_3 - S_2 A_{23} = 0$$

Solving this SMPE by minimization methods, we obtain

$$R_1 = \begin{bmatrix} 0.6001 & 0.3381 \\ 0.6106 & 0.3276 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.0922 & 1.3475 & 0.5264 \\ 0.0032 & 1.3475 & 0.4374 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0.6722 & -0.3969 \\ 1.3716 & -1.0963 \end{bmatrix}.$$

Eigenvalue with maximal module of matrix R_1 equals 0.9382. Since eigenvalue λ_m of $A_a \in R^{40\times 40}$ also has the same value, we conclude that solvent R_1 is maximal solvent. Applying *Theorem 12*, we arrive at condition $\rho\left(R_{1m}\right)=0.9382<1$ wherefrom we conclude that the observed closed loop large-scale time-delay system is asymptotically stable.

3. Conclusion

In this paper we have presented necessary and sufficient conditions for the asymptotic stability of a particular class of linear continuous and discrete time-delay systems. These results have been extended to the large scale continuous and discrete time-delay systems covering the cases of two and multiple existing subsystems. The delay dependent criteria are derived by Lyapunov's direct method and are exclusively based on the solvents of particular matrix equation and Lyapunov equation for non delay systems. Obtained stability conditions do not possess conservatism. For discrete time-delay systems the dominant solvent of given polynomial matrix equation can be calculated using generalized Traub's or Bernoulli's algorithm which possess significantly smaller number of computation than the standard algorithm.

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References

1. Kamen E.W., (1983), Linear systems with commensurate time-delays: Stability and stabilization independent of delay, *IEEE Transactions on Automatic Control*, Vol. 27, 367-375; corrections in *IEEE Transactions on Automatic Control,* Vol. 28, 1983, pp. 248-249.
2. Hertz D., E. I. Jury, E. Zeheb, (1984), Stability independent and dependent of delay for delay differential systems, *Journal of Franklin Institute*, 318, 1984, pp. 143-150.
3. Niculescu S.I., Souza de C. E., L.Dugard, Dion J.M., (1995), Robust exponential stability of uncertain linear systems with time-varying delays, *Proceedings of 3rd European Control Conference*, Rome, Italy, 1995, pp. 1802-1807.
4. Richard J.P., A., Goubet- Bartholomeus, Tchangani P. A., M. Dambrine, (1997), Nonlinear delay systems: Tools for quantitative approach to stabilization, *Stability and Control of Time-Delay Systems*, (Eds: Dugard L. and Verriest E.I.), Springer-Verlag, 1997, pp. 218-240.
5. Hale J.K., Lunel S.M., (1993), *Introduction to Functional Differential Equations*, Applied Mathematics Sciences Series, 99, Springer-Verlag, 1993.
6. Kolmanovskii V.B., Nosov V.R., (1986), *Stability of Functional Differential Equations*, New York: Academic Press, 1986.
7. Su J.H., (1994), Further results on the robust stability of linear systems with a single delay, *Systems and Control Letters*, Vol. 23, 1994, pp. 375-379.

8. Niculescu S.I., de Souza C. E., Dion J.M., Dugard L., (1994), Robust stability and stabilization for uncertain linear systems with state delay: Single delay case (I). *Proceedings of IFAC Workshop on Robust Control Design*, Rio de Janeiro, Brazil, 1994, pp. 469-474.
9. Boyd S., El Ghaoui L., Feron E., Balakrishnan V., (1994), *Linear matrix inequalities in system and control theory*, SIAM Studies in Applied Mathematics, 15., Philadelphia, USA, 1994.
10. Li X., de Souza C., (1995), LMI approach to delay dependent robust stability and stabilization of uncertain linear delay systems, *Proceedings of 34th IEEE Conference on Decision and Control*, New Orleans, Louisiana, USA, 1995, pp. 3614-3619.
11. Niculescu S.I., Trofino-Neto A., Dion J.M., and Dugard L., (1995), Delay-dependent stability of linear systems with delayed state: An LMI approach, *Proceedings of 34th IEEE Conference on Decision and Control*, New Orleans, Louisiana, USA, 1995, pp. 1495-1497.
12. Gu K., (1997), Discretized LMI set in the stability problem of linear uncertain time-delay systems, *International Journal of Control*, Vol. 68, 1997, 923-934.
13. Su J.H., (1995), The asymptotic stability of linear autonomous systems with commensurate delays, *IEEE Transactions on Automatic Control*, Vol. 40, 1995, pp. 1114-1118.
14. Chen J., Gu G., Nett C.N., (1994), A new method for computing delay margins for stability of linear delay systems, *Proceedings of 33rd IEEE Conference on Decision and Control*, Lake Buena Vista, Florida, USA, 1994, pp. 433-437.
15. Niculescu S.I., Dion J.M., Dugard L., Li H., (1997), Stability of linear systems with delayed state: An LMI approach, *JESA*, special issue on “Analysis and control of time-delay systems”, 31, 1997, pp. 955-970.
16. Niculescu S.I., Verriest E.I., Dugard L., Dion J.M., (1997), Stability and robust stability of time-delay systems: A guided tour, *Lecture notes in control and information sciences*, Vol. 228, 1997, pp. 1-71, London, Springer.
17. Kharitonov V., (1998), Robust stability analysis of time delay systems: A survey, *Proceedings of 4th IFAC System Structure and Control Conference*, Nantes, France, July, 1998.
18. Richard J.P., (1998), Some trends and tools for the study of time delay systems, *Proceedings of CESA98 IMACS/IEEE Multi conference*, Hammamet, Tunisia, 1998, pp. 27-43.
19. Niculescu S.I., Richard J.P., (2002), Analysis and design of delay and propagation systems, *IMA Journal of Mathematical Control and Information*, 19 (1-2), 2002, pp. 1-227.
20. Richard J.P., (2003), Time-delay systems: an overview of some recent advances and open problems, *Automatica*, Vol. 39, 2003, pp. 1667-1694.
21. Kolmanovskii V., Richard J.P., (1999), Stability of some linear systems with delays, *IEEE Transactions on Automatic Control*, Vil. 44, 1999, pp. 984-989.
22. Malek-Zavarei M., Jamshidi M., (1987), *Time-Delay Systems, Analysis, Optimization and Applications*, Systems and Control Series, Vol. 9, 1987.
23. Verriest E., Niculescu S.I., (1998), Delay-independent stability of linear neutral systems: a Riccati equation approach, *Stability and Control of Time-Delay Systems* (Eds: Dugard L. and Verriest E.), Vol. 227 Springer-Verlag, 1998, pp. 92-100.
24. Li X., de Souza C., (1997), Criteria for robust stability and stabilization of uncertain linear systems with state delay, *Automatica*, Vol. 33, 1997, pp. 1657-1662.
25. Kolmanovskii V., Niculescu S.I., Richard J.P., (1999), On the Lyapunov–Krasovskii functionals for stability analysis of linear delay systems, *International Journal of Control*, Vol. 72, 1999, pp. 374-384.
26. Park P., (1999), A delay-dependent stability criterion for systems with uncertain time-invariant delays, *IEEE Transactions on Automatic Control*, Vol. 44, 1999, pp. 876-877.
27. Lien C.H., Yu K.W., Hsieh J.G., (2000), Stability conditions for a class of neutral systems with multiple time delays, *Journal of Mathematical Analysis and Applications*, 245, 2000, pp. 20-27.

28. Niculescu S.I., (2001), On delay-dependent stability under model transformations of some neutral linear systems, *International Journal of Control*, Vol. 74, 2001, pp. 609-617.
29. Fu M., Li H., Niculescu S.I., (1997), Robust stability and stabilization of time-delay system via integral quadratic constraint approach, *Stability and Control of Time-Delay Systems* (Eds: Dugard L. and Verriest E.I.), Springer-Verlag, 1997, pp. 101-116.
30. Chen J., On computing the maximal delay intervals for stability of linear delay systems, *IEEE Transactions on Automatic Control*, Vol. 40, 1995, pp. 1087-1093.
31. Park P., Moon Y.S., Kwon W.H., (1998), A delay-dependent robust stability criterion for uncertain time-delay systems, *Proceedings of the American control conference*, USA, 1998, pp. 1963-1964.
32. Moon Y.S., Park P.G., Kwon W.H., Lee Y.S., (2001), Delay-dependent robust stabilization of uncertain state-delayed systems, *International Journal of Control*, Vol. 74 (14), 2001, pp. 1447-1455.
33. Fridman E., Shaked U., (2003), Delay-dependent stability and H∞ control: Constant and time-varying delays, *International Journal of Control*, Vol. 76, 2003, pp. 48-60.
34. Fridman E., Shaked U., (2002), A descriptor system approach to H∞ control of linear time-delay systems, *IEEE Transactions on Automatic Control*, Vol. 47, 2002, pp. 253-270.
35. Han Q.L., (2005), On stability of linear neutral systems with mixed time-delays: A discretized Lyapunov functional approach, *Automatica*, Vol. 41, 2005, pp. 1209-1218.
36. Han Q.L., (2005), A new delay-dependent stability criterion for linear neutral systems with norm-bounded uncertainties in all system matrices, *International Journal of Systems Science*, Vol. 36, 2005, pp. 469–475.
37. Fridman E., Shaked U., (2002), An improved stabilization method for linear time-delay systems, *IEEE Transactions on Automatic Control*, Vol. 47, 2002, pp. 1931–1937.
38. Lee T.N., Diant S., (1981), Stability of time-delay systems, *IEEE Transactions on Automatic Control*, Vol. 26, No. 4, 1981, pp. 951-953.
39. Xu S., Lam J., Yang C., H∞ and positive real control for linear neutral delay systems, *IEEE Transactions on Automatic Control*, Vol. 46, 2001, pp. 1321–1326.
40. Verriest E., Ivanov A., (1995), Robust stability of delay difference equations, *Proceedings IEEEE Conf. on Dec. and Control*, New Orleans, LA, 1995, pp. 386-391.
41. Kapila V., Haddad W., (1998), Memoryless H∞ controllers for discrete-time systems with time delay, *Automatica*, Vol. 34, 1998, pp. 1141-1144.
42. Song S., Kim J., Yim C., Kim H., (1999), H∞ control of discrete-time linear systems with time-varying delays in state, *Automatica,* Vol. 35, 1999, pp. 1587-1591.
43. Mahmoud M., (2000), Robust H∞ control of discrete systems with uncertain parameters and unknown delays, *Automatica*, Vol. 36, 2000, pp. 627-635.
44. Lee Y.S., Kwon W.H., (2002), Delay-dependent robust stabilization of uncertain discrete-time state-delayed systems, *Preprints of the 15th IFAC World Congress*, Barcelona, Spain, 2002.
45. Shi P., Agarwal R.K., Boukas E.K., Shue S.P., (2000), Robust H∞ state feedback control of discrete time delay linear systems with norm-bounded uncertainty, *International Journal of Systems Science*, Vol. 31, 2000, pp. 409- 415.
46. Gorecki H., Fuksa S., Grabovski P., Korytowski A., (1989), *Analysis and synthesis of time delay systems*, John Wiley & Sons, 1989.
47. Debeljkovic D.Lj.,, Stojanovic S.B., (2008), Asymptotic Stability Analysis of Linear Time Delay Systems: Delay Dependent Approach, *Systems, Structure and Control* (Eds: Petr Husek), I–Tech, Vienna, 2008, pp. 29-60.
48. Lancaster P., Tismenetsky M., *The theory of matrices, 2nd Edition*, Academic press, New York, 1985.

49. Stojanovic S.B., Debeljkovic D.Lj., (2005), Necessary and Sufficient Conditions for Delay-Dependent Asymptotic Stability of Linear Continuous Large Scale Time Delay Autonomous Systems, *Asian Journal of Control*, Vol. 7, No. 4, 2005, pp. 414-418.
50. Stojanovic S.B., Debeljkovic D.Lj., (2008), Necessary and Sufficient Conditions for Delay-Dependent Asymptotic Stability of Linear Discrete Time Delay Autonomous Systems, *Proceedings of 17th IFAC World Congress*, Seoul, Korea, July 06–10, 2008, pp. 2613-2618.
51. Dennis J.E., Traub J.F., Weber R.P., (1976), The algebraic theory of matrix polynomials, *SIAM J. Numer. Anal.*, Vol. 13 (6), 1976, pp. 831-845.
52. Dennis J.E., Traub J.F., Weber R.P., (1978), Algorithms for solvents of matrix polynomials, *SIAM J. Numer. Anal.*, Vol. 15 (3), 1978, pp. 523-533.
53. Stojanovic S.B., Debeljkovic D.Lj., (2008), Delay–Dependent Stability of Linear Discrete Large Scale Time Delay Systems: Necessary and Sufficient Conditions, *International Journal of Information & System Science*, Vol. 4, No. 2, 2008, pp. 241-250.
54. Stojanovic S.B., Debeljkovic D.Lj., (2008), Quadratic stability and stabilization of uncertain linear discrete-time systems with state delay: a LMI approach, *Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications & Algorithms*, Vol. 15, 2008, pp. 195-206.

STABILNOST LINEARNIH SISTEMA SA KAŠNJENJEM KOJA ZAVISI OD VREMENSKOG KAŠNJENJA

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Rad se bavi problemom stabilnosti linearnih običnih i velikih sistema koja zavisi od vremenskog kašnjenja. Izvedeni su potrebni i dovoljni uslovi asimptotske stabilnosti kontinualnih i diskretnih linearnih sistema sa kašnjenjem. Ovi rezultati su prošireni na klasu velikih sistema sa kašnjenjem pri čemu je razmatran slučaj sa dva i više podsistema. Kriterijumi stabilnosti su izvedeni koristeći Ljapunov direktni metod a zasnivaju se na rešenju posebnih klasa matričnih jednačina i Ljapunove jednačine za sisteme bez kašnjenja. Izvedeni uslovi stabilnosti ne poseduju konzervativizam. Nekoliko numeričkih primera je urađeno kako bi se pokazala primenljivost izvedenih rezultata.

Key words: *kontinualni sistemi sa kašnjenjem, diskretni sistemi sa kašnjenjem, veliki sistemi sa kašnjenjem, stsbilnost zavisna od kašnjenja, Ljapunova stabilnost, potrebni i dovoljni uslovi*

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