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ADVANCES IN CLASSICAL AND ANALYTICAL MECHANICS: A REVIEWS OF AUTHOR'S RESULTS

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Abstract*. A review, in subjective choice, of author's scientific results in area of: classical mechanics, analytical mechanics of discrete hereditary systems, analytical mechanics of discrete fractional order system vibrations, elastodynamics, nonlinear dynamics and hybrid system dynamics is presented. Main original author's results were presented through the mathematical methods of mechanics with examples of applications for solving problems of mechanical real system dynamics abstracted to the theoretical models of mechanical discrete or continuum systems, as well as hybrid systems. Paper, also, presents serries of methods and scientific results authored by professors Mitropolyski, Andjelić and Rašković, as well as author's of this paper original scientific research results obtained by methods of her professors. Vector method based on mass inertia moment vectors and corresponding deviational vector components for pole and oriented axis, defined in 1991 by K. Hedrih, is presented. Results in construction of analytical dynamics of hereditary discrete system obtained in collaboration with O. A. Gorosho are presented. Also, some selections of results author's postgraduate students and doctorantes in area of nonlinear dynamics are presented. A list of scientific projects headed by author of this paper is presented with a list of doctoral dissertation and magister of sciences thesis which contain scientific research results obtained under the supervision by author of this paper or their fist doctoral candidates.*

Keywords: *Review, vector method, mass moment vectors, deviational mass moment vector, rotator, coupled rotations, no intersecting axes, basic vectors of position vector tangent space, angular velocity of the tangent space basic vectors, rheonomic constraint, rheonomic coordinate, mobility, angular velocity of basic vector rotation, velocity of basic vector extension, asymptotic approximation of solution, Krilov-Bogolyubov-Mitropolyski asymptotic averaged method, method of variation of constants, hereditary system, rheological and relaxational kernels, standard hereditary element, integro-differential equation, fractional order derivative,*

covariant coordinate, contravariant coordinate, Physical coordinate, discrete continuum method, space fractional order structure, chains, eigen main plane nets, eigen main chains, fractional order oscillator, fractional order properties characteristic number, transfer of signals, multi-frequency, material particles, rigid body, gyrorotor, deformable body, multi bdy system, transversal, longitudinal, multi-plate system, multi-belt system, stochastic stability.

1. INTRODUCTION

Main author's research results, presented in this paper, are:

**** Advances in classical mechanics.***

Mass moment vectors connected to pole and axis, allowed the author to give a new perspective onto rotation of bodies around stationary axis and a stationary point, and on dynamics of rotors and coupled rotors in general. By introducing definitions of mass moment vectors connected to pole and axis, and by proving their properties, and also by introducing purely kinematical rotator vectors, which he used to represent short and elegant expressions for kinetic pressures and kinetic impacts on the rotor shaft bearings, the author made a contribution to classical mechanics, as well as a contribution to the methodology of university teaching of rotor kinematics. In a monographic paper published in 1998, and a monograph published in Serbia in 2001, as well as in a series of published papers in the period 1992-2010, beginning with a paper at ICTAM in Israel (1992), and later in a series of papers published in Japan, Germany, China, Ukraine, Russia and Greece, the author shows definitions and properties, as well as applications of mass moment vectors connected to pole and axis for analyzing mass moment states and properties of kinetic parameters of rotor dynamics, dynamics of rigid body coupled rotation around no intersecting axes and dynamics of coupled rotors. (see References I [1-20]).

Angular velocity of the basic vectors rotation of a tangent space of the vector positions of material particles of mechanical system dynamics with geometrical, stationary and rheonomic constraints are obtained. Extensions of dimensions of tangent space of the vector positions of material particles of mechanical system dynamics from three dimensional real spaces to configuration space of independent generalized curvilinear coordinate systems is identified. Reductions of numbers of coordinates and extensions of tangent space of vector passions are analyzed (see References II [21-34]).

**** Advances in Analytical Mechanics.***

- Analytical Mechanics of Discrete Hereditary Systems.

Foundation and construction of analytical mechanics of discrete hereditary systems was the work of two authors – Oleg Aleksandrovich Goroshko and Katica R. (Stevanović) Hedrih. Their original contribution to modern analytical mechanics, the authors published in their monograph of the same name, which came into existence in the period of their cooperation between 1996-1999, and was published in 2001. The contents of this monograph represents the first, in the world published integral theory of analytical mechanics of discrete hereditary systems. Through a short review of the contents of the monograph published in Serbia, as well as a series of presented results and/or published

papers in the period 1995-2009 in Serbia, Ukraine, Russia, China and US, we shall point out the main contributions of these two authors in this area. (see References III [35-54])

- Analytical mechanics of discrete fractional order systems

Through a series of papers published or presented in the period 2005-2009 in France, Portugal, Turkey, Germany, Ukraine, China and Romania, as well as in monograph publications and international journals, the author contributed to the development of analytical mechanics of discrete systems, of fractional order with special focus of the results on oscillatory systems of fractional order. Prominent among these contributions are the results relating to homogenous chain systems of fractional order and homogenous couples chain systems of fractional order. The author introduced new terms, such as eigen main chains, main coordinates of eigen chain systems of homogenous coupled chains into hybrid systems of fractional order, as well as main partial oscillators of fractional order with corresponding main coordinates and corresponding oscillatory modes of fractional order with creep properties. (see References III [35-54])

***Advances in Elastodynamics, Nonlinear Dynamics and Hybrid System Dynamics**

Among the contents of a series of papers published in international journals (2003-2010) or journals of prestigious scientific institutes in the world (1970-2009), as well as in monographs published by Kluwer and Springer, contributions of author to linear and nonlinear dynamics of deformable bodies (rods, plates, moving strips), systems of coupled deformable bodies, especially stand out and can be classifies as a single scientific area of Elastodynamics and the newly established area of hybrid system dynamics. A number of results are on the energy analysis of complex hybrid system dynamics. Five theorems on characteristic equations of complex systems, obtained by coupling deformable bodies and discrete systems with finite number of degrees of freedom, static or dynamic or combined couples have been defined and proven.

A number of original results are about nonlinear properties of systems with coupled rotation motions. A number of theorems on coupled singularities and homoclinic orbits in the form of number eight has also been defined and proven. (see References IV. [35-68], V [69-88])

2. ADVANCES IN CLASSICAL MECHANICS.

2.1* Vector method and applications

Vector method [4], based on mass moment vectors and vector rotators coupled for pole and oriented axes, is used for obtaining vector expressions for kinetic pressures on the shaft bearings of a rigid body dynamics with coupled rotations around no intersecting axes [16-19]. This method is very effective and suitable in applications. Mass inertia moment vectors and corresponding deviational vector components for pole and oriented axis are defined by K. Hedrih in 1991 [1]. A complete analysis of obtained vector expressions for derivatives of linear momentum and angular momentum give us a series of the kinematical vectors rotators around both directions determined by axes of the rigid body coupled rotations around no intersecting axes[16-19]. These kinematical vectors rotators are defined for a system with two degrees of freedom as well as for rheonomic system with two degrees of mobility and one degree of freedom and coupled

rotations around two coupled no intersecting axes as well as their angular velocities and intensity.

As an example of defined dynamics [16-19], we take into consideration a heavy gyrorotor-disk with one degree of freedom and coupled rotations when one component of rotation is programmed by constant angular velocity. For this system with nonlinear dynamics, series of graphical presentation of three parameter transformations in relations with changes of eccentricity and angle of inclination (skew position) of heavy rigid body in relation to self rotation axis are presented, as well as in relation with changing orthogonal distance between no intersecting axes of coupled rotations. Some graphical visualization of vector rotators properties are presented, too.

Using K. Hedrih's (See Refs. [1-9]) mass moment vectors and vector rotators, some characteristics members of the vector expressions of derivatives of linear momentum and angular momentum for the gyro rotor coupled rotations around two no intersecting axes obtain physical and dynamical visible properties of the complex system dynamics [16-18].

Between them there are vector terms that present deviational couple effect containing vector rotators which directions are same as kinetic pressure components on corresponding gyro rotor shaft bearings [10-15] and [18-20].

2.1.1. Mass moment vectors for the axis to the pole

The monograph [4], IUTAM extended abstract [1] and monograph paper [5] contain definitions of three mass moment vectors coupled to a axis passing through a certain point as a reference pole. Now, we start with necessary definitions of mass momentum vectors.

Definitions of selected mass moment vectors for the axis and the pole, which are used in this paper are:

1* Vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass linear moment for the axis, oriented by the unit vector $\vec{n}$, through the point – pole O, in the form:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} \overset{def}{=} \iiint_V [\vec{n}, \vec{\rho}]dm = [\vec{n}, \vec{\rho}_C]M \ , \qquad dm = \sigma dV \ ; \tag{1}$$

where $\vec{\rho}$ is the position vector of the elementary body mass particle dm in point N, between pole O and mass particle position N.

2* Vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment for the axis, oriented by the unit vector $\vec{n}$, through the point – pole O, in the form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \overset{def}{=} \iiint_V [\vec{\rho}, [\vec{n}, \vec{\rho}]]dm \tag{2}$$

For special cases, the details can be seen in [1-9]. In the previously cited references, the spherical and deviational parts of the mass inertia moment vector and the inertia tensor are analyzed. In monograph [4] knowledge about the change (rate) in time and, the derivatives of the mass moment vectors of the body mass linear moment, the body mass inertia moment for the pole and a corresponding axis for different properties of the body, is shown, on the basis of results from the first author's References [6-9].

The relation

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O_1)} + \left[\vec{\rho}_O, \vec{\mathfrak{S}}_{\vec{n}}^{(O_1)}\right] + \left[\vec{\mathrm{M}}_C^{(O_1)}, [\vec{n}, \vec{\rho}_O]\right] + [\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]]\mathrm{M} \tag{3}$$

is ***the vector form of the theorem*** *for the relation of material body mass inertia moment vectors,* $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ *and* $\vec{\mathfrak{J}}_{\vec{n}}^{(O_1)}$*, for two parallel axes through two corresponding points, pole* O *and pole* O_1 *(for details detail Refs. [ ] by K. Hedrih).* We can see that all the members in the last expression-relation (3) have the similar structure. These structures are: $[\vec{\rho}_o, [\vec{n}, \vec{r}_C]]\mathrm{M}$, $[\vec{r}_C, [\vec{n}, \vec{\rho}_o]]\mathrm{M}$ and $[\vec{\rho}_o, [\vec{n}, \vec{\rho}_o]]\mathrm{M}$.

In the case when the pole O_1 is the centre C of the body mass, the vector $\vec{r}_C$ (the position vector of the mass centre with respect to the pole O_1) is equal to zero, whereas the vector $\vec{\rho}_O$ turns into $\vec{\rho}_C$ so that the last expression (3) can be written in the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\mathrm{M} \tag{4}$$

This expression (4) *represents the vector form of the theorem of the rate change of the mass inertia moment vector for the axis and the pole, when the axis is translated from the pole at the mass centre* C *to the arbitrary point, pole* O.

The *Huygens-Steiner theorems* (see Refs. [4] and [5]) for the body mass axial inertia moments, as well as for the mass deviational moments, emerged *from this theorem* (4) on the change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at point O for the axis oriented by the unit vector $\vec{n}$ passing trough the mass center C, and when the axis is moved by translate to the other point O.

Mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ for the axis to the pole is possible to decompose in two parts: first $\vec{n}\left(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}\right)$ collinear with axis and second $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ normal to the axis. So we can write:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{n}\left(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}\right) + \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = J_{\vec{n}}^{(O)}\vec{n} + \vec{\mathfrak{D}}_{\vec{n}}^{(O)} \tag{5}$$

Collinear component $\vec{n}\left(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}\right)$ to the axis corresponds to the axial mass inertia moment $J_{\vec{n}}^{(O)}$ of the body. Second component, $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$, orthogonal to the axis, we denote by the $\vec{\mathfrak{D}}_O^{(\vec{n})}$, and it is possible to obtain by both side double vector products by unit vector $\vec{n}$ with mass moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ in the following form:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \left[\vec{n}, \left[\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, n\right]\right] = \vec{\mathfrak{J}}_{\vec{n}}^{(O)}(\vec{n}, \vec{n}) - \vec{n}\left(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}\right) = \vec{\mathfrak{J}}_{\vec{n}}^{(O)} - J_O^{\vec{n}}\vec{n} \tag{6}$$

In case when rigid body is balanced with respect to the axis the mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ is collinear to the axis and there is no deviational part. In this case axis of rotation is main axis of body inertia. When axis of rotation is not main axis then mass inertial moment vector for the axis contains deviation part $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$. That is case of rotation unbalanced rotor according to axis and bodies skew positioned to the axis of rotation.

2.1.2. Model of a rigid body coupled multi-rotation around multi-axes without intersections

Let us consider rigid body coupled multi-rotations around axes without intersections, first oriented by unit vector $\vec{n}_1$ with fixed position and second and next oriented by unit vectors $\vec{n}_j$, $j = 2,3,...,K$, which are rotating around fixed axis as well as around series of previous axes and with corresponding angular velocities $\vec{\omega}_j = \omega_j \vec{n}_j$, $j = 1,2,3,...,K$. See Figure 1. Axes of rotations are without intersections. Rigid body is positioned on the moving rotating axis oriented by unit vector $\vec{n}_K$. Rigid body rotates around rotating self rotation axis with angular velocity $\vec{\omega}_K = \omega_K \vec{n}_K$ and around series of the previous axes in order and in whole around fixed axis oriented by unit vector $\vec{n}_1$ with angular velocity $\vec{\omega}_1 = \omega_1 \vec{n}_1$. The shortest orthogonal distances between axes are defined by length $\overline{O_{(j)}\tilde{O}_{(j+1)}}$, $j = 1,2,3,...,K$ and each of these is perpendicular to both close axes that each is to the direction of component angular velocities $\vec{\omega}_j = \omega_j \vec{n}_j$ and $\vec{\omega}_{j+1} = \omega_{j+1}\vec{n}_{j+1}$. These vectors are $\vec{r}_{0(j)(j+1)} = \overrightarrow{O_{(j)}O_{(j+1)}}$:

$$\vec{r}_{0(j)(j+1)} = \frac{r_{0(j)(j+1)}}{\sin\alpha_{(j)(j+1)}}\left[\vec{n}_{(j)},\vec{n}_{(j+1)}\right] = r_0 \frac{\left[\vec{n}_{(j)},\vec{n}_{(j+1)}\right]}{\left|\left[\vec{n}_{(j)},\vec{n}_{(j+1)}\right]\right|} = r_{0(j)(j+1)}\vec{u}_{0(j)(j+1)} \tag{7}$$

and it can be seen on Fig.1.

In the considered rigid body coupled rotations around no intersecting numerous axes, an elementary mass around point N is denoted as dm, with position vector $\vec{\rho}$, and with origin in the point O_K on the movable self rotation axis, and with $\vec{r}$ vector positions of the same body elementary mass with origin in the point O_1, where point O_1 is fixed on the axis oriented by unit $\vec{n}_1$. Both points are on the ends of the corresponding shortest orthogonal distance between two in the neighborhood axes of body coupled multi-rotations. Position vector of elementary mass with origin in pole O_1 and its velocity are in the following forms:

$$\vec{r}_k = \sum_{k=1}^{K-1}\left(\vec{r}_{0(k),(k+1)} + \vec{r}_{0(k+1),(k+1)}\right) + \vec{\rho}, \quad \vec{v}_K = \sum_{k=1}^{K-1}\left[\sum_{j=1}^{k}\vec{\omega}_j, \vec{r}_{0(k),(k+1)} + \vec{r}_{0(k+1),(k+1)}\right] + \left[\sum_{j=1}^{K}\vec{\omega}_j, \vec{\rho}\right] \tag{8}$$

For the case of three coupled rotations around three axes without intersections position vector of elementary mass with origin in pole O_1 and its velocity are in the following forms (see Fig.1):

$$\vec{r} = \vec{r}_{012} + \vec{r}_{022} + \vec{r}_{023} + \vec{\rho} \text{ and } \vec{v} = \left[\vec{\omega}_1, \vec{r}_{012} + \vec{r}_{022}\right] + \left[\vec{\omega}_1 + \vec{\omega}_2,, \vec{r}_{023}\right] + \left[\vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3, \vec{\rho}\right]. \tag{9}$$

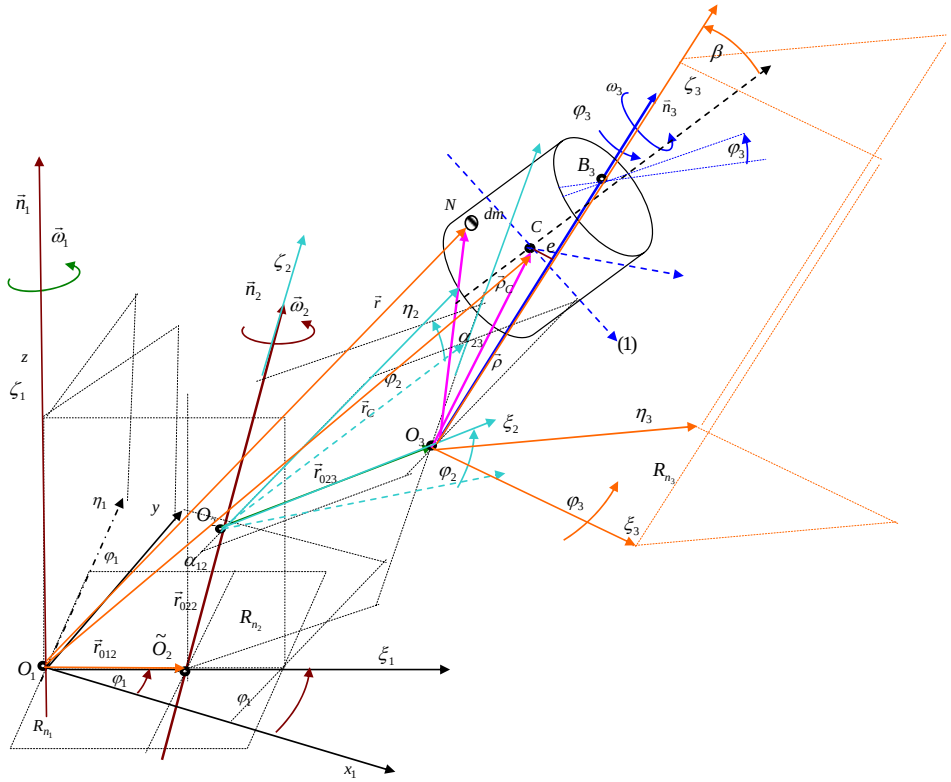


Figure 1. Arbitrary position of rigid body multi-coupled rotations around finite numbers of axes without intersections.

2.1.3. Linear momentum of a rigid body coupled multi-rotations around axes without intersections

By using basic definition of linear momentum and expression for velocity of elementary body mass (9), we can write linear momentum in the following vector form:

$$\vec{\mathfrak{K}} = \sum_{k=1}^{K-1}\sum_{j=1}^{k} \omega_j \left[\vec{n}_j, \vec{r}_{0(k),(k+1)} + \vec{r}_{0(k+1),(k+1)}\right] M + \sum_{j=1}^{K} \omega_j \vec{\mathfrak{S}}_{\vec{n}_j}^{(O_K)}. \qquad (10)$$

where $\vec{\mathfrak{S}}_{\vec{n}_j}^{(O_K)} = \iiint_V \left[\vec{n}_j, \vec{\rho}\right] dm$, $j = 1,2,3,...,K$, are corresponding body mass linear moments of the rigid body for the axes oriented by direction of component angular velocities of coupled multi-rotations through the movable pole O_K on self rotating axis. First terms in the form of the first sum in expression (10) presents translation part of linear momentum. This part is equal to zero in case when axes intersect in one point. Second sum in expression (10) for linear momentum present linear momentum of pure rotation,

as relative motion around all axes with intersection in the pole O_K on self rotation axis. These K terms are different from zero in all cases.

Example 1: Expression of linear momentum of a rigid body coupled rotations around two no intersecting axes, we can write in the following form:

$$\vec{\mathfrak{K}} = \vec{\mathfrak{K}}_{\vec{n}_1}^{(O_{1-2})} + \vec{\mathfrak{K}}_{\vec{n}_1}^{(O_2)} + \vec{\mathfrak{K}}_{\vec{n}_2}^{(O_2)} = [\vec{\omega}_1, \vec{r}_{012}]M + \omega_1 \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_2)} + \omega_2 \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_2)} \tag{11}$$

Example 2: Expression of linear momentum of a rigid body coupled rotations around three axes without intersections, we can write in the following form:

$$\vec{\mathfrak{K}} = \omega_1[\vec{n}_1, \vec{r}_{012} + \vec{r}_{022} + \vec{r}_{023}]M + \omega_2[\vec{n}_2, \vec{r}_{023}]M + \omega_1 \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_3)} + \omega_2 \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_3)} + \omega_3 \vec{\mathfrak{E}}_{\vec{n}_3}^{(O_3)} \tag{12}$$

2.1.3. Angular momentum of a rigid body coupled multi-rotations around axes without intersections

By using basic definition of angular momentum and expression for velocity of rotation of elementary body mass and its position vector (9), we can write vector expression for angular momentum.

Example 1: Expression of angular momentum of a rigid body coupled rotations around two axes without intersection, we can write in the following form:

$$\vec{\mathfrak{L}}_{O_1} = \omega_1 \vec{n}_1 r_0^2 M + \omega_1[\vec{\rho}_C, [\vec{n}_1, \vec{r}_0]]M + \omega_1\left[\vec{r}_0, \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_2)}\right] + \omega_2\left[\vec{r}_0, \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_2)}\right] + \omega_1 \vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)} + \omega_2 \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)} \tag{13}$$

Example 2: Expresion of angular momentum of a rigid body coupled rotations around three axes without intersections, we can write in the following form:

$$\begin{aligned}\vec{\mathfrak{L}}_{O_1} = {} & \omega_1\left[\vec{r}_{O_3}, \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_{1-2-2})}\right] + \omega_1\left[\vec{r}_{O_3}, \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_{2-3})}\right] + \omega_2\left[\vec{r}_{O_3}, \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_{2-3})}\right] + \omega_1\left[\vec{\rho}_C, \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_{1-2-2})}\right] + \omega_1 \vec{\mathfrak{J}}_{\vec{n}_1}^{(O_3)} + \omega_2 \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_3)} + \omega_3 \vec{\mathfrak{J}}_{\vec{n}_3}^{(O_3)} + \\ & + \omega_1\left[\vec{\rho}_C, \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_{2-3})}\right] + \omega_2\left[\vec{\rho}_C, \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_{2-3})}\right] + \omega_1\left[\vec{r}_{O_3}, \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_3)}\right] + \omega_2\left[\vec{r}_{O_3}, \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_3)}\right] + \omega_3\left[\vec{r}_{O_3}, \vec{\mathfrak{E}}_{\vec{n}_3}^{(O_3)}\right]\end{aligned} \tag{14}$$

2.1.4. Derivative of linear momentum and angular momentum of rigid body coupled rotations around two axes without intersection

Example 1. By using expressions for linear momentum (13), the derivative of linear momentum of rigid body coupled rotations around two axes without intersection, we can write the following vector expression:

$$\begin{aligned}\frac{d\vec{\mathfrak{K}}}{dt} = {} & \dot{\omega}_1[\vec{n}_1, \vec{r}_0]M + \omega_1^2[\vec{n}_1, [\vec{n}_1, \vec{r}_0]]M + \dot{\omega}_1 \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_2)} + \omega_1^2\left[\vec{n}_1, \vec{\mathfrak{E}}_{\vec{n}_1}^{(O_2)}\right] + \\ & + \dot{\omega}_2 \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_2)} + \omega_2^2\left[\vec{n}_2, \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_2)}\right] + 2\omega_1\omega_2\left[\vec{n}_1, \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_2)}\right]\end{aligned} \tag{15}$$

After analysis structure of linear momentum derivative terms, we can see that there is possibility to introduce pure kinematic vectors, depending on component angular velocities and component angular accelerations of component coupled rotations, that are useful to express derivatives of linear moment in following form

$$\frac{d\vec{\mathfrak{K}}}{dt} = \vec{\mathfrak{R}}_{01}[\vec{n}_1, \vec{r}_0]M + \vec{\mathfrak{R}}_{011}\left|\vec{\mathfrak{E}}_{\vec{n}_1}^{(O_2)}\right| + \vec{\mathfrak{R}}_{022}\left|\vec{\mathfrak{E}}_{\vec{n}_2}^{(O_2)}\right| + 2\omega_1\omega_2\left[\vec{n}_1, \vec{\mathfrak{E}}_{\vec{n}_2}^{(O_2)}\right]. \tag{16}$$

We can see that in previous vector expression (16), for derivative of linear momentum, are introduced the following three vector rotators:

$$\vec{\mathfrak{R}}_{01} = \dot{\omega}_1 \vec{u}_{01} + \omega_1^2 \vec{v}_{01}, \quad \vec{\mathfrak{R}}_{01} = \dot{\omega}_1\left[\vec{n}_1, \frac{\vec{r}_0}{r_0}\right] + \omega_1^2\left[\vec{n}_1, \left[\vec{n}_1, \frac{\vec{r}_0}{r_0}\right]\right], \qquad \vec{\mathfrak{R}}_{011} = \dot{\omega}_1 \vec{u}_{011} + \omega_1^2 \vec{v}_{011},$$

$$\vec{\mathfrak{R}}_{011}=\dot{\omega}_1\frac{\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}\right|}+\omega_1^2\left[\vec{n}_1,\frac{\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}\right|}\right],\tag{17}$$

$$\vec{\mathfrak{R}}_{022}=\dot{\omega}_1\vec{u}_{022}+\omega_1^2\vec{v}_{022},\qquad \vec{\mathfrak{R}}_{022}=\dot{\omega}_2\frac{\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}\right|}+\omega_2^2\left[\vec{n}_2,\frac{\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}\right|}\right].$$

Also, we can see that in vector expression for derivative of angular momentum appear the following vector rotators: $\vec{\mathfrak{R}}_1=\dot{\omega}_1\vec{u}_1+\omega_1^2\vec{v}_1$ and $\vec{\mathfrak{R}}_2=\dot{\omega}_1\vec{u}_2+\omega_1^2\vec{v}_2$ expressed by following expressions:

$$\vec{\mathfrak{R}}_1=\dot{\omega}_1\frac{\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}\right|}+\omega_1^2\left[\vec{n}_1,\frac{\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}\right|}\right]=\dot{\omega}_1\vec{u}_1+\omega_1^2\vec{v}_1,\quad \vec{\mathfrak{R}}_2=\dot{\omega}_2\frac{\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}\right|}+\omega_2^2\left[\vec{n}_2,\frac{\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}\right|}\right]=\dot{\omega}_2\vec{u}_2+\omega_2^2\vec{v}_2\tag{18}$$

Example 2. By using expressions for linear momentum the derivative of linear momentum of rigid body coupled rotations around three axes without intersections, not difficult to obtain corresponding expression. Also, as in previous example, after analysis structure of linear momentum derivative terms, we can see that there is possibility to introduce pure kinematic vectors, depending on component angular velocities and component angular accelerations of component coupled three rotations. We can see that in vector expression for derivative of linear momentum series of the vector rotators appear. Some of these vector rotators are listed here:

$$\vec{\mathfrak{R}}_{011}=\dot{\omega}_1\vec{u}_{011}+\omega_1^2\vec{v}_{011},\qquad \vec{\mathfrak{R}}_{011}=\dot{\omega}_1\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}\right|}+\omega_1^2\left[\vec{n}_1,\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}\right|}\right],$$

$$\vec{\mathfrak{R}}_{022}=\dot{\omega}_2\vec{u}_{022}+\omega_2^2\vec{v}_{022},$$

$$\vec{\mathfrak{R}}_{022}=\dot{\omega}_2\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_2}\right|}+\omega_2^2\left[\vec{n}_2,\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O3)}_{\vec{n}_2}\right|}\right].\tag{19}$$

$$\vec{\mathfrak{R}}_{033}=\dot{\omega}_3\vec{u}_{033}+\omega_1^2\vec{v}_{033},\qquad \vec{\mathfrak{R}}_{033}=\dot{\omega}_3\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}\right|}+\omega_3^2\left[\vec{n}_2,\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}\right|}\right]$$

Also, we can see that in vector expression for derivative of angular momentum of rigid body coupled rotations around three axes without intersections appear the following vector rotators: $\vec{\mathfrak{R}}_1=\dot{\omega}_1\vec{u}_1+\omega_1^2\vec{v}_1$, $\vec{\mathfrak{R}}_2=\dot{\omega}_2\vec{u}_2+\omega_2^2\vec{v}_2$ and $\vec{\mathfrak{R}}_3=\dot{\omega}_3\vec{u}_3+\omega_1^2\vec{v}_3$ expressed by following expressions:

$$\vec{\mathfrak{R}}_1=\dot{\omega}_1\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}\right|}+\omega_1^2\left[\vec{n}_1,\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_1}}{\left|\vec{\mathfrak{S}}^{(O3)}_{\vec{n}_1}\right|}\right]=\dot{\omega}_1\vec{u}_1+\omega_1^2\vec{v}_1,$$

$$\vec{\mathfrak{R}}_2=\dot{\omega}_2\frac{\vec{\mathfrak{S}}^{(O3)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_2}\right|}+\omega_2^2\left[\vec{n}_2,\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_2}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_2}\right|}\right]=\dot{\omega}_2\vec{u}_2+\omega_2^2\vec{v}_2$$

$$\vec{\mathfrak{R}}_3=\dot{\omega}_3\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}\right|}+\omega_3^2\left[\vec{n}_3,\frac{\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}}{\left|\vec{\mathfrak{S}}^{(O_3)}_{\vec{n}_3}\right|}\right]=\dot{\omega}_3\vec{u}_3+\omega_3^2\vec{v}_3\tag{20}$$

2.1.5. Concluding remarks

By using theorems of changes of linear momentum and angular momentum with respect to time, one may write two vector equations of dynamic equilibrium of rigid body coupled multi-rotations about axes without intersection as the follows:

$$\frac{d\vec{\mathfrak{K}}}{dt} = \vec{G} + \vec{F}_{AN} + \vec{F}_{BN} + \vec{F}_{Am} + \sum_{i=1}^{i=P} \vec{F}_i \tag{21}$$

$$\frac{d\vec{\mathfrak{L}}_{O_1}}{dt} = \left[\vec{r}_0 + \vec{\rho}_{C,}\vec{G}\right] + \left[\vec{r}_0 + \vec{\rho}_{A,}, \vec{F}_{AN}\right] + \left[\vec{r}_0 + \vec{\rho}_{B,}, \vec{F}_{BN}\right] + \left[\vec{r}_0 + \vec{\rho}_{A,}, \vec{F}_{Am}\right] + \sum_{i=1}^{i=P}\left[\vec{r}_0 + \vec{\rho}_{i,}, \vec{F}_i\right] \tag{22}$$

where $\vec{F}_i, i = 1,2,3,..., p$ are external active forces, $\vec{G}$ is weight of a rotor, $\vec{F}_A$ and $\vec{F}_B$ are forces of bearing reactions at fixed axis. From previous analysis, we can conclude that vector rotators appear into expressions of the kinetic reactions of the shaft bearings of the structures of the rigid body multi-coupled rotations and that is very important to analyze their intensity as well as their relative angular velocity and angular acceleration around axes of coupled multi-rotations.

Recommendation foe next research and for solving three main mathematical tasks: a* Generalization of the expressions for derivatives of linear momentum and angular momentum for rigid body coupled multi-rotations around finite numbers axes without intersections; b* expressions for kinetic pressures on bearing to series of the axes of coupled rotations and corresponding numbers of coupled nonlinear differential equations depending of number of system degree of freedom with corresponding solutions and c* build a algorithm for using obtained results as a standard software program for analysis nonlinear dynamic phenomena in rigid body coupled rotation around finite number axes without intersections. These defined tasks need a team interdisciplinary research, and will be very useful for engineering practice in analysis and simulation numerous engineering system dynamics with coupled rotation and for vibro-diagnostic.

2.2. Tangent spaces of position vectors and angular velocities of their basic vectors in different coordinate systems

Angular velocities of the basic vectors of tangent spaces of the position vectors of mass particles of the discrete rheonomic mechanical system are obtained in different coordinate systems [22]. Starting from real three dimensional coordinate systems of Descartes orthogonal three dimensional system type with fixed coordinates axis as a reference, by different coordinate transformations for each position vector of corresponding mass particle in discrete rheonomic mechanical system, basic vectors of position vector tangent three dimensional spaces are obtained in different curvilinear coordinate systems suitable to the corresponding geometrical scleronomic or rheonomic constraints applied to the considered rheonomic system. For each basic vector of the basic triedar of position vector tangent space of each mass particle of the discrete rheonomic mechanical system, angular velocity vectors of basic vector rotations are determined.

Then, after consideration and analysis of the number and properties of the geometrical scleronomic and rheonomic constraints applied to the mass particles of the

considered discrete rheonomic mechanical system, number of system degree of mobility as well as number of system degree of freedom are determined. Corresponding number of independent coordinates are chosen and corresponding rheonomic coordinates are introduced. By use extended set of the generalized coordinates contained corresponding number of independent coordinates and corresponding number of rheonomic coordinates, position vectors of the mass particles of the discrete rheonomic mechanical system, are separated into two subsets.

First subset contain position vectors of the mass particle, keep their three dimensional tangent space each with three basic vectors.

Second subset contain position vectors of the mass particle, each depending, in general case, of the all generalized coordinates, independent and rheonomic. Then, each of the position vectors are with $n+R$ -dimensional tangent spaces and with basic vectors.

2.2.1. Introduction

Let us consider a discrete system with N mass particles with mass m_α, and with corresponding position in real three dimensional space determined by geometrical points $N_{(\alpha)}, \alpha = 1,2,3,...,N$ (see Figure 2). For beginning we take that positions of the material points, as well as corresponding geometrical points coordinates are determined by coordinates in fixed orthogonal Descartes coordinate system with three coordinates as denoted by $N_{(\alpha)}\left(x_{(\alpha)}, y_{(\alpha)}, z_{(\alpha)}\right). \alpha = 1,2,3,...,N$, where O is fixed coordinate origin, and Ox, Oy and Oz fixed oriented coordinate strain lines-coordinate axes. Coordinates of the position vector of each material point are equal to coordinate of the geometrical point which determine mass particle position in the space. For Descartes coordinate system for position of the each mass particle we can write: $\vec{\rho}_{(\alpha)}\left(x_{(\alpha)}, y_{(\alpha)}, z_{(\alpha)}\right) = x_{(\alpha)}\vec{i} + y_{(\alpha)}\vec{j} + z_{(\alpha)}\vec{k}$, $\alpha = 1,2,3,...,N$.

Let us, now, consider previous discrete system with N mass particles wirh mass m_α, and with corresponding position in real three dimensional space determined by same geometrical points $N_{(\alpha)}, \alpha = 1,2,3,...,N$ in generalized coordinate system of curvilinear coordinates $\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right)$ $\alpha = 1,2,3,...,N$ corresponding to mass particle positions. For same geometrical points coordinates in considered three coordinate systems are: $N_{(\alpha)}\left(x_{(\alpha)}, y_{(\alpha)}, z_{(\alpha)}\right). \alpha = 1,2,3,...,N$ and $N_{(\alpha)}\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right)$, $\alpha = 1,2,3,...,N$ .. Formulae of coordinate transformation from previous coordinate system with fixed axes and new curvilinear coordinate system are:

$$\begin{aligned} x_{(\alpha)} &= x_{(\alpha)}\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right) \\ y_{(\alpha)} &= y_{(\alpha)}\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right) \\ z_{(\alpha)} &= z_{(\alpha)}\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right) \end{aligned} \tag{1}$$

Position vectors of each mass particle and corresponding geometrical points are invariant geometrical objects in both coordinate systems, but their coordinates in

considered coordinate systems are not equal to coordinates of the corresponding geometrical point. In generalized coordinate system geometrical points $N_{(\alpha)}, \alpha = 1,2,3,...,N$ have following coordinates: $\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right)$, $\alpha = 1,2,3,...,N$ and coordinate of position vectors of these geometrical points are $\left(\rho_{(\alpha)}{}^1, \rho_{(\alpha)}{}^2, \rho_{(\alpha)}{}^3\right), \alpha = 1,2,3,...,N$. For position vectors we can write:

$$\vec{\rho}_{(\alpha)}\left(x_{(\alpha)}, y_{(\alpha)}, z_{(\alpha)}\right) = x_{(\alpha)}\vec{i} + y_{(\alpha)}\vec{j} + z_{(\alpha)}\vec{k} \overset{def=inv.vektor}{=} \vec{\rho}_{(\alpha)}\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right) \tag{2}$$

$$\vec{\rho}_{(\alpha)}\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right) =$$
$$= \rho_{(\alpha)}{}^1\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right)\vec{g}_{(\alpha)1} + \rho_{(\alpha)}{}^2\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right)\vec{g}_{(\alpha)2} + \rho_{(\alpha)}{}^3\left(q_{(\alpha)}{}^1, q_{(\alpha)}{}^2, q_{(\alpha)}{}^3\right)\vec{g}_{(\alpha)3}$$
$$\alpha = 1,2,3,...,N \tag{3}$$

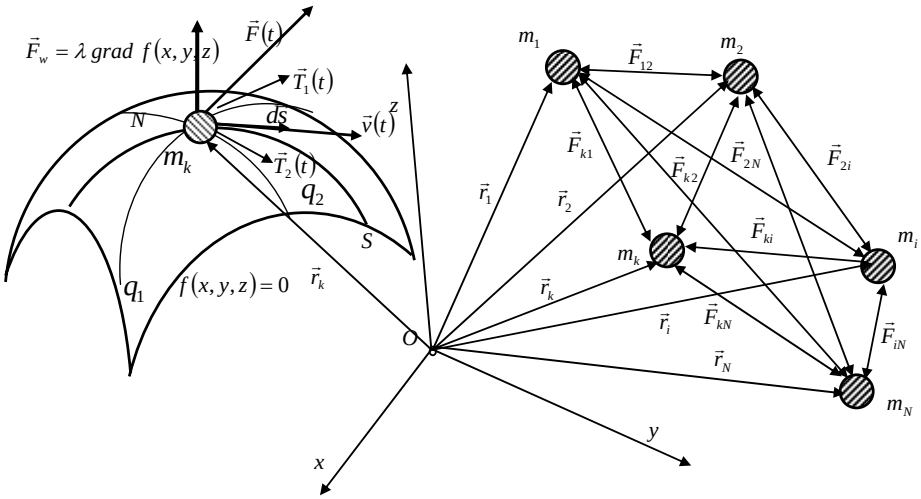


Figure 2. Discrete material system with N mass particles and geometrical rheonomic constraints

For first example in polar-cylindrical coordinate system geometrical points have the following coordinates: $N_{(\alpha)}\left(r_{(\alpha)}, \varphi_{(\alpha)}, z_{(\alpha)}\right)$ $\alpha = 1,2,3,...,N$ and position vectors $\vec{\rho}_{(\alpha)}\left(r_{(\alpha)}, \varphi_{(\alpha)}, z_{(\alpha)}\right)$ of corresponding geometrical point are: $r_{(\alpha)}, 0, z_{(\alpha)}$ and we can write:

$$\vec{\rho}_{(\alpha)}\left(r_{(\alpha)}, \varphi_{(\alpha)}, z_{(\alpha)}\right) = r_{(\alpha)}\vec{r}_{o(\alpha)} + 0 \cdot \vec{c}_{0(\alpha)} + z_{(\alpha)}\vec{k} \equiv r_{(\alpha)}\vec{r}_{o(\alpha)} + z_{(\alpha)}\vec{k} \tag{4}$$
$$\alpha = 1,2,3,...,N$$

where $\vec{r}_{o(\alpha)}, \vec{c}_{0(\alpha)}$ and $\vec{k}$, $\alpha = 1,2,3,...,N$ are basic unit vectors of tangent space of corresponding position vector in polar-cylindrical coordinate system.

For second example in spherical coordinate system geometrical points have the following coordinates: $N_{(\alpha)}\left(\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)}\right)\ \ \alpha=1,2,3,...,N$ and position vectors $\vec{\rho}_{(\alpha)}\left(\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)}\right)$ of corresponding geometrical point are: $\rho_{(\alpha)},0,0$ and we can write:

$$\vec{\rho}_{(\alpha)}\left(\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)}\right)=\rho_{(\alpha)}\vec{\rho}_{0(\alpha)}+0\cdot\vec{c}_{0(\alpha)}+0\cdot\vec{\nu}_{0(\alpha)}\equiv\rho_{(\alpha)}\vec{\rho}_{0(\alpha)} \qquad (5)$$
$$\alpha=1,2,3,...,N$$

where $\vec{\rho}_{0(\alpha)},\vec{c}_{0(\alpha)}$ and $\vec{\nu}_{0(\alpha)},\alpha=1,2,3,...,N$ are basic unit vectors of tangent space of corresponding position vector in polar-cylindrical coordinate system.

2.2.2. Basic vectors of the position vector three-dimensional tangent space in generalized curvilinear coordinate systems

In real two-dimensional coordinate systems, position vector tangent spaces are three-dimensional and the basic vectors of the tangent spaces of each position vector of each mass particle we denote with $\vec{g}_{(\alpha)i}$, $\alpha=1,2,3,...,N$, $i=1,2,3$ (see Figure 3). These vectors are in tangent directions to the corresponding curvilinear coordinate line and in general are not unit vectors. Basic vectors it is possible to obtain by following way (for detail see Refs. [22], [23], [24], [25], [26], [27], [28], [33] and [34]):

$$\vec{g}_{(\alpha)i}=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial q_{(\alpha)}{}^{i}},\ \alpha=1,2,3,...,N,\ \ i=1,2,3 \qquad (6)$$

or by formula coordinate transformation and by following expressions:

$$\begin{aligned}
\vec{g}_{(\alpha)1}&=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial q_{(\alpha)}{}^{1}}=\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\vec{i}+\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\vec{j}+\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\vec{k}\\
\vec{g}_{(\alpha)2}&=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial q_{(\alpha)}{}^{2}}=\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\vec{i}+\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\vec{j}+\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\vec{k} \qquad (7)\\
\vec{g}_{(\alpha)3}&=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial q_{(\alpha)}{}^{3}}=\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\vec{i}+\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\vec{j}+\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\vec{k}
\end{aligned}$$

Contravariant coordinates of the position vectors it is possible to obtain by following formulas:

$$\begin{aligned}
\rho_{(\alpha)}{}^{1}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)=&\frac{1}{\Delta}\left[\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}-\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\right]x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)-\\
&-\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}-\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\right]y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)-\\
&-\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}-\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\right]z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)
\end{aligned}$$

$$\rho_{(\alpha)}{}^{2}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)=-\frac{1}{\Delta}\left[\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}-\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\right]x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)+$$
$$+\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}-\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\right]y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)-$$
$$-\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}-\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{3}}\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\right]z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)$$

$$\rho_{(\alpha)}{}^{3}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)=\frac{1}{\Delta}\left[\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}-\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\right]x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)-$$
$$-\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}-\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\right]y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)+$$
$$+\frac{1}{\Delta}\left[\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}-\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{2}}\frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q_{(\alpha)}{}^{1}}\right]z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)$$

where

$$\Delta_{(\alpha)}=\begin{vmatrix}\frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{1}} & \frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{1}} & \frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{1}}\\ \frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{2}} & \frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{2}} & \frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{2}}\\ \frac{\partial x_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{3}} & \frac{\partial y_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{3}} & \frac{\partial z_{(\alpha)}\left(q_{(\alpha)}{}^{1},q_{(\alpha)}{}^{2},q_{(\alpha)}{}^{3}\right)}{\partial q^{3}}\end{vmatrix}\neq 0$$

2.2.3. Change of the basic vectors of the position vector three-dimensional tangent space in generalized curvilinear coordinate systems

Without losing generality, we consider change of basic vectors of a position vector of one mass particle during mass particle motion through real space and described in three-dimensional space. Also, we focused our attention to the orthogonal curvilinear coordinate system. For that case change (first derivative with respect to time) with time of the basic vectors of tangent space of a position vector are (see Figure 3.):

$$\frac{d\vec{g}_1}{dt}=\overset{*}{\vec{g}}_1+$$
$$=\vec{g}_1\left(\Gamma^1_{11}\dot{q}^1+\Gamma^1_{12}\dot{q}^2+\Gamma^1_{13}\dot{q}^3\right)+\vec{g}_2\left(\Gamma^2_{11}\dot{q}^1+\Gamma^2_{12}\dot{q}^2+\Gamma^2_{13}\dot{q}^3\right)+\vec{g}_3\left(\Gamma^3_{11}\dot{q}^1+\Gamma^3_{12}\dot{q}^2+\Gamma^3_{13}\dot{q}^3\right)$$

$$\frac{d\vec{g}_2}{dt}=\overset{*}{\vec{g}}_2+\left[\vec{\omega}_{p2},\vec{g}_2\right]=$$
$$=\left(\Gamma^1_{21}\vec{g}_1+\Gamma^2_{21}\vec{g}_2+\Gamma^3_{21}\vec{g}_3\right)\dot{q}^1+\left(\Gamma^1_{22}\vec{g}_1+\Gamma^2_{22}\vec{g}_2+\Gamma^3_{22}\vec{g}_3\right)\dot{q}^2+\left(\Gamma^1_{23}\vec{g}_1+\Gamma^2_{23}\vec{g}_2+\Gamma^3_{23}\vec{g}_3\right)\dot{q}^3 \qquad (8)$$
$$\frac{d\vec{g}_3}{dt}=\overset{*}{\vec{g}}_3+\left[\vec{\omega}_{p3},\vec{g}_3\right]=$$
$$=\left(\Gamma^1_{31}\vec{g}_1+\Gamma^2_{31}\vec{g}_2+\Gamma^3_{31}\vec{g}_3\right)\dot{q}^1+\left(\Gamma^1_{32}\vec{g}_1+\Gamma^2_{32}\vec{g}_2+\Gamma^3_{32}\vec{g}_3\right)\dot{q}^2+\left(\Gamma^1_{33}\vec{g}_1+\Gamma^2_{33}\vec{g}_2+\Gamma^3_{33}\vec{g}_3\right)\dot{q}^3$$

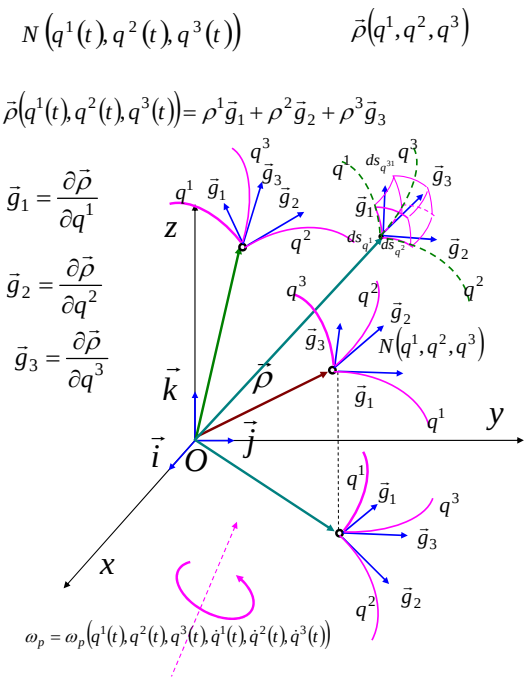


Figure 3. A position vectors and its three-dimensional space with corresponding curvilinear coordinate system and tangent space with corresponding three basic vectors of the position vector tangent spaces along mass particle motion through time

After analysis of the obtained derivatives of the basic vectors of position vector tangent spaces in three-dimensional orthogonal curvilinear coordinate systems we can separate two sets of the terms in obtained expressions (8). First set correspond to the relative derivative of the corresponding basic vectors in the following forms:

$$\begin{aligned}
\overset{*}{\vec{g}}_1 &= \vec{g}_1\left(\Gamma^1_{11}\dot{q}^1 + \Gamma^1_{12}\dot{q}^2 + \Gamma^1_{13}\dot{q}^3\right)\\
\overset{*}{\vec{g}}_2 &= \vec{g}_2\left(\Gamma^2_{21}\dot{q}^1 + \Gamma^2_{22}\dot{q}^2 + \Gamma^2_{23}\dot{q}^3\right)\\
\overset{*}{\vec{g}}_3 &= \vec{g}_3\left(\Gamma^3_{31}\dot{q}^1 + \Gamma^3_{32}\dot{q}^2 + \Gamma^3_{33}\dot{q}^3\right)
\end{aligned} \tag{9}$$

These vectors present vector forms of extensions of the corresponding basic vectors and in scalar form it is possible to express relative change of the intensity – dilatation of the basic vectors in direction of its previous kinetic state. In differential form is possible to write:

$$\begin{aligned}
d\varepsilon_1 &= \frac{d|\vec{g}_1|}{|\vec{g}_1|} = \Gamma^1_{11}dq^1 + \Gamma^1_{12}dq^2 + \Gamma^1_{13}dq^3\\
d\varepsilon_2 &= \frac{d|\vec{g}_2|}{|\vec{g}_2|} = \Gamma^2_{31}dq^1 + \Gamma^2_{32}dq^2 + \Gamma^2_{33}dq^3
\end{aligned} \tag{10}$$

$$d\varepsilon_3 = \frac{d\left|\vec{g}_{31}\right|}{\left|\vec{g}_3\right|} = \Gamma^3_{31}dq^1 + \Gamma^3_{32}dq^2 + \Gamma^3_{33}dq^3$$

From analysis of the obtained derivatives of the basic vectors of position vector tangent spaces in three-dimensional orthogonal curvilinear coordinate systems we can separate second set of the terms in obtained expressions (8). Second set correspond to the rotation change of the corresponding basic vectors in the following forms:

$$\begin{aligned}
\left[\vec{\omega}_{p1}, \vec{g}_1\right] &= \vec{g}_2\left(\Gamma^2_{11}\dot{q}^1 + \Gamma^2_{12}\dot{q}^2 + \Gamma^2_{13}\dot{q}^3\right) + \vec{g}_3\left(\Gamma^3_{11}\dot{q}^1 + \Gamma^3_{12}\dot{q}^2 + \Gamma^3_{13}\dot{q}^3\right) \\
\left[\vec{\omega}_{p2}, \vec{g}_2\right] &= \vec{g}_1\left(\Gamma^1_{21}\dot{q}^1 + \Gamma^1_{22}\dot{q}^2 + \Gamma^1_{23}\dot{q}^3\right) + \vec{g}_3\left(\Gamma^3_{21}\dot{q}^1 + \Gamma^3_{22}\dot{q}^2 + \Gamma^3_{23}\dot{q}^3\right) \\
\left[\vec{\omega}_{p3}, \vec{g}_3\right] &= \vec{g}_1\left(\Gamma^1_{31}\dot{q}^1 + \Gamma^1_{32}\dot{q}^2 + \Gamma^1_{33}\dot{q}^3\right) + \vec{g}_2\left(\Gamma^2_{31}\dot{q}^1 + \Gamma^2_{32}\dot{q}^2 + \Gamma^2_{33}\dot{q}^3\right)
\end{aligned} \qquad (11)$$

where we introduce notation $\vec{\omega}_{p1}$, $\vec{\omega}_{p2}$ and $\vec{\omega}_{p3}$ for vectors of the angular velocities of the corresponding basic vectors of the position vector tangent space. When curvilinear coordinate system is not orthogonal and angles between three basic vectors are changeable with time these angular velocities are different for each basic vector. When basic vectors are orthogonal and without change orthogonal relation, all three angular velocity are same.

For the case of the discrete mechanical system N mass particles for each vector position of each mass particle is necessary, by analogous way as presented in previous part, is possible to determine change of the basic vectors of tangent space of position vectors.

After analysis of the obtained derivatives of the basic vectors of position vector tangent spaces for each mass particle, in three-dimensional orthogonal curvilinear coordinate systems, we can separate two sets of the terms in obtained expression and corresponding for other two sets of the basic vectors. First set correspond to the relative derivative of the corresponding basic vectors. These vectors present vector forms of extensions of the corresponding basic vectors and in scalar form it is possible to express relative changes of the intensities – dilatations of the basic vectors in direction of their previous kinetic state.

From analysis of the obtained derivatives of the basic vectors of position vector tangent spaces for each mass particle in three-dimensional orthogonal curvilinear coordinate systems, we can separate second sets of the terms in obtained expressions. Second set correspond to the rotation change of the corresponding basic vectors. We introduce notation $\vec{\omega}_{(\alpha)p1}$, $\vec{\omega}_{(\alpha)p2}$ and $\vec{\omega}_{(\alpha)p3}$ for vectors of the angular velocities of the corresponding basic vectors of the position vector tangent spaces. When basic vectors are orthogonal and without change orthogonal relation, all three angular velocity are same, for each vector position.

For example 1*: in polar-cylindrical curvilinear coordinate system by expressions (8), (9), (10) and (11) we can write (see Figure 4.a*):

$$\frac{d\vec{g}_1}{dt} = \frac{d\vec{g}_r}{dt} = \dot{\varphi}\left(-\vec{i}\sin\varphi + \vec{j}\cos\varphi\right) = \frac{d\vec{r}_0}{dt} = \dot{\varphi}\vec{c}_0 = \frac{1}{r}\dot{\varphi}\vec{g}_\varphi = \left[\vec{\omega}_{\mathrm{Pr}}, \vec{g}_r\right]$$

$$\frac{d\vec{g}_2}{dt} = \frac{d\vec{g}_\varphi}{dt} = \dot{r}\vec{c}_0 + r\frac{d\vec{c}_0}{dt} = \dot{r}\vec{c}_0 - r\dot{\varphi}\,\vec{r}_0 = \overset{*}{\vec{g}}_\varphi + \left[\vec{\omega}_{P\varphi}, \vec{g}_\varphi\right]$$

$$\frac{d\vec{g}_3}{dt}=\frac{d\vec{g}_z}{dt}=\frac{d\vec{k}}{dt}=0$$

$$\overset{*}{\vec{g}}_r=0\,,\qquad d\varepsilon_r=\frac{d\left|\vec{g}_\varphi\right|}{\left|\vec{g}_\varphi\right|}=\frac{dr}{r}\,,\qquad \vec{\omega}_{\mathrm{Pr}}=\dot{\varphi}\vec{k}$$

$$\overset{*}{\vec{g}}_\varphi=\dot{r}\vec{c}_0=\frac{\dot{r}}{r}\vec{g}_\varphi\,,\qquad \vec{\omega}_{\mathrm{P}\varphi}=\dot{\varphi}\vec{k}\,,\qquad \overset{*}{\vec{g}}_z=0\,,\qquad \vec{\omega}_{\mathrm{Pz}}=0$$

Angular velocities of the basic vectors of each position vector tangent space of mass particle motion in polar-cylindrical curvilinear coordinate systems are: $\vec{\omega}_{(\alpha)\mathrm{P}}=\dot{\varphi}_{(\alpha)}\vec{k}$, $\alpha=1,2,3,...,N$.

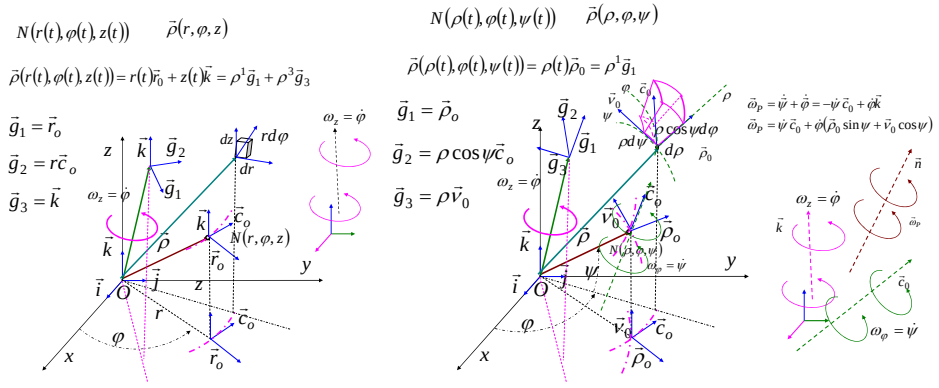


Figure 4. A position vectors and its three-dimensional spaces with corresponding curvilinear coordinate system and tangent space with corresponding three basic vectors of the position vector tangent spaces along mass particle motion through time
a* polar-cylindrical curvilinear coordinate system; b* spherical curvilinear coordinate system

For example 2*: in spherical curvilinear coordinate system by expressions (8), (9), (10) and (11), we can write (see Figure 4.b*):

$$\frac{d\vec{g}_1}{dt}=\frac{d\vec{g}_\rho}{dt}=\dot{\varphi}\vec{c}_0\cos\psi+\dot{\psi}\vec{v}_0=\frac{1}{\rho}\dot{\varphi}\vec{g}_\varphi+\frac{1}{\rho}\dot{\psi}\vec{g}_\psi$$

$$\frac{d\vec{g}_2}{dt}=\frac{d\vec{g}_\varphi}{dt}=\vec{c}_0\left(\dot{\rho}\cos\psi-\rho\dot{\psi}\sin\psi\right)-\dot{\varphi}\left(\vec{\rho}_0\cos\psi-\vec{v}_0\sin\psi\right)\rho\cos\psi$$

$$\frac{d\vec{g}_3}{dt}=\frac{d\vec{g}_\psi}{dt}=\dot{\rho}\vec{v}_0+\rho\left(-\dot{\varphi}\vec{c}_0\sin\psi-\dot{\psi}\vec{\rho}_0\right)$$

$$\overset{*}{\vec{\rho}}_\rho=0\,,\qquad \left[\vec{\omega}_{\mathrm{P}\rho},\vec{\rho}_\rho\right]=\dot{\psi}\vec{v}_0+\dot{\varphi}\vec{c}_0\cos\psi$$

$$\vec{\omega}_{\mathrm{P}\rho}=\dot{\psi}\,\vec{c}_0+\dot{\varphi}\vec{k}=-\frac{\dot{\psi}}{\rho\cos\psi}\vec{g}_\varphi+\dot{\varphi}\left(\vec{g}_\rho\sin\psi+\frac{1}{\rho}\vec{g}_\psi\cos\psi\right)$$

$$\overset{*}{\vec{g}}_{\varphi}=\vec{c}_0\left(\dot{\rho}\cos\psi-\rho\dot{\psi}\sin\psi\right)=\frac{1}{\rho\cos\psi}\left(\dot{\rho}\cos\psi-\rho\dot{\psi}\sin\psi\right)\vec{g}_{\varphi}$$

$$\left[\vec{\omega}_{P\varphi},\vec{g}_{\varphi}\right]=\left[-\dot{\psi}\,\vec{c}_0+\dot{\varphi}\left(\vec{\rho}_0\sin\psi+\vec{\nu}_0\cos\psi\right),\vec{c}_0\rho\cos\psi\right]=-\dot{\varphi}\left(\vec{\rho}_0\cos\psi-\vec{\nu}_0\sin\psi\right)\rho\cos\psi$$

$$\vec{\omega}_{P\varphi}=-\dot{\vec{\psi}}+\dot{\vec{\varphi}}=\dot{\psi}\,\vec{c}_0+\dot{\varphi}\vec{k}=-\frac{\dot{\psi}}{\rho\cos\psi}\vec{g}_{\varphi}+\dot{\varphi}\left(\vec{g}_{\rho}\sin\psi+\frac{1}{\rho}\vec{g}_{\psi}\cos\psi\right)$$

$$\frac{d\vec{g}_3}{dt}=\frac{d\vec{g}_{\psi}}{dt}=\dot{\rho}\,\vec{\nu}_0+\rho\left(-\dot{\varphi}\vec{c}_0\sin\psi-\dot{\psi}\vec{\rho}_0\right)$$

$$\overset{*}{\vec{g}}_{\psi}=\dot{\rho}\,\vec{\nu}_0=\frac{\dot{\rho}}{\rho}\vec{g}_{\psi},$$

$$\left[\vec{\omega}_{P\psi},\vec{g}_{\psi}\right]=-\dot{\varphi}\vec{g}_{\varphi}tg\,\psi-\dot{\psi}\vec{g}_{\psi}=\left[-\dot{\psi}\,\vec{c}_0+\dot{\varphi}\left(\vec{\rho}_0\sin\psi+\vec{\nu}_0\cos\psi\right),\rho\vec{\nu}_0\right]$$

$$\vec{\omega}_{P\psi}=-\dot{\vec{\psi}}+\dot{\vec{\varphi}}=\dot{\psi}\,\vec{c}_0+\dot{\varphi}\vec{k}=-\frac{\dot{\psi}}{\rho\cos\psi}\vec{g}_{\varphi}+\dot{\varphi}\left(\vec{g}_{\rho}\sin\psi+\frac{1}{\rho}\vec{g}_{\psi}\cos\psi\right)$$

Angular velocities of the basic vectors of each position vector tangent space of mass particle motion in spherical curvilinear coordinate systems are:

$$\vec{\omega}_{(\alpha)P}=-\dot{\vec{\psi}}_{(\alpha)}+\dot{\vec{\varphi}}_{(\alpha)}=\dot{\psi}_{(\alpha)}\vec{c}_{0(\alpha)}+\dot{\varphi}_{(\alpha)}\vec{k}$$

$$\vec{\omega}_{(\alpha)P}=-\dot{\psi}_{(\alpha)}\vec{c}_{0(\alpha)}+\dot{\varphi}_{(\alpha)}\left(\vec{\rho}_{0(\alpha)}\sin\psi_{(\alpha)}+\vec{\nu}_{0(\alpha)}\cos\psi_{(\alpha)}\right),$$

$$\alpha=1,2,3,...,N$$

2.2.4. Dimensional extension of the position vector tangent spaces of the reheonomic mechanical system in generalized curvilinear coordinate systems

Considered discrete mechanical system is constrained by G geometrical stationary constraints in the form:

$$f_{\beta}\left(q_{(1)}{}^{1},q_{(1)}{}^{2},q_{(1)}{}^{3},q_{(2)}{}^{1},q_{(2)}{}^{2},q_{(2)}{}^{3},.....,q_{(N)}{}^{1},q_{(N)}{}^{2},q_{(N)}{}^{3}\right)=0\,,\ \beta=1,2,3....G \quad (12)$$

and by R geometrical rheonomic constraints in the form (see Ref. [23]):

$$f_{\gamma}\left(q_{(1)}{}^{1},q_{(1)}{}^{2},q_{(1)}{}^{3},q_{(2)}{}^{1},q_{(2)}{}^{2},q_{(2)}{}^{3},.....,q_{(N)}{}^{1},q_{(N)}{}^{2},q_{(N)}{}^{3},\phi_{\gamma}(t)\right)=0\,,$$

$$\gamma=1,2,3....R \quad (13)$$

Considered system is rheonomic system with $p=3N-G$ degree of the system mobility, and with $n=3N-G-R$ degrees of the freedom. For the n generalized independent coordinates we take q^{i}, $i=1,2,3,....,n$. Also, we introduce additional subsystem of the R rheonomic coordinates $q^{0\gamma}=q^{n+\gamma}=\phi_{\gamma}(t)$, $\gamma=1,2,3....R$ which correspond to number of rheonomic constraints. Then we have extended system of the generalized curvilinear coordinates q^{i}, $i=1,2,3,....,n,...,n+\gamma,.....,n+R$. Then we know that subsystem of R rheonomic coordinates $q^{0\gamma}=q^{n+\gamma}=\phi_{\gamma}(t)$, $\gamma=1,2,3....R$ contain known rheonomic coordinates as functions of the time. But, force of the rheonomic constraints change are unknown (see Ref. [21]).

Let us now take into account that first n coordinates of the position vectors of the mass particles are independent generalized coordinates. Extended system of the

generalized coordinates containing independent coordinates q^i, $i=1,2,3,....,n$ and rheonomic coordinates $q^{0\gamma}=q^{n+\gamma}=\phi_\gamma(t)$, $\gamma=1,2,3....R$, then it is possible to list in the form:

$$
\begin{aligned}
&q^1=q_{(1)}{}^1, && .\\
&q^2=q_{(1)}{}^2, && .....,\\
&q^3=q_{(1)}{}^3, && q^{n-5}=q_{(n-1)}{}^1,\\
&q^4=q_{(2)}{}^1, && q^{n-4}=q_{(n-1)}{}^2,\\
&q^5=q_{(2)}{}^2, && q^{n-3}=q_{(n-1)}{}^3\\
&q^6=q_{(2)}{}^3,. && q^{n-2}=q_{(n)}{}^1,\\
&..... && q^{n-1}=q_{(n)}{}^2,\\
& && q^n=q_{(n)}{}^3
\end{aligned}
$$

$$q^{0\gamma}=q^{n+\gamma}=\phi_\gamma(t),\ \gamma=1,2,3....R \tag{14}$$

On the basic of the listed system (14), we can conclude that in considered case, we use coordinates of the positions vectors of the first $K\le\frac{n}{3}=\frac{1}{3}(3N-G-R)$ mass particle as generalized independent coordinates.

Then on the basis of previous for the coordinates of the geometrical point which correspond to the mass particle positions at arbitrary moment of the motion, we can write:

$$
\begin{aligned}
&N_1\left(q^1=q_{(1)}{}^1,q^2=q_{(1)}{}^2,q^3=q_{(1)}{}^3\right),\\
&N_2\left(q^4=q_{(2)}{}^1,q^5=q_{(2)}{}^2,q^6=q_{(2)}{}^3\right),\\
&.....,\\
&N_K\left(q^{3K-2}=q_{(K)}{}^1,q^{3K-1}=q_{(K)}{}^2,q^{3K}=q_{(K)}{}^3\right)\\
&N_{K+j}\left(q_{(K+j)}{}^1\left(q^1,q^2,.....q^n,q_{n+1},.....q^{n+R}\right),q_{(K+j)}{}^2\left(q^1,q^2,.....q^n,q_{n+1},.....q^{n+R}\right),q_{(K+j)}{}^3\left(q^1,q^2,.....q^n,q_{n+1},.....q^{n+R}\right)\right)\\
&j=1,2,3,...,(N-K)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\vec\rho_1\left(q^1=q_{(1)}{}^1,q^2=q_{(1)}{}^2,q^3=q_{(1)}{}^3\right),\\
&\vec\rho_2\left(q^4=q_{(2)}{}^1,q^5=q_{(2)}{}^2,q^6=q_{(2)}{}^3\right),\\
&.....,\\
&\vec\rho_K\left(q^{3K-2}=q_{(K)}{}^1,q^{3K-1}=q_{(K)}{}^2,q^{3K}=q_{(K)}{}^3\right)\\
&\vec\rho_{K+j}\left(q^1,q^2,.....q^n,q_{n+1},.....,q^{n+R}\right),\quad j=1,2,3,...,(N-K)
\end{aligned}
\tag{16}
$$

2.2.5. Concluding remarks

We can see that in extended system of generalized coordinates, we can identified two sets of the position vectors of the mass particles: one set (15) contain K, $K\le\frac{n}{3}=\frac{1}{3}(3N-G-R)$ position vectors of the mass particles depending of three

generalized coordinates, and second set (16) contain the $N-K\geq\frac{G+R}{3}$, $K\leq\frac{n}{3}=\frac{1}{3}(3N-G-R)$ position vectors of the mass particles depending of all $p=3N-G$ generalized coordinates in general case, or more then of three generalized coordinates.

Also we can conclude that in extended system of generalized coordinates, we can identified two sets of the position vectors of the mass particles, one set (15) contain the K, $K\leq\frac{n}{3}=\frac{1}{3}(3N-G-R)$ position vectors of the mass particles with three-dimensional tangent space and each with three basic vectors of this tangent spaces, and second set (16) contain the $N-K\geq\frac{G+R}{3}$, $K\leq\frac{n}{3}=\frac{1}{3}(3N-G-R)$ position vectors of the mass particles with extended dimension of the tangent space and to each tangent space correspond $p=3N-G$ basic vectors in general case, or more than three basic vectors of the tangent space.

Open directions for next research and applications. As a possible open directions for next research and application are: an analysis of expressions for generalized Corilis forces introduced by changing position of motion observer from fixed coordinate system to rotate curvilinear coordinate system correspond to vector position tangent space; applications of the previous results for solving problems of the numerous coordinate system properties used in astro-dynamics; extension and proof of extension Lagrange differential equations to the description of the rheonomic system dynamics and necessary generalizations.

3. ADVANCES IN ANALYTICAL MECHANICS.

3.1. Analytical mechanics of hereditary discrete system vibrations

3.1.1. Introduction

Integro-differential equations and their applications in development of analytical mechanics of discrete hereditary systems are used by *Gorosko and Hedrih (Stevanović)* (see References III [35-54])

Research results in area of mechanics of hereditary discrete systems, obtained by Gorosko and Hedrih (Stevanović) (see References IV [35-54]) are generalized and presented in the monograph [35] which contains first completed presentation of the analytical dynamics of hereditary discrete systems. Two classes of dynamically defined and undefined hereditary systems are defined and considered by introducing corresponding restrictions. Main results of mechanics of hereditary discrete systems are presented with new applications important to engineering.

Approximation of expressions for the coefficients of damping and corresponding decrements as well as for circular frequency of oscillations of hereditary oscillatory systems are obtained with high accuracy in the first and second approximations.

Analogy between hereditary interactions and reactive forces in systems of automatic control is identified and a possibility to extend theory of analytical dynamics of hereditary systems to mechanical systems with automatic control is pointed out (see References [36] and [37] by Gorosko and Hedrih (Stevanović)).

The Lagrange's mechanics of hereditary systems are extended and generalized to thermo-rheological [35,52] and piezo-rheological [35] discrete mechanical systems as well as to discrete mechanical systems with standard light creep elements. .

Analytical dynamics as general science of mechanical system motions was founded by Lagrange (*Joseph Luis Lagrange* (1735-1813)) in the period of his work at Berlin Academy. The Lagrange's book "*Mècanique Analytique*" contains basic analytical methods of mechanics and was published in France in 1788. Introduced analytical methods in Mechanics by Lagrange are main and first base of analytical mechanics in general. Lagrange's equations of second kind and Lagrange's equations of first kind with unwoven Lagrange's multipliplicators of constraints are main fundament of Analytical Dynamics.

Analytical dynamics is largely applied and used in engineering system dynamics and in natural sciences as well as for investigation of mechanical system dynamics and in physics of the microworld.

Mechanics of hereditary continuum is presented by series of fundamental publications and monographs. In current literature term "*hereditary*" and "*rheological*" systems are equivalent. Mechanics of discrete hereditary systems up to a ten years before was presented only by separate single papers and containing only solutions of partial problems.

Research results in area of mechanics of hereditary discrete systems, obtained by authors of this paper, are generalized and presented in the monograph [35], published in 2001 by Gorosko and Hedrih (Stevanović)), which contains first presentation of analytical dynamics of hereditary discrete systems. We can conclude that this monograph contains *complete foundation of analytical dynamics theory of discrete hereditary systems* and by using these results, numerous examples are obtained and solved (see Refs. [35-54]). In this analytical mechanics of hereditary discrete systems, modified Lagrange's differential equations second kind in the form differential and integro-differential forms with kernels of relaxation or rhelogy are derived.

3.1.2. Models of hereditary elements in analytical dynamics of hereditary discrete systems.

Hereditary system is each system which contains mutual hereditary interaction between material particles in the form of one or more coupling constraints with hereditary properties.

Simple visco-elastic element is Voight's type element (Woldemar Voigt, 1859-1919). In the state of extension resultant force appears by two components, one by visco and one by elastic properties in the deformation of visco-elastic element and constitutive stress-strain relation given as relation between force and extension of element in the following form:

$$P(t) = cy(t) + \mu\dot{y}(t) \tag{1}$$

In Mechanics of hereditary continuum in the case of axial (in one direction) stressed and deformed Voight's type body stress strain constitutive relation is expressed by following relation: $\sigma(t) = E\varepsilon(t) + \mu\dot{\varepsilon}(t)$. More acceptable and precise and better compatible with experimental data with real hereditary body properties is model of the standard visco-elastic body (Kelvin and Poyting-Thompson's body). Constitutive stress strain relation given as relation between force and extension of element in the following form:

$$n\dot{P}(t) + P(t) = nc\dot{y}(t) + \tilde{c}y(t) \tag{2}$$

In mechanics of hereditary system constants n, c and $\tilde{c}$ obtain special names: time of relaxation, rigidity coefficients, one momenteneous and prologues one.

For generalized hereditary element model relation between force and deformation is possible to describe by differential equation high order derivative in the following form:

$$\sum_{r=1}^{m} a_k \frac{d^k P}{dt_k} + P(t) = b_0 x + \sum_{k=1}^{m} b_k \frac{d^k x}{dt^k} \tag{3}$$

For more complex viscose elements (represented by the Jeffreys' bidy (J-body) and Lethersich's body) stress-strain state is described by differential equation in the form:

$$n\dot{P}(t) + P(t) = b_1\dot{y}(t) + nb_2\ddot{y}(t) \tag{4}$$

Equivalency and analogy of hereditary interactions and reactive forces in systems of automatic control gives possibility to extend theory of analytical dynamics of hereditary systems to mechanical systems with automatic control. For example, automaton with transfer function presented in the following form (see References [36] and [37] by Gorosko and Hedrih (Stevanović)):

$$W(p) = \frac{b_0 + b_1 p + ... + b_n p^n}{1 + a_1 p + ... + a_n p^n} \tag{5}$$

presents a hereditary interaction (3) between material particles of the discrete mechanical system with one degree of freedom.

Parameters of the automaton of arbitrary structures are defined in an experimental way and it is possible to obtain amplitude-phase characteristic. In our opinion there are real possibilities and perspective to use method of amplitude-phase characteristic for experimental obtaining of mechanical characteristic of the hereditary discrete mechanical systems. It is possible to solve some difficulties with identification coefficient of the momenteneous rigidity which appear in the mechanical investigation of the hereditary forms and shortened longtime experiments.

3.1. 3. Integral models of the stress-strain state of the hereditary elements.

There are three mathematical forms for description of constitutive relations of hereditary properties of hereditary interaction [35], in the building of hereditary system's mechanics. These forms are(see References [36] amd [37] by Gorosko and Hedrih (Stevanović)):

1* Differential equation, expressed in the form of dependence reaction force P of the rheological coordinate x, usually presented as deformation or relative displacement of the hereditary constraint in the form (3).

2* Integral equation, expressed in the form of dependence reaction force P of the rheological coordinate y, usually presented as deformation or relative displacement of the hereditary constraint:

$$P(t) = c\left[y(t) - \int_0^t \mathcal{H}(t-\tau) y(\tau) d\tau \right] \tag{6}$$

where $\mathcal{H}(t-\tau) = \frac{c-\tilde{c}}{nc} e^{-\frac{1}{n}(t-\tau)}$ is relaxation kernel, and $\beta = \frac{1}{n}$ is coefficient of the element relaxation.

This integral relation (6) can be obtained by solving equation (2) with respect to the force P. By this integral equation, the relaxation of the reaction force P depending on the rheological coordinate y, is presented and expressed.

For the case of the generalized standard hereditary element (3) integral equation is possible to obtain in the form (6) in which relaxation kernel $\mathcal{H}(t-\tau)$ presents sum by sum of exponents.

3* Integral equation, expressed in the form of dependence rheological coordinate y, usually presented deformation or relative displacement of the hereditary constraint and reaction force P:

$$y(t) = \frac{1}{c}\left[P(t) + \int_0^t \mathfrak{K}(t-\tau) P(\tau) d\tau \right] \tag{7}$$

where $\mathfrak{K}(t-\tau) = \frac{c-\tilde{c}}{nc} e^{-\frac{\tilde{c}}{nc}(t-\tau)}$ is kernel of rheology and $\beta_1 = \frac{\tilde{c}}{nc}$ is the coefficient of the creep or retardation or rheology.

3.1. 4. Three forms of equations of motions of a hereditary oscillator.

Simple model of a hereditary discrete system is hereditary oscillator with one degree of freedom which contains one material particle with mass m and one standard hereditary element P with material visco-elastic properties defined by following coefficients: n, c and $\tilde{c}$ constitutive stress-strain relation expressed by relation (2) between force $P(t)$ and generalized and rheological coordinate $y(t)$. Then by using principle of dynamical equilibrium of the oscillator it is possible to obtain equation of the oscillator motion in the following form:

$$m\ddot{y}(t) + P(t) = F(t) \tag{8}$$

where $P(t)$ is resistive reaction of the rheological element, $F(t)$ external forced excitation. Using constitutive relation (2) or (10) for stressed and deformed standard hereditary (rheological) element for eliminating resistive reaction of the rheological

element $P(t)$ from last equation (8) we obtain three corresponding forms of the equation of motion of the rheological – hereditary oscillator with one degree of freedom listed as follow: one in differential form:

$$nm\dddot{y}(t)+m\ddot{y}(t)+nc\dot{y}(t)+\tilde{c}y(t)=n\dot{F}(t)+F(t) \tag{9}$$

and two in integro-differential forms:

$$m\ddot{y}(t)+c\left[y(t)-\int_{-\infty}^{t}\mathfrak{R}(t-\tau)y(\tau)d\tau\right]=F(t) \tag{10}$$

$$m\ddot{y}(t)+c\left[y(t)+\int_{-\infty}^{t}\mathfrak{K}(t-\tau)m\ddot{y}(\tau)d\tau\right]=F(t)+c\int_{-\infty}^{t}\mathfrak{K}(t-\tau)F(\tau)d\tau \tag{11}$$

For the case of the weak singular hereditary oscillator equation of the dynamic equilibrium (oscillator motion) in the differential form is not possible to obtain, but in the integro-differential forms is possible.

3.1. 5. Thermo-rheological pendulum

3.1. 5.1. Light standard thermo-rheological hereditary element

When standard hereditary element is modified by two temperatures $T_K(t)$ and $T_M(t)$, which are introduced by thermo-modification of visco-elastic properties by temperature $T_K(t)$, and by thermo-modification of elasto-viscosic properties by temperature $T_M(t)$, than constitutive relation between stress and strain state of the thermo-rheological hereditary element (see Ref. [35]) is:

$$n\dot{P}(t)+P(t)+n\dot{F}_M(t)+F_K(t)=nc\dot{\rho}(t)+\tilde{c}\left[\rho(t)-\rho_0\right] \tag{12}$$

in which

$$F_M(t)=c_M\alpha_M T_M(t),\ F_K(t)=c_K\alpha_K T_K(t) \tag{13}$$

are thermo-elastic forces, and $\rho(t)$ is rheological coordinate, c_M, c_K are coefficients of thermo-elastic rigidity, α_M, α_K are coefficients of thermo-elastic dilatations, n is time of relaxation, and $c, \tilde{c}$ an instantaneous rigidity and a prolonged one of an element.

Constitutive relation (12) of the thermo-rhelogical hereditary element from differential form, we can rewrite in two integro-differential forms.

3.1. 5. 2. Light standard piezo-and thermo-rheological hereditary element

When standard hereditary element is modified by two polarization voltages $U_K(t)$ and $U_M(t)$, which are introduced by piezo-modification of visco-elastic properties of subelement of piezoceramics, by $U_K(t)$ and by piezo-modification of elasto-viscosic properties by $U_M(t)$, and thermo-modified by two temperatures $T_K(t)$ and $T_M(t)$, than constitutive relation between stress and strain state of the piezo-rheological hereditary hybrid element is in the form (12) in which

$$\begin{aligned} F_M(t)&=c_{UM}\alpha_{UM}U_M(t)+c_{TM}\alpha_{TM}T_M(t)\\ F_K(t)&=c_{UK}\alpha_{UK}U_K(t)+c_{TK}\alpha_{TK}T_K(t) \end{aligned} \tag{14}$$

are thermoelastic and piezo-elastic forces, and $\rho(t)$ is rheological coordinate, c_{TM}, c_{TK} are coefficients of thermo-elastic rigidity, α_{TM}, α_{TK} are coefficients of thermo-elastic dilatations, c_{UM}, c_{UK} are coefficients of piezo-elastic rigidity, α_{UM}, α_{UK} are coefficients of piezo-elastic dilatations n is time of relaxation, and $c, \tilde{c}$ an instantaneous rigidity and a prolonged one of an hybrid element.

3.1. 5.3. Pendulum with standard thermo-rheological hereditary element

The thermo-rheological hereditary pendulum has two degrees of freedom, one degree of motion freedom defined by angular coordinate ϑ and one degree of deformations freedom defined by changeable length of thread as a coordinate $\rho(t)$.

Let us compose the equations of the thermo-rheological pendulum dynamics with thread in which the standard thermo-rheological hereditary element with constitutive stress-strain relation (12) is incorporated. Now, by introducing force $P(t)$ of the extension of the thermorheological hereditary thread from constitutive relation (12) presented into integral form, the equations of the pendulum motion are in the forms (for detail see Reference [52] by Hedrih (Stevanović)):

$$\ddot{\rho} - (\rho_o + \rho(t))\dot{\theta}^2 + g\cos\theta + \frac{c}{m}\left[\rho(t) - \int_0^t \rho(\tau)\mathrm{R}(t-\tau)d\tau\right] = \\ = \frac{1}{m}F_M(t) - \frac{c}{m(c-\tilde{c})}\int_0^t [F_M(\tau) - F_K(\tau)]\mathrm{R}(t-\tau)d\tau P(t) \tag{15}$$

$$(\rho_0 + \rho(t))^2\ddot{\theta} + 2(\rho_0 + \rho(t))\dot{\theta}\dot{\rho}(t) + g(\rho_0 + \rho(t))\sin\theta = \mathfrak{M}(t) \tag{16}$$

This system is a system with one integro-differential and one differential equation of the thermo-rheological hereditary pendulum with motion in vertical plane.

If the thermo-rheological pendulum is in the horizontal plane, from second differential equation of the previous system, we can obtain the relation between the length of the pendulum thread and of the angular velocity in the following form:

$$\dot{\theta}(t) = \dot{\theta}(0)\left[\frac{\rho_0 + \rho(0)}{\rho_0 + \rho(t)}\right]^2 \tag{17}$$

By introducing this previous expression (17) in the first equation of the system (16) (for the case of horizontal plane) the following integro-differential equation for the pendulum length thread is obtained:

$$\ddot{\rho}(t) - [\dot{\theta}(0)]^2\frac{[\rho_0 + \rho(0)]^4}{[\rho_0 + \rho(t)]^3} + \frac{c}{m}\left[\rho(t) - \int_0^t \rho(\tau)\mathrm{R}(t-\tau)d\tau\right] = \\ = \frac{1}{m}F_M(t) - \frac{c}{m(c-\tilde{c})}\int_0^t [F_M(\tau) - F_K(\tau)]\mathrm{R}(t-\tau)d\tau P(t) \tag{18}$$

3.1..6. Concluding remarks

Solution of obtained integrdo-differential equation (18) is mathematical problem in analytical mechanics of hereditary discrete system dynamics.

Also, solutions of the similar integro-differential equations are tasks for mathematics in the function of applications in mechanics and engineering system dynamics with hereditary properties.

In the basis of the construction of Lagrange's mechanics of hereditary discrete systems, the classical mechanics principles are used [35]. These principles are: Principle of the work of forces along corresponding possible system displacements, as well as Principle of dynamical equilibrium.

Initial conditions of hereditary system dynamics are very important, containing the history of rheological interactions of the system. Then, it is important to take into account stress-strain history of viscoelastic elements – interactions between hereditary system material particles.

Analogy between hereditary interactions and reactive forces in the systems of automatic control gives possibility to extend theory of analytical dynamics of hereditary systems to mechanical systems with automatic control.

For description of properties of dynamics of a hereditary system by using relaxational or rheological kernel (resolvent), these kernels are expressed by exponential or fractional-exponential forms [35] . Descriptions of hereditary properties of the system by using differential forms (2) and integral form (6) and (7) with exponential kernels are equivalent. For the case of fractional-exponential forms of the kernel (6) and (7) in the integral form corresponding equivalent differential forms not exist.

The Lagrange's mechanics of hereditary systems is extended and generalized to the thermo-rheological [35, 52] and piezo-rheological [35] mechanical systems.

Open directions for next research and applications. Directions for next research in area of mechanics of hereditary discrete system must be focused to find analytical forms of solutions or approximations of solutions of integro-differential equations and to build mathematical theory of the material memory of the history of previous stress and strains in the material before starting system motion and its observation. Mathematical theory for slowing problems with determinations of the initial conditions of the hereditary system is second main task in this area. Present in science, there are numerous numerical approach and numerical experiments over the integro-differential equations and numerical procedure expressed by software tools but for advances in area of analytical dynamics of hereditary systems it is necessary analytical approach, solutions and qualitative methods for evaluations of the system solution stability.

For practical applications in mechanics and engineering system dynamics analytical forms of the approximations of solutions of intgro-differential equations are necessary for easier quantitative estimation larger class of the dynamic phenomena hereditary system behavior. All real constructions and engineering structures are with hereditary properties.

3.2. Analytical mechanics of fractional order discrete system vibrations

3. 2.1. Introduction

Differential fractional order equations and theirs applications in development of analytical dynamics of discrete and continuous fractional order systems with like single and multi-frequency modes, or fractional order modes are important results for applications in different area of science and practice (see References (see References III [39-46]).

Discrete continuum method is based on the continuum discretization and coupling by standard light hereditary, or fractional order elements (see References [49] and [50]). Main nets, main chains and mail fractional order oscillators with main fractional order modes of plane as well as space coupled mechanical chains (cases with ideal elastic, hereditary and fractional order). Fractional order standard light elements have applications in mechanics of continuum (models of longitudinal and transversal oscillations of beams), in biomechanics (mechanical models of double helix DNA chains [45]), to systems with coupled pendulums, as well is in signals transfer (see ReferencesIII.).

3.2.2. Standard light fractional order element

Standard light coupling element of negligible mass is in the form of axially stressed rod without bending, and which has the ability to resist deformation under static and dynamic conditions. ***Standard light fractional order creep element*** for which the constitutive stress-strain relation for the restitution force as the function of element elongation is given by fractional order derivatives in the form (see References [40] and [41] by Hedrih (Stevanović)):

$$P(t) = -\left\{c_0 x(t) + c_\alpha \mathfrak{D}_t^\alpha [x(t)]\right\} \tag{1}$$

where $\mathfrak{D}_t^\alpha[\bullet]$ is operator of the α^{th} derivative with respect to time t in the following form:

$$\mathfrak{D}_t^\alpha [x(t)] = \frac{d^\alpha x(t)}{dt^\alpha} = x^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau \tag{2}$$

where c, c_α are rigidity coefficients – momentary and prolonged one, and α a rational number between 0 and 1, $0 < \alpha < 1$.

3.2.3. Governing equations of the fractional order multi-chain plane system model

Coupled governing fractional order differential equations of the multi chain fractional order plane system vibrations, according notation in Figures 5.a* and b*, and determined standard light fractional elements by constitutive relation (1) and (2) , used for coupling of the mass particles, are in the following form:

$$
\begin{aligned}
m_{k,j}\ddot{x}_{k,j} = &-c_{k-1,j}\left(x_{k,j}-x_{k-1,j}\right)+c_{k,j}\left(x_{k+1,j}-x_{k,j}\right)-c_{\alpha,k-1,j}\mathfrak{D}_t^{\alpha}\left[x_{k,j}-x_{k-1,j}\right]+\\
&+c_{\alpha,k,j}\mathfrak{D}_t^{\alpha}\left[x_{k+1,j}-x_{k,j}\right]-\tilde{c}_{k,(j-1,j)}\left(x_{k,j}-x_{k,j-1}\right)+\tilde{c}_{k,(j,j+1)}\left(x_{k,j+1}-x_{k,j}\right)-\\
&-\tilde{c}_{\alpha,k,(j-1,j)}\mathfrak{D}_t^{\alpha}\left[x_{k,j}-x_{k,j-1}\right]+\tilde{c}_{\alpha,k,(j,j+1)}\mathfrak{D}_t^{\alpha}\left[x_{k,j+1}-x_{k,j}\right]
\end{aligned}
\qquad (3)
$$

$k=1,2,3,4,....,N$, $j=1,2,3,4,....,M$.

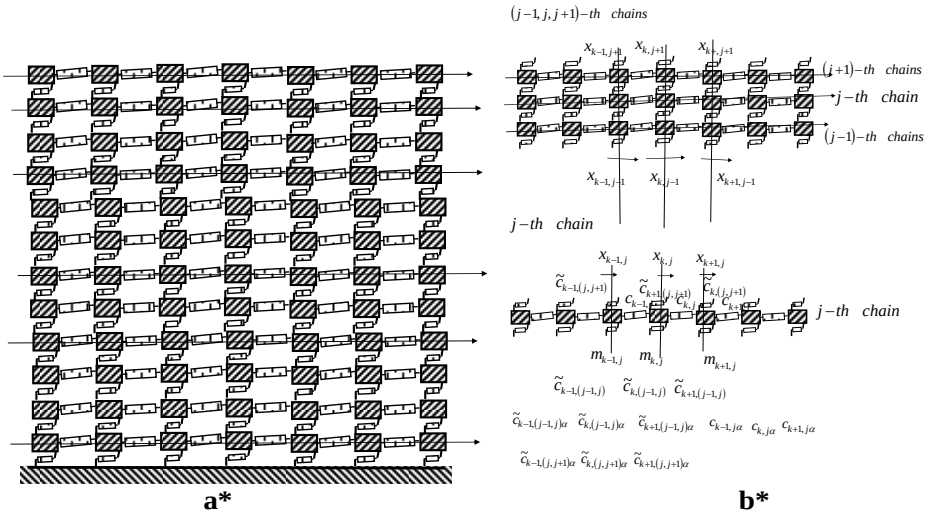


Figure 5. Discrete continuum fractional order model in a plane - Hybrid multi chain fractional order plane system. **(a*)** Hybrid multi-chain system, in one plane and in the form of coupled chains by standard light fractional order elements and in the cantilever form of boundary conditions; **(b*)**. Three coupled chains $(j-1)$-th , j-th and $(j+1)$-th chains, $j=1,2,3,4,....,M$, as part – subsystem of the hybrid multi chain system with coupling elements and kinetic parameters: masses, stiffnesses and fractional order parameter of the fractional order element and generalized coordinates of the system. Attached a separate j-th chain of the hybrid chain system with notation of the generalized coordinates $x_{i,j}$, $k=1,2,3,4,....,N$, $j=1,2,3,4,....,M$.

For the homogeneous plane system corresponding to the system (3) of fractional order differential equations let us introduce the coordinate transformation, in accordance with trigonometric method (see References [54] by Rašković and [43], [44] and [40] by Hedrih (Stevanović)) in the following form: :

$$
x_{k,j}=\sum_{s=1}^{s=M}\xi_{k(s)}\sin j\varphi_s\ ,\ k=1,2,3,4,....,N\ ,\ j=1,2,3,4,....,M\ . \qquad (4)
$$

where $\xi_{k(s)}$ are normal coordinates of the main chains of the hybrid plane system as well as generalized coordinates of the s-th main chain from the sets and for the corresponding linear system are in the form: $\xi_{k(s)}=C_{k(s)}\cos\left(\tilde{\omega}_{k(s)}t+\alpha_{k(s)}\right)$ and

$\tilde{\omega}^2_{k(s)} = \tilde{u}_{k(s)} \frac{\kappa c}{m} = 4 \frac{\kappa c}{m} \sin^2 \frac{\varphi_s}{2}$, $\tilde{u}_{k(s)} \equiv \tilde{u}_{(s)}$ and φ_s are eigen characteristic numbers of the hybrid system obtained by using the trigonometric method (see Refs. [54] by Rašković (1965) and [40], [43], [44], [45], [46] and [48] by Hedrih (Stevanović) (2002, (2004), (2007), (2009) and (2010)) depending of boundary conditions of the transversal chains in the form of the longitudinal chain connections.

$$\frac{m}{c} \ddot{\xi}_{k(s)} - \xi_{k-1(s)} + \left(2 + \kappa \tilde{u}_{k(s)}\right) \xi_{k(s)} - \xi_{k+1(s)} + \mathfrak{D}_t^{\alpha} \left[-\xi_{k-1(s)} + \left(2\kappa_{\alpha} + \tilde{\kappa}_{\alpha} \tilde{u}_{k(s)}\right) \xi_{k(s)} - \xi_{k+1(s)} \right] = 0$$

$$k = 1,2,3,4,....,N \, , s = 1,2,3,....,M \tag{5}$$

Let us introduce the following coordinate transformation:

$$\xi_{k(s)} = \sum_{r=1}^{r=N} \xi_{k(s)(r)} = \sum_{r=1}^{r=N} \eta_{(s)(r)} \sin k\vartheta_r \, , \; s = 1,2,3,....,M \tag{6}$$

Taking into account that $\sin k\vartheta_r$ is different them zero in arbitrary cases from (3), we can obtain the transformed system of the governing fractional order differential equations (5) of the eigen main chains as well as the transformed basic governing system of fractional order differential equations (3), with respect to the new introduced coordinates $\eta_{(s)(r)}$ containing $M \times N$ independent subsystems for each pair of the $(s)(r)$ from the sets $s = 1,2,3,....,M$ and $r = 1,2,3,4,....,N$ in the following forms:

$$\ddot{\eta}_{(s)(r)} + \omega^2_{(s)(r)} \eta_{(s)(r)} + \omega^2_{\alpha(s)(r)} \mathfrak{D}_t^{\alpha} \left[\eta_{(s)(r)} \right] = 0 \, , \tag{7}$$

$s = 1,2,3,....,M \, , r = 1,2,3,4,....,N$

where

$$\omega^2_{\alpha(s)(r)} = \frac{c}{m} \left\langle 2\left(\kappa_{\alpha} - 1\right) + u_{(s)(r)} + \left(\tilde{\kappa}_{\alpha} - \kappa\right) \tilde{u}_{k(s)} \right\rangle \, , \; s = 1,2,3,....,M \, , r = 1,2,3,4,....,N \tag{8}$$

This last system of fractional order differential equations (7) represents $M \times N$ independent partial fractional order differential equations describing independent fractional order oscillators each with one degree of freedom and eigen normal coordinate $\eta_{(s)(r)}$, $s = 1,2,3,....,M \, , r = 1,2,3,4,....,N$ of the considered fractional order hybrid system and containing N sets of the M eigen main chains normal coordinates $\eta_{(s)(r)} \, , s = 1,2,3,....,M$, $r = 1,2,3,4,....,N$. Then, we can conclude that simultaneously with determination of the normal coordinates of the eigen main chains, we determine as well as normal coordinates of the considered hybrid fractional order plane system vibrations with $M \times N$ degrees of freedom. Also, we can conclude that normal coordinates for the linear system, correspond to the normal coordinates of the corresponding fractional order system and expressions for generalized coordinate transformation to the eigen normal coordinates of the basic linear system vibrations, we can use for the corresponding coordinate transformation of the corresponding fractional order hybrid system vibrations to the eigen normal coordinates.

3.2.4. Eigen factional order signals and eigen main chain signals in the fractional order multi-chain plane system model

Type of the obtained fractional order differential equations in the system (7) is same as in numerous author's papers, but with different coefficients. These coefficients

are sets of eigen circular frequencies $\omega^2_{(s)(r)}$ and fractional order material properties characteristic numbers $\omega^2_{\alpha(s)(r)}$ in the form:

$$\omega^2_{(s)(r)} = \frac{c}{m}\left[4\kappa \sin^2\frac{\varphi_s}{2} + 2(1-\cos\vartheta_r)\right], \; s = 1,2,3,....,M \, , r = 1,2,3,4,...., N \tag{9}$$

$$\omega^2_{\alpha(s)(r)} = \frac{c}{m}\left\langle 2(\kappa_\alpha - 1) + 2(1-\cos\vartheta_r) + 4\tilde{\kappa}_\alpha \sin^2\frac{\varphi_s}{2}\right\rangle, \quad s = 1,2,3,....,M \, , r = 1,2,3,4,...., N \tag{10}$$

which depend of boundary multi-chain system conditions determining characteristic numbers: φ_s and ϑ_r (see Refs. [54] by Rašković (1965) and [40], [43], [44], [45], [46] and [48] by Hedrih (Stevanović)).

Now, taking into account solutions of the fractional order differential equation (see Reference [38] by Bačlić and Atanacković (2000)), for the system of the fractional order differential equations (7), we can write corresponding solutions of $\eta_{(s)(r)}$, $s = 1,2,3,....,M \, , r = 1,2,3,4,...., N$, in the form of fractional order like one frequency time functions which are eigen main modes and normal coordinate $\eta_{(s)(r)}$ of the fractional order plane system in the following expansion:

$$\begin{aligned}\eta_{(s)(r)}(t) = \eta_{(s)(r)}(0)\sum_{k=0}^{\infty}(-1)^k \omega^{2k}_{\alpha(s)(r)} t^{2k} \sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j \omega^{2j}_{\alpha(s)(r)} t^{-\alpha j}}{\omega^{2j}_{(s)(r)}\Gamma(2k+1-\alpha j)} + \\ + \dot{\eta}_{(s)(r)}(0)\sum_{k=0}^{\infty}(-1)^k \omega^{2k}_{\alpha(s)(r)} t^{2k+1} \sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j \omega^{-2j}_{\alpha(s)(r)} t^{-\alpha j}}{\omega^{2j}_{(s)(r)}\Gamma(2k+2-\alpha j)}\end{aligned} \tag{11}$$

$s = 1,2,3,....,M \, , r = 1,2,3,4,...., N$

This last expressions (11) determines eigen normal fraction order like one frequency modes of the hybrid fractional order system corresponding to one eigen circular frequency and corresponding eigen fractional order properties characteristic number (10) of material fractional order properties of standard light elements. Also, we can conclude that the expressions (11) are mathematical descriptions of the main normal fractional order like one frequency signals.

Now, taking into account coordinate transformation:

$$\xi_{k(s)} = \sum_{r=1}^{r=N}\xi_{k(s)(r)} = \sum_{r=1}^{r=N}\eta_{(s)(r)}\sin k\vartheta_r \tag{12}$$

$$x_{k,j} = \sum_{s=1}^{s=M}\xi_{k(s)}\sin j\varphi_s = \sum_{s=1}^{s=M}\sum_{r=1}^{r=N}\eta_{(s)(r)}\sin k\vartheta_r \sin j\varphi_s \tag{13}$$

and solutions (11), we obtain expressions for the like multi-frequency fractional order generalized coordinate in the following form:

1* eigen normal coordinates for obtaining eigen main chains and generalized coordinate of the eigen main chains,

$$\begin{aligned}\xi_{p(s)} = \sum_{r=1}^{r=N}\eta_{(s)(r)}(0)\sin p\vartheta_r \sum_{k=0}^{\infty}(-1)^k \omega^{2k}_{\alpha(s)(r)} t^{2k} \sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j \omega^{2j}_{\alpha(s)(r)} t^{-\alpha j}}{\omega^{2j}_{(s)(r)}\Gamma(2k+1-\alpha j)} + \\ + \sum_{r=1}^{r=N}\dot{\eta}_{(s)(r)}(0)\sin p\vartheta_r \sum_{k=0}^{\infty}(-1)^k \omega^{2k}_{\alpha(s)(r)} t^{2k+1} \sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j \omega^{-2j}_{\alpha(s)(r)} t^{-\alpha j}}{\omega^{2j}_{(s)(r)}\Gamma(2k+2-\alpha j)}\end{aligned} \tag{14}$$

$p = 1,2,3,...,N$, $s = 1,2,3,....,M$

2* generalized coordinate of the hybrid fractional order system:

$$x_{p,q} = \sum_{s=1}^{s=M}\sum_{r=1}^{r=N} \sin p\vartheta_r \sin q\varphi_s \sum_{r=1}^{r=N} \eta_{(s)(r)}(0) \sin p\vartheta_r \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha(s)(r)}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha(s)(r)}^{2j} t^{-\alpha j}}{\omega_{(s)(r)}^{2j} \Gamma(2k+1-\alpha j)} +$$
$$+ \sum_{s=1}^{s=M}\sum_{r=1}^{r=N} \sin p\vartheta_r \sin q\varphi_s \sum_{r=1}^{r=N} \dot{\eta}_{(s)(r)}(0) \sin p\vartheta_r \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha(s)(r)}^{2k} t^{2k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha(s)(r)}^{-2j} t^{-\alpha j}}{\omega_{(s)(r)}^{2j} \Gamma(2k+2-\alpha j)}$$
$$p = 1,2,3,...,N \text{ , } Q = 1,2,3,....,M \qquad (15)$$

3.2.5. Eigen factional order signals and eigen main chain signals in the fractional order multi-chain space system model

For the homogeneous fractional order multi-chain space system dynamics, presented in Figure 6, by the similar way as in previous chapter III.2.4. is possible to write system of coverning fractional order differential equations by use coordinate notation from Figure 6, and corresponding material system coordinates. Then let introduce the following coordinate transformation:

$$x_{(i),(j),(k)} = \sum_{s=1}^{s=K} \xi_{(i),(j)}^{(s)} \sin k\varphi_s \text{ , } i = 1,2,3,4,....,N \text{ , } j = 1,2,3,4,....,M \text{ , } k = 1,2,3,....,K \text{ .} \qquad (16)$$

where $\xi_{(i),(j)}^{(s)}$, $i = 1,2,3,4,....,N$, $j = 1,2,3,4,....,M$ for each $s = 1,2,3,....,K$, are normal coordinates of the main plane subsystems $R^{(s)}$, $s = 1,2,3,....,K$ in the form of the independent K plane nets each consisting of the coupled M chains each with N degrees of freedom.

φ_s , $s = 1,2,3,....,K$ are eigen characteristic numbers of the hybrid system and according trigonometric method (see Refs. [54] by Rašković (1965) and [40], [43], [44], [45], [46] and [48] by Hedrih (Stevanović)) depending on boundary conditions of the transversal coupled chains in the form of the normal direction of chain connections between parallel plane nets, determined by direction of increasing indices $k = 1,2,3,....,K$. Each of these K main and independent plane nets are with $N \times M$ degree of freedom with $N \times M$ normal coordinates $\xi_{(i),(j)}^{(s)}$, $i = 1,2,3,4,....,N$, $j = 1,2,3,4,....,M$ for each $s = 1,2,3,....,K$ and are like multi frequency fractional order time functions form K independent subsets of circular frequencies $\omega_{(i),(j)}^{(s)}$ and corresponding fractional order characteristic numbers $\omega_{\alpha,(i),(j)}^{(s)}$ $i = 1,2,3,4,....,N$, $j = 1,2,3,4,....,M$ for each $s = 1,2,3,....,K$.

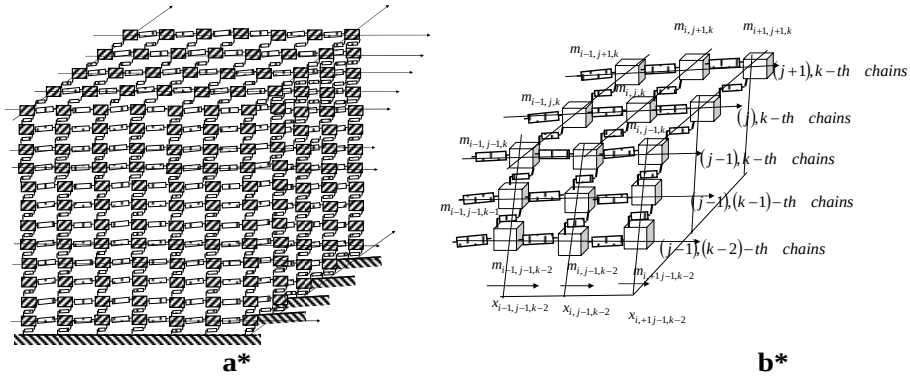


Figure 6. Discrete continuum fractional order space model - Hybrid multi chain fractional order space system. **(a*)** Hybrid multi-chain system, in space and in the form of coupled chains by standard light fractional order elements and in the cantilever form of boundary conditions; **(b*)**. The coupled chains $(j-1),k$ -th , j,k -th and $(j+1),k$ -th chains, and $(j-1),k$ -th , $(j-1),(k-1)$ -th , and $(j-1),(k-2)$ -th, $j=1,2,3,4,....,M$, $k=1,2,3,....,K$ as part – subsystem of the hybrid multi chain system with coupling elements and kinetic parameters: masses, stiffnesses and fractional order parameter of the fractional order element and generalized coordinates of the system, with notation of the generalized coordinates $x_{i,j,k}$, $k=1,2,3,4,....,N$, $j=1,2,3,4,....,M$, $k=1,2,3,....,K$.

Taking into account that $\sin k\varphi_s$ is different them zero in arbitrary cases from system of governing fractional order differential equations, we can obtain the transformed basic governing system of fractional order differential equations with respect to the coordinates $\xi^{(s)}_{(i),(j)}$, $i=1,2,3,4,....,N$, $j=1,2,3,4,....,M$ containing K independent subsystems of coupled fractional order differential equations of like multi-frequency $N\times M$ -frequency main plane nets for each s from the set of $s=1,2,3,....,K$ in the following forms:

$$\frac{m}{c}\ddot{\xi}^{(s)}_{(i),(j)}-\xi^{(s)}_{(i-1),(j)}+\left(2+\kappa\tilde{u}^{(s)}_{(i),(j)}\right)\xi^{(s)}_{(i),(j)}-\xi^{(s)}_{(i+1),(j)}+\kappa\left[-\xi^{(s)}_{(i),(j-1)}+2\xi^{(s)}_{(i),(j)}-\xi^{(s)}_{(i),(j+1)}\right]+$$
$$+\kappa_\alpha \mathfrak{D}^\alpha_t\left[-\xi^{(s)}_{(i-1),(j)}+\left(2+\kappa\tilde{u}^{(s)}_{(i),(j)}\right)\xi^{(s)}_{(i),(j)}-\xi^{(s)}_{(i+1),(j)}\right]+\kappa_\alpha \mathfrak{D}^\alpha_t\left[-\xi^{(s)}_{(i),(j-1)}+2\xi^{(s)}_{(i),(j)}-\xi^{(s)}_{(i),(j+1)}\right]=0$$
$$i=1,2,3,4,....,N\ ,\ j=1,2,3,4,....,M\ ,\ s=1,2,3,....,K \qquad (17)$$

Previous obtained K subsets of fractional order differential equations describing dynamics of the main subsystems is expressed by new coordinates $\xi^{(s)}_{(i),(j)}$, $i=1,2,3,4,....,N$, $j=1,2,3,4,....,M$ containing K independent subsystems for each s from the set of $s=1,2,3,....,K$.

These subsystems present K mathematical descriptions of dynamics of independent eigen main plane (or surface) nets containing coupled chains with corresponding subset of the eigen circular frequencies and corresponding fractional order characteristic

numbers of main plane (surface) nets. Through these eigen main plane nets is possible to transfer subset of the signals with frequencies from the corresponding subset of the eigen circular frequencies. These signals are fractional order like $N \times M$ frequency signals.
Next approach is similar as in the previous chapter III.2.4. for system containing coupled chains in one plane. Then, due to the limited length of the paper, we will not to present all derivatives and suppose writer to follow previpus chapter to obtain independent subsystems of the fractional order differential equations describing main independent fractional order oscillators each with one degree of freedom, in the form:

$$\ddot{\zeta}^{(s)(r)(p)} + \omega^2_{(s)(r)(p)}\zeta^{(s)(r)(p)} + \omega^2_{\alpha(s)(r)(p)}\mathfrak{D}^{\alpha}_t\left[\zeta^{(s)(r)(p)}\right] = 0\,, \tag{18}$$

$s = 1,2,3,....,K$, $r = 1,2,3,....,M$, $p = 1,2,3,....,N$

where

$$\xi^{(s)}_{(i),(j)} = \sum_{r=1}^{r=M} \eta^{(s)(r)}_{(i)} \sin j\vartheta_r \text{ , } i = 1,2,3,4,....,N \text{ , } j = 1,2,3,4,....,M \text{ , } s = 1,2,3,....,K \tag{19}$$

$$\eta^{(s)(r)}_{(i)} = \sum_{p=1}^{p=N} \zeta^{(s)(r)(p)} \eta^{(s)(r)}_{(i)} \sin i\psi_p \quad \text{, } i = 1,2,3,4,....,N \text{ . } s = 1,2,3,....,K \text{ ,}$$

$$r = 1,2,3,....,M \tag{20}$$

Previous system (18) contains $N \times M \times K$ independent fractional order differential equations each only along one coordinate $\zeta^{(s)(r)(p)}$, $s = 1,2,3,....,K$, $r = 1,2,3,....,M$, $p = 1,2,3,....,N$ . . These coordinates $\zeta^{(s)(r)(p)}$, $s = 1,2,3,....,K$, $r = 1,2,3,....,M$, $p = 1,2,3,....,N$, are normal coordinates of the hybrid discrete fractional order space system containing parallel coupled chains in the parallel planes and in the parallel lines in these planes.

Number of fractional order partial oscillators is equal to the product $N \times M \times K$ and equal to the number of the system degrees of freedom.

3.2.6. Concluding remarks

Then we can conclude that through eigen main plane (surface) nets $R^{(s)}$, $s = 1,2,3,....,K$, it is possible to transfer like $N \times M$ -eigen frequency fractional order signals as independent on other subsets of plane like $N \times M$ eigen frequency fractional order signals in other eigen main plane nets $R^{(s)}$, $s = 1,2,3,....,K$.

Each eigen main fractional order plane nets $R^{(s)}$, $s = 1,2,3,....,K$ is possible to decompose into M independent eigen chains, in total there are $M \times K$ main independent chains of all space system, with normal coordinates $\eta^{(s)(r)}_{(i)}$ $i = 1,2,3,4,....,N$, $s = 1,2,3,....,K$, $r = 1,2,3,....,M$ of these independent eigen chains . Then we can conclude that through each independent eigen main chain is possible to transfer like N - frequency fractional order signal, as well as that coordinate $\eta^{(s)(r)}_{(i)}$ are N - frequency fractional order time functions with corresponding main chains sub set of N - frequencies and corresponding characteristic fractional order properties numbers.

From the last obtained system (18) containing $N \times M \times K$ independent fractional order differential equations each along one normal coordinate $\zeta^{(s)(r)(p)}$, $s = 1,2,3,....,K$, $r = 1,2,3,....,M$, $p = 1,2,3,....,N$ we can conclude that each of these normal coordinates of the system is like one frequency fractional order time function with eigen circular frequency from the set $\omega^2_{(s)(r)(p)}$, $s = 1,2,3,....,K$, $r = 1,2,3,....,M$, $p = 1,2,3,....,N$ and fractional order characteristic numbers $\omega^2_{\alpha(s)(r)(p)}$, $s = 1,2,3,....,K$, $r = 1,2,3,....,M$, $p = 1,2,3,....,N$ describing fractional order properties of the system eigen vibrations. Number of fractional order partial oscillators is equal to the product $N \times M \times K$ and equal to the number of the system's degrees of freedom. Then we obtain normal coordinates for the transfer of one frequency fractional order signals through space fractional order vibration structure.

Then, we can conclude that simultaneously with determination of the normal coordinates of the eigen main nets and eigen main chains we determine as well as normal coordinates of the considered hybrid fractional order system with $N \times M \times K$ degree of freedom. Also, we can conclude that normal coordinates for the linear system, corresponding to normal coordinates of the corresponding fractional order space system and expressions for generalized coordinate transformations to the eigen normal coordinates of the basic linear system we can use for the corresponding coordinate transformation of the corresponding fractional order space hybrid system to the eigen normal coordinates for the considered hybrid system.

Open directions for next research and applications. Directions for next research in area of mechanics of fractional order discrete system must be focused to find analytical forms of solutions or approximations of solutions of fractional order differential equations different types and integrals.

Also, applications of the fractional derivatives and fractional integrals for describing constitutive relations of different types and sources of material . Research in this area nust be focused also to the experimental investigation of the material constants and parameters defined by fractional order derivatives and operators.

For practical applications in mechanics and engineering system dynamics analytical forms of the approximations of solutions of fractional order differential equations are necessary for easier quantitative estimation larger class of the dynamic phenomena fractional order system behavior. All real constructions and engineering structures are with plastic properties.

4. Advances in elastodynamics, nonlinear dynamics and hybrid system dynamics

4.1. Krilov-Bogolyubov-Mitropolyski asymptotic method of nonlinear mechanics, method of constant variation and averaging method

The different first approximations of solutions of nonlinear differential equations have very large applications in engineering practice for fast evaluations of the kinetic parameters of engineering dynamics (see Reference [55-57] by Hedrih (Stevanović) and [58-60] by by Hedrih (Stevanović) and Simonović). Some time these

first approximation are used for engineering practice with enough precisions and not necessary to use second and higher approximation. One of the main reason that in this part we take into consideration a comparison between first approximation obtained by different method, as well as used different starting known solution for obtaining first approximation.

Let compare three first approximations of the solution of a nonlinear differential equation with small nonlinearity, describing dynamics of nonlinear oscillator with one degree of freedom (see Figure 5.a*) , in the form (see Reference [55-57] by Hedrih (Stevanović)):

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=-\tilde{\omega}_{N1}^2 x_1^3(t),\ \text{for}\ \delta_1\neq 0,\ \varepsilon\neq 0,\ \omega_1^2>\delta_1^2 \tag{1}$$

in which $\delta_1=\varepsilon_1\tilde{\delta}_1$ and $\tilde{\omega}_{N1}^2=\varepsilon\tilde{\tilde{\omega}}_{N1}^2$, and ε and ε_1 are small parameters.

We use three different approach and three methods for obtaining first approximation of the previous nonlinear differential equation (1). First is starting known analytical solution of a corresponding linearized differential equation which correspond to nonlinear differential equation (1).

IV.1.1* In first case, for starting known solution, we can take solution of the linear differential equation in the following form:

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2 x_1(t)=0,\ \text{for}\ \delta_1\neq 0,\ \varepsilon=0,\ \omega_1^2>\delta_1^2 \tag{2}$$

with known analytical solution in the form:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos(p_1 t+\alpha_{01}),\ \text{for}\ \delta_1\neq 0,\ \varepsilon=0,\ \omega_1^2>\delta_1^2 \tag{3}$$

in which circular frequency of damped vibration is in the form $p_1=\sqrt{\omega_1^2-\delta_1^2}$ and, R_{01} and α_{01} are integral constant depending of initial conditions. Amplitude of this oscillation is in the form $R_{01}e^{-\delta_1 t}$ and decreasing with time.

4.1.2* For finding first approximation of the nonlinear differential equation (1), we take starting known analytical solution (3) of linearized differential equation in the form (2) and as a possible firs approximation of the solution we take into consideration the following

$$x_1(t)=R_1(t)e^{-\delta_1 t}\cos\Phi_1(t)\ ,\ \text{for}\ \delta_1\neq 0,\ \varepsilon\neq 0,\ \omega_1^2>\delta_1^2 \tag{4}$$

in which $a(t)=R_1(t)e^{-\delta_1 t}$ amplitude and full phase $\Phi_1(t)=p_1 t+\phi_1(t)$ contain unknown functions of time $R_1(t)$ and $\phi_1(t)$ which need to determine. For this fist approach, we applied Lagrange method of variation of constants to the known solution (3) of the linearized differential equation corresponding to nonlinear differential equation (see Reference [55-56] by Hedrih (Stevanović)). After obtaining system of differential equation along unknown functions of time $R_1(t)$ and $\phi_1(t)$ we applied average to the obtained members along one period of the $T_a=\dfrac{2\pi}{p_1}=\dfrac{2\pi}{\sqrt{\omega_1^2-\delta_1^2}}$ damping vibrations.

Then, after differentiation along time of the proposed approximation of the solution (4) we obtain:

$$\dot{x}_1(t) = -\delta_1 R_1(t) e^{-\delta_1 t}\cos\Phi_1(t) - R_1(t) e^{-\delta_1 t} p_1 \sin\Phi_1(t) + \\ + \dot{R}_1(t)\, e^{-\delta_1 t}\cos\Phi_1(t) - R_1(t) e^{-\delta_1 t}\dot{\phi}_1(t)\sin\Phi_1(t) \tag{5}$$

to which we introduce condition that this first derivative (5) of the proposed first approximation of the solution (4) have same form as solution (3) of the corresponding linearized differential equation (2), by other words, this condition express the following: that this first derivative (5) of the proposed first approximation of the solution (4) have same form as in the case that unknown function of time $R_1(t)$ and $\phi_1(t)$ are constant.

After applying introduced previous condition we obtain first derivative of the proposed first approximation (4) of the solution in the following form:

$$\dot{x}_1(t) = -\delta_1 R_1(t) e^{-\delta_1 t}\cos\Phi_1(t) - R_1(t) e^{-\delta_1 t} p_1 \sin\Phi_1(t) \tag{6}$$

and the following condition

$$\dot{R}_1(t)\cos\Phi_1(t) - R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t) = 0 \tag{7}$$

that unknown functions of time $R_1(t)$ and $\phi_1(t)$ must to satisfy.

Second derivative of the proposed first approximation (4) of the solution is in the following form:

$$\ddot{x}_1(t) = \left(\delta_1^2 R_1(t) - R_1(t) p_1^2 - \delta_1 \dot{R}_1(t) - R_1(t) p_1 \dot{\phi}_1\right) e^{-\delta_1 t}\cos\Phi_1(t) + \\ + \left(\delta_1 R_1(t)\dot{\phi}_1 + 2\delta_1 R_1(t)\, p_1 - \dot{R}_1(t)\, p_1\right) e^{-\delta_1 t}\sin\Phi_1(t) \tag{8}$$

Alter introducing first (6) and second (8) derivatives of the proposed first approximation (4) of the solution into nonlinear differential equation (1) and taking into account condition (7) we obtain the system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ in following form:

$$\dot{R}_1(t)\cos\Phi_1(t) - R_1(t)\dot{\phi}_1(t)\sin\Phi_1(t) = 0$$

$$R_1(t) p_1 \dot{\phi}_1 \cos\Phi_1(t) + \dot{R}_1(t)\, p_1 \sin\Phi_1(t) = \tilde{\omega}_{N1}^2 e^{-2\delta_1 t}\left[R_1(t)\right]^3 \cos^3\Phi_1(t) \tag{9}$$

Previous obtained system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ present a non homogeneous algebra system along derivatives of unknown function of time $\dot{R}_1(t)$ and $\dot{\phi}_1(t)$ with determinate in the form:

$$\Delta = \begin{vmatrix} \cos\Phi_1(t) & -R_1(t)\sin\Phi_1(t) \\ p_1 \sin\Phi_1(t) & R_1(t) p_1 \cos\Phi_1(t) \end{vmatrix} = R_1(t) p_1 \left[\cos^2\Phi_1(t) + \sin^2\Phi_1(t)\right] = R_1(t) p_1 \tag{10}$$

with following solutions:

$$\dot{R}_1(t) = \frac{\Delta_1}{\Delta} = \frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t}\left[R_1(t)\right]^3 \cos^3\Phi_1(t)\sin\Phi_1(t)$$

$$\dot{\phi}_1(t) = \frac{\Delta_2}{\Delta} = \frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t}\left[R_1(t)\right]^2 \cos^4\Phi_1(t), \quad \text{for} \quad \delta_1 \neq 0\,, \quad \varepsilon \neq 0\,, \quad \omega_1^2 > \delta_1^2 \tag{11}$$

Then, after obtaining previous system of differential equation (11) along unknown functions of time, $R_1(t)$ and $\phi_1(t)$, we applied average to the obtained

members along full phase $\Phi_1(t) = p_1 t + \phi_1(t)$ in interval $\Phi \in [-,2\pi]$ correspond to one period of the $T_a = \dfrac{2\pi}{p_1} = \dfrac{2\pi}{\sqrt{\omega_1^2 - \delta_1^2}}$ damping vibrations:

$$\dot{R}_1(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t}[R_1(t)]^3 \cos^3\Phi_1(t)\sin\Phi_1(t)d\Phi_1(t)$$

$$\dot{\phi}_1(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t}[R_1(t)]^2 \cos^4\Phi_1(t)d\Phi_1(t), \text{ for } \delta_1 \neq 0, \tag{12}$$

$$\varepsilon = 0,\ \omega_1^2 > \delta_1^2$$

Then, we obtain a system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ in first averaged approximation:

$$\dot{R}_1(t) = 0$$

$$\dot{\phi}_1(t) = \frac{3}{8}\frac{\tilde{\omega}_{N1}^2}{p_1} e^{-2\delta_1 t}[R_1(t)]^2 \text{ for } \delta_1 \neq 0,\ \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2 \tag{13}$$

After integration of the previous system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ in first averaged approximation for known initial values in first approximation $t = 0$, $R_1(0) = R_{01}$ и $\phi_1(0) = \phi_{01} = -\dfrac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 + \alpha_{01}$ we obtain:

$$R_1(t) = R_{01} = const$$

$$\phi_1(t) = -\frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2\left(e^{-2\delta_1 t} - 1\right) + \phi_{01} = -\frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}, \tag{14}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$

where $\alpha_{01} = \phi_{01} + \dfrac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2$, and full phase is in the form:

$$\Phi_1(t) = p_1 t + \phi_1(t) = p_1 t - \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2\left(e^{-2\delta_1 t} - 1\right) + \phi_{01} = p_1 t - \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01},$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$ (15)

Then first averaged approximation of the solution of the nonlinear differential equation with hard cubic small nonlinearity (1)

$$x_1(t) = R_{01}e^{-\delta_1 t}\cos\left(p_1 t - \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}\right),$$

for $\delta_1 \neq 0$, $\varepsilon = 0$, $\omega_1^2 > \delta_1^2$ (16)

In the case that, we have a nonlinear differential equation with sogt cubic small nonlinearity in the following form:

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2x_1(t)=+\tilde{\omega}_{N1}^2x_1^3(t) \text{ for } \delta_1\neq 0,\ \varepsilon\neq 0,\ \omega_1^2>\delta_1^2 \tag{17}$$

on the basis of the previous obtained firs averaged approximation of the solution we can write:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1t+\frac{3}{16\delta_1p_1}\tilde{\omega}_{N1}^2R_{01}^2e^{-2\delta_1 t}+\alpha_{01}\right),$$

$$\text{for } \delta_1\neq 0,\ \varepsilon\neq 0,\ \omega_1^2>\delta_1^2 \tag{18}$$

where for known initial values in first approximation $t=0$, $R_1(0)=R_{01}$ и and $\phi_1(0)=\phi_{01}=\frac{3}{16\delta_1p_1}\tilde{\omega}_{N1}^2R_{01}^2+\alpha_{01}$ we obtain:

$$\alpha_{01}=\phi_{01}-\frac{3}{16\delta_1p_1}\tilde{\omega}_{N1}^2R_{01}^2. \tag{19}$$

For the case that for $\delta_1=0$ we can use the system of differential equations along unknown functions of time $R_1(t)$ and $\phi_1(t)$ in first averaged approximation (13) and before integration put $\delta_1=0$, and after that applied integration, or find limes of the solutions (16) and 18) for $\delta_1\to 0$, and taking into account that is $\lim\limits_{\delta_1\to 0}\frac{\left(e^{-2\delta_1 t}-1\right)}{\delta_1}=-2t$, obtain first averaged approximation of the solution of nonlinear differential equations (1) as well as (17)

$$\ddot{x}_1(t)+2\delta_1\dot{x}_1(t)+\omega_1^2x_1(t)=\mp\tilde{\omega}_{N1}^2x_1^3(t) \tag{20}$$

in the following form:

$$x_1(t)=R_{01}e^{-\delta_1 t}\cos\left(p_1t\mp\frac{3}{16\delta_1p_1}\tilde{\omega}_{N1}^2R_{01}^2e^{-2\delta_1 t}+\alpha_{01}\right)$$

$$\text{for } \delta_1\neq 0,\ \varepsilon\neq 0,\ \omega_1^2>\delta_1^2 \tag{21}$$

where $\alpha_{01}=\phi_{01}\pm\frac{3}{16\delta_1p_1}\tilde{\omega}_{N1}^2R_{01}^2$

$$x_1(t)=R_{01}\cos\left\langle\left(\omega_1\pm\frac{3}{8\omega_1}\tilde{\omega}_{N1}^2R_{01}^2\right)t+\phi_{01}\right\rangle,$$

$$\text{for } \delta_1=0,\varepsilon_1=0\ \ \varepsilon\neq 0,\ \omega_1^2>\delta_1^2 \tag{22}$$

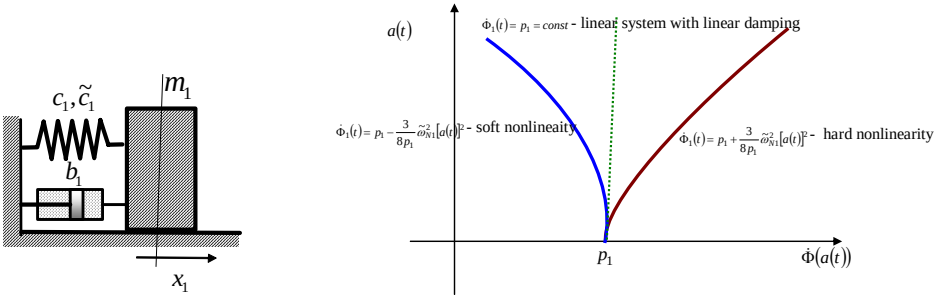
4.1. 2.1* In second case, for taking starting known solution for obtaining approximation of solution of the nonlinear differential equation (1), we can take solution of the linear differential equation in the following form:

$$\ddot{\tilde{x}}(t)+p^2\tilde{x}(t)=0 \tag{23}$$

with known analytical solution in the form:

$$\tilde{x}(t)=a\cos(p_1t+\alpha_0) \tag{24}$$

in which circular frequency of harmonic vibration is in the form $p=\sqrt{\omega_1^2-\delta_1^2}$ and, a and α_0 are integral constant depending of initial conditions. Amplitude of this oscillation is in the form a and is constant and no depending of time.



a* b*

Figure 7. a* Nonlinear system with one degree of freedom. b* Amplitude-frequency characteristic for free vibrations of the system damped nonlinear dynamics with of soft and hard nonlinearity

4.1. 2.2* For that case, we must transform nonlinear differential equation (1) taking into account the following generalized coordinate transformation:

$$\tilde{x}=xe^{\delta_1 t} \text{ or } x=\tilde{x}e^{-\delta_1 t}. \tag{25}$$

After generalized coordinate transformation an transformation of differential nonlinear equation (1), we obtain:

$$\frac{d^2\tilde{x}}{dt^2}+p^2\tilde{x}=-\tilde{\omega}_{N1}^2\tilde{x}^3(t)e^{-2\delta_1 t} \tag{26}$$

where

$$\varepsilon\tilde{f}\left(\tilde{x}e^{-\delta_1 t}\right)e^{\delta_1 t}=-\tilde{\omega}_{N1}^2\tilde{x}^3(t)e^{-2\delta_1 t}. \tag{27}$$

Let start with general form of the nonlinear differential equation in the form:

$$\frac{d^2x}{dt^2}+2\delta\frac{dx}{dt}+\omega^2x=\varepsilon f\left(x,\frac{dx}{dt}\right) \tag{28}$$

For small parameter $\varepsilon=0$ we obtain linear differential equation

$$\frac{d^2x}{dt^2}+2\delta\frac{dx}{dt}+\omega^2x=0 \tag{29}$$

with solution:

$$x=ae^{-\delta t}\cos\psi=e^{-\delta t}\tilde{x} \tag{30}$$

with amplitude $ae^{-\delta t}$ with phase $\psi=pt+\alpha$, where $p=\sqrt{\omega^2-\delta^2}$ and also:

$$\frac{da}{dt}=0 \qquad \frac{d\psi}{dt}=p=const \tag{31}$$

In which a and α are determined by their initial values. By generalized coordinate transformation (25) nonlinear differential equation (28) take the following form:

$$\left(\frac{d^2\widetilde{x}}{dt^2}+\left\langle\omega^2-\delta^2\right\rangle\widetilde{x}\right)=\varepsilon\widetilde{f}\left(\widetilde{x}e^{-\delta t},\frac{d\widetilde{x}}{dt}e^{-\delta t}\right)e^{\delta t} \tag{32}$$

or

$$\frac{d^2\widetilde{x}}{dt^2}+p^2\widetilde{x}=\varepsilon\widetilde{f}\left(\widetilde{x}e^{-\delta t},\frac{d\widetilde{x}}{dt}e^{-\delta t}\right)e^{\delta t}=\varepsilon\widetilde{\widetilde{f}}\left(\widetilde{x},\frac{d\widetilde{x}}{dt},\tau\right) \tag{33}$$

In beginning, we supposed that $\delta=\varepsilon_1\widetilde{\delta}$, and that ε_1 is same order of small value as ε and that $\tau=\varepsilon t$ is slow changing time and that for one period $T_a=\dfrac{2\pi}{p}$ change of the system dynamics is small, and that function $\varepsilon\widetilde{f}\left(\widetilde{x}e^{-\delta t},\dfrac{d\widetilde{x}}{dt}e^{-\delta t}\right)=\varepsilon\widetilde{\widetilde{f}}\left(\widetilde{x},\dfrac{d\widetilde{x}}{dt},\tau\right)$ satisfy all necessary conditions for application of the asymptotic method Ktilov-Bogolyubov-Mitropolyski for application f the method with slow changing system dynamics parameters and with slow changing time (see Reference [61-68] by Yu. A. Mitropolyskiy).

Then, the n-the asymptotic approximation of the two parametric family of a one frequency solution of differential equation (33) we suppose in the form:

$$xe^{\delta t}=\widetilde{x}=a\cos\psi+\varepsilon U_1(a,\psi,\tau)+\varepsilon^2U_2(a,\psi,\tau)+\ldots \tag{34}$$

where $U_1(a,\psi,\tau),U_2(a,\psi,\tau)$, …. periodic functions of $\psi=pt+\phi(t)$, with period 2π, and no containing first harmonic of ψ, and where amplitude and phase a and ψ are unknown functions which are determined by system of differential equations corresponding order n-th asymptotic approximation along amplitude and phase in the form:

$$\begin{aligned}\frac{da}{dt}&=\varepsilon A_1(a,\tau)+\varepsilon^2A_2(a,\tau)+\ldots\\ \frac{d\psi}{dt}&=p+\varepsilon B_1(a,\tau)+\varepsilon^2B_2(a,v)+\ldots\end{aligned} \tag{35}$$

where $A_1(a,\tau)$, $A_2(a,\tau)$, …., and $B_1(a,\tau),B_2(a,v),\ldots$. Are unknown functions of amplitude and slow changing time.

Introducing, on the basis of previous formulated condition we can write:

$$\int_0^{2\pi}U_j(a,\psi,\tau)e^{i\psi}d\psi=0 \qquad\qquad j=1,2,\ldots m \tag{36}$$

Then, we calculate first and second derivatives of the n-th supposed asymptotic approximation ot the solution in the following forms:

$$\begin{aligned}\frac{d\widetilde{x}}{dt}&=-ap\sin\psi+\varepsilon\left\{A_1(a,\tau)\cos\psi-aB_1(a,\tau)\sin\psi+\varepsilon\frac{\partial U_1}{\partial\tau}+p\frac{\partial U_1}{\partial\psi}\right\}+\\ &+\varepsilon^2\left\{A_1(a,\tau)\cos\psi-aB_2(a,\tau)\sin\psi+A_1(a,\tau)\frac{\partial U_1}{\partial a}+B_1(a,\tau)\frac{\partial U_1}{\partial\psi}+p\frac{\partial U_2}{\partial\psi}+\varepsilon\frac{\partial U_2}{\partial\tau}\right\}+\varepsilon^3\ldots\end{aligned} \tag{36}$$

$$\frac{d^2\tilde{x}}{dt^2} = -ap^2\cos\psi + \\ + \varepsilon\left\{-2pA_1\sin\psi - 2paB_1\cos\psi + \frac{dA_1(a,\tau)}{d\tau}\frac{d\tau}{dt}\cos\psi - a\frac{dB_1(a,\tau)}{d\tau}\frac{d\tau}{dt}\sin\psi + p^2\frac{\partial^2 U_1}{\partial\psi^2} + \frac{\partial^2 U_1}{\partial t^2} + p\frac{\partial^2 U_1}{\partial t\partial\psi}\right\} + \\ + \varepsilon^2\left\{\left(A_1\frac{dA_1}{da} - aB_1^2 - 2paB_2\right)\cos\psi - \left(2pA_2 + 2A_1B_1 + A_1a\frac{dB_1}{da}\right)\sin\psi\right\} + \\ + \varepsilon^2\left\{2pA_1\frac{\partial^2 U_1}{\partial a\partial\psi} + 2pB_1\frac{\partial^2 U_1}{\partial\psi^2} + p^2\frac{\partial^2 U_2}{\partial\psi^2} + \frac{\partial^2 U_2}{\partial t^2} + \varepsilon p\frac{\partial^2 U_2}{\partial\tau\partial\psi} + \varepsilon A_1\frac{\partial^2 U_1}{\partial\tau\partial a} + \ldots\right\} + \\ + \varepsilon^3\ldots \qquad (37)$$

After introducing previous asysmptotic aproximation of the solution (34) and their first and second derivatives into nonlinear differential equation (33) and applying method of equal coefficient of the small parameters un left and right side of transformation of nonlinear differential equation, we obtain series of the relation between unknown functions $U_1(a,\psi,\tau), U_2(a,\psi,\tau)$, …., . $A_1(a,\tau)$, $A_2(a,\tau)$, …., and $B_1(a,\tau)$, $B_2(a,v)$,….. For the reason that we need only first asymptotic approximation of the solution, we take into account the following relation obtained from coefficients with first step of the small parameter ε

$$-2pA_1(a.\tau)\sin^2\psi = \tilde{f}\left(e^{-\delta t}\{a\cos\psi\}, e^{-\delta t}\{-ap\sin\psi\}\right)e^{\delta t}\sin\psi \\ -2paB_1(a.\tau)\cos^2\psi = \tilde{f}\left(e^{-\delta t}\{a\cos\psi\}, e^{-\delta t}\{-ap\sin\psi\}\right)e^{\delta t}\cos\psi \qquad (38)$$

Taking into account development of the previous expressions along full phase $\psi = pt + \phi(t)$ we obtain relations – equations for obtaining unknown functions $A_1(a,\tau)$ and $B_1(a,\tau)$ in the following form (see Reference [61-68] by Yu. A. Mitropolyskiy).:

$$A_1(a.\tau) = -\frac{1}{2\pi p}\int_0^{2\pi}\tilde{\tilde{f}}\left(e^{-\tilde{\delta}\tau}\{a\cos\psi\}, e^{-\tilde{\delta}\tau}\{-ap\sin\psi\}\right)e^{\tilde{\delta}\tau}\sin\psi d\psi \\ B_1(a.\tau) = -\frac{1}{2\pi pa}\int_0^{2\pi}\tilde{\tilde{f}}\left(e^{-\tilde{\delta}\tau}\{a\cos\psi\}, e^{-\tilde{\delta}\tau}\{-ap\sin\psi\}\right)e^{\tilde{\delta}\tau}\cos\psi d\psi \qquad (39)$$

Then taking into account that $\varepsilon\tilde{f}\left(\tilde{x}e^{-\delta t}, \frac{d\tilde{x}}{dt}e^{-\delta t}\right) = \varepsilon\tilde{\tilde{f}}\left(\tilde{x}, \frac{d\tilde{x}}{dt}, \tau\right)$ and differential equation in the form (33) and introducing (27) in previous obtained expression (33) for obtaining functions $A_1(a,\tau)$ and $B_1(a,\tau)$ we can write:

$$\varepsilon A_1(a.t) = 0 \\ \varepsilon B_1(a.\tau) = \frac{1}{2\pi p}\tilde{\omega}_{N1}^2 a^2 e^{-2\tilde{\delta}\tau}\frac{3}{4} = \frac{3}{8\pi p}\tilde{\omega}_{N1}^2 a^2 e^{-2\delta t} \qquad (40)$$

where δ_1 ($\delta_1 = \varepsilon_1\tilde{\delta}_1$ and $\tilde{\omega}_{N1}^2 = \varepsilon\tilde{\tilde{\omega}}_{N1}^2$, for ε and ε_1 same order small values.

Then, system of differential equation (35) along a and ψ in the first asymptotic approximation is possible to write in the following form:

$$\frac{da}{dt} = 0 \\ \frac{d\psi}{dt} = p + \frac{3}{8\pi p}\tilde{\omega}_{N1}^2 a^2 e^{-2\delta t} \qquad (41)$$

and first asymptotic approximation of the solution in the form

$$xe^{\delta t} = \widetilde{x} = a\cos\psi \text{ or in the form: } x = \widetilde{x}e^{-\delta t} = ae^{-\delta t}\cos\psi \tag{42}$$

In the previous first asymptotic approximation full phase is in the form:

$$\psi(t) = pt - \frac{3}{16\delta_1 p}\widetilde{\omega}_{N1}^2 R_{01}^2\left(e^{-2\delta_1 t} - 1\right) + \psi_{01} = p_1 t - \frac{3}{16\delta_1 p}\widetilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}, \tag{43}$$

for $\delta_1 \neq 0$, $\varepsilon \neq 0$, $\omega_1^2 > \delta_1^2$, $p = \sqrt{\omega_1^2 - \delta_1^2}$.

We can see and conclude that first approximation of the solution of considered nonlinear differential equation (1) obtained by application different methods, first method of variation constant with average along full phase, and asymptotic method by Krilov – Bogolyubov-Mitropolyski (see Reference [61-68] by Yu. A. Mitropolyskiy) give us same results, but with different methods and proof.

4.1.3.1* In third case, for taking starting known solution for obtaining approximation of solution of the nonlinear differential equation (1), we can take solution of the linear differential equation in the following form:

$$\ddot{x} + \omega_1^2 x = 0 \tag{44}$$

with known analytical solution in the form:

$$x(t) = a\cos(\omega_1 t + \alpha_9). \tag{45}$$

in which circular frequency of harmonic vibration is in the form ω_1 and, a and α_0 are integral constant depending of initial conditions. Amplitude of this oscillation is in the form a and is constant and no depending of time.

4.1.3.2* For that case, for finding first approximation of the nonlinear differential equation (1), we take starting known analytical solution (45) of linearized differential equation in the form (44) and as a possible firs approximation of the solution we take into consideration the following

$$x(t) = a(t)\cos\Phi(t)., \text{ for } \delta_1 \neq 0,\ \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2 \tag{46}$$

in which $a(t)$ amplitude and full phase $\Phi_1(t) = \omega_1 t + \phi_1(t)$ contain unknown functions of time $a(t)$ and $\phi_1(t)$ which must to determine. For this third approach, we applied known Krilov-Bogolyubov-Mitropolyski asymptotic method of average to find first asymptotic approximation of the solution of nonlinear differential equation (1).

Then we start with nonlinear differential equation

$$\ddot{x} + \omega_1^2 x = \varepsilon f(x, \dot{x}), \tag{47}$$

and suppose first asymptotic approximation in the form:

$$x(t) = a(t)\cos\Phi(t). \tag{48}$$

where unknown functions $a(t)$ and $\Phi(t)$ are determined from the system of differential equations of first asymptotic approximation (see Reference [61-68] by Yu. A. Mitropolyskiy) in the following form:

$$\frac{da(t)}{dt} = \varepsilon A(a),$$

$$\frac{d\Phi(t)}{dt} = \omega_1 + \varepsilon B(a), \tag{48}$$

where

$$A(a) = -\frac{1}{2\pi\omega_1}\int_0^{2\pi} f\left(a\cos\Phi, -a\omega_1\sin\Phi\right)\sin\Phi\, d\Phi,$$

$$B(a) = -\frac{1}{2\pi\, a\omega_1}\int_0^{2\pi} f\left(a\cos\Phi, -a\omega_1\sin\Phi\right)\cos\Phi\, d\Phi. \tag{49}$$

For our nonlinear differential equation (1)

$$f(x,\dot{x}) = -\left(2\delta\dot{x} + \omega_N^2 x^3\right), \tag{50}$$

where je $\delta_1 = \varepsilon_1\delta,\ \tilde{\omega}_{N1}^2 = \varepsilon\omega_N^2$.

Taking into account initial values in first approximation $t = 0 : a(0) = a_o,\ \Phi(0) = \Phi_o$ we obtain that

$$a(t) = a_o e^{-\varepsilon\delta t} = a_o e^{-\delta_1 t},$$

$$\Phi(t) = \omega_1 t - \frac{3}{16\delta\omega_1}\omega_N^2 a_o^2\left(e^{-2\varepsilon\delta t} - 1\right) + \Phi_o = \omega_1 t - \frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t} - 1\right) + \Phi_o. \tag{51}$$

and first asymptotic approximation of the solution of a nonlinear differential equation (1) around harmonic starting known analytical solution, we can write in the following form:

$$x(t) = a_o e^{-\delta_1 t}\cos\left[\omega_1 t - \frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t} - 1\right) + \Phi_o\right]. \tag{52}$$

From this obtained first asymptotic approximation (52) of the solutions of nonlinear differential equation (1) with starting known analytical harmonic solution (45) in the case for damping coefficient tends to zero $\delta_1 \to 0$, and taking into account that is $\lim_{\delta_1\to 0}\frac{\left(e^{-2\delta_1 t} - 1\right)}{\delta_1} = -2t$, we obtain first asymptotic approximation of the solution for conservative nonlinear system vibrations in the form as in the previous two case obtained first approximation of solution of same nonlinear differential equation (1) by use different method and different starting known analytical solution.

4.1.4. Concluding remarks

Let we made a general review of the obtained results for approximately solving of the nonlinear differential equation with small cubic nonlinearity in the form: :

$$\ddot{x}_1(t) + 2\delta_1\dot{x}_1(t) + \omega_1^2 x_1(t) = \mp\tilde{\omega}_{N1}^2 x_1^3(t) \tag{53}$$

in which hard or soft, refers to $\mp$ sign approximately, $\delta_1 = \varepsilon_1\tilde{\delta}_1$ and $\tilde{\omega}_{N1}^2 = \varepsilon\tilde{\tilde{\omega}}_{N1}^2$, and ε and ε_1 are small parameters (see Figure 7)..

By use first two methods, starting known analytical solutions in the form (3) and we obtained same first approximation of the solution in the following form:

$$x_1(t) = R_{01} e^{-\delta_1 t} \cos\left(p_1 t \mp \frac{3}{16\delta_1 p_1} \tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01} \right),$$

$$\text{for } \delta_1 \neq 0,\ \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2 \tag{54}$$

where $\alpha_{01} = \phi_{01} \pm \frac{3}{16\delta_1 p_1} \tilde{\omega}_{N1}^2 R_{01}^2$. For the case that damping coefficient tends to zero, from this first approximation (54), we obtain first approximation of the solution for conservative nonlinear system dynamics in the following form:

$$x_1(t) = R_{01} \cos\left\langle \left(\omega_1 \pm \frac{3}{8\omega_1} \tilde{\omega}_{N1}^2 R_{01}^2 \right) t + \phi_{01} \right\rangle,$$

$$\text{for } \delta_1 = 0, \varepsilon_1 = 0 \quad \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2 \tag{55}$$

We can see that circular frequency of nonlinear dynamic of conservative system is not isochroous and depends of initial conditions – initial amplitude.

For the case that coefficient of the cubic nonlinearity tends to zero, from this first approximation (54), we obtain known analytical solution of the linear no conservative system dynamics in the following form:

$$x_1(t) = R_{01} e^{-\delta_1 t} \cos(p_1 t + \alpha_{01}),$$

$$\text{for } \delta_1 \neq 0, \varepsilon = 0, \omega_1^2 > \delta_1^2 \quad \tilde{\omega}_{N1}^2 = o \tag{56}$$

From the third case we start by harmonic known analytical solution in the form (45), we obtain the following first asymptotic approximation of the solution of same nonlinear differential equation:

$$x(t) = a_o e^{-\delta_1 t} \cos\left[\omega_1 t \mp \frac{3}{16\delta_1 \omega_1} \omega_{N1}^2 a_o^2 \left(e^{-2\delta_1 t} - 1 \right) + \Phi_o \right].$$

$$\text{for } \delta_1 \neq 0,\ \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2 \tag{57}$$

This asymptotical approximation is different them in previous case (54) and this is normally because we take different starting analytical known solution if different basic linear differential equations as a two different linearizations of the considered same nonlinear differential equation.

For the case that damping coefficient tends to zero, from this first approximation (57), we obtain first approximation of the solution for conservative nonlinear system dynamics in the following form:

$$x(t) = a_o e^{-\delta_1 t} \cos\left[\left(\omega_1 \pm \frac{3}{16\delta_1 \omega_1} \omega_{N1}^2 a_o^2 \right) t + \Phi_o \right],$$

$$\text{for } \delta_1 = 0, \varepsilon_1 = 0 \quad \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2 \tag{58}$$

same as in the previous cases (55).

For the case that coefficient of the cubic nonlinearity tends to zero, from this first approximation (57), we cannot obtain known analytical solution of the linear no conservative system dynamics in the form (56) but we obtain:

$$x_1(t) = R_{01}e^{-\delta_1 t}\cos(\omega_1 t + \alpha_{01}),$$
$$\text{for } \delta_1 \neq 0,\ \varepsilon = 0,\ \omega_1^2 > \delta_1^2\ \ \tilde{\omega}_{N1}^2 = o \tag{59}$$

not acceptable, because in this case starting solution was harmonic. In this case if we need harmonic solution we must annulled parameters of cubic nonlinearity and of dumping in the same time.

Then we can conclude that first two approach (54) for obtaining first approximation are more general and more suitable for use in the considered approximation of the solution them (59).

Open directions for next research and applications. Directions for next research in area of approximation must be focused to find analytical forms of approximations of solutions of nonlinear differential equations. Present in science, there are numerous numerical approach and numerical experiments over the nonlinear differential equations for numerical slowing nonlinear one, or coupled system of nonlinear differential equations, but these are only particular solutions without proof that these solutions are right, and general.

For practical applications in mechanics and engineering system dynamics analytical forms of the approximations of solutions are necessary for easier quantitative estimation larger class of the nonlinear dynamic phenomena and nonlinear dynamics of the stem behavior.

4.2. Hybrid system dynamics with complex structures and transfer energy

4.2. 1. Governing coupled partial differential equations of transversal vibrations of coupled axially moving double belt system

The sandwich belt system contain two belts coupled by distributed discrete light, neglected mass, ideally elastic belts with stiffness c m as a elastic layer. The both belts are represented by area of the constant cross sections A along length ℓ between rolling and fixed bearings A and B, and by ρ the density of the belt material. Let suppose that sandwich double belt system is moving in the axial directions x with an axial velocity $v(t)$. The transversal vibrations of the sandwich double belts are represented by the transverse displacements $w_1(x,t)$ of upper belt and $w_2(x,t)$ of lower belt. b is damping coefficient of the damping force distributed along belts. Also, let suppose that displacements are small, and that cross sections during the transverse vibration haven't deplanations. Also, it is supposed that both belts are loaded by active axial force, due to the belts' tension, and external distributed excitations $q_{(i)}(x,t), i = 1,2$ perpendicular to the x-axis, than in stressed state in the belt's cross section appear normal stresses with intensity σ, almost sure constant intensity during the time vibrations and along the length of belt between bearings. Than we can conclude that normal stress σ in belts of sandwich double belt system for a cross section during

vibrations change only direction. For both belts in this double belt system, let's accept a string (wave)-like type model between two rolling bearings.

In Figure 8. a* the kinetic parameters of the transversal forced vibrations of the axially moving sandwich belts are presented, and in 8.b* the elementary segment, with length dx, of the axially moving sandwich belt system excited by external transversal distributed forces and notations of the kinetics parameters are pointed out. $q_1(x,t)$ and $q_2(x,t)$ are external transversal excitations distributed along upper and lower belts between rolling bearings and are function of the coordinate x directed in the direction of the axially moving belt system.

Using d'Alambert principle of dynamical equilibrium and applying to the transversal forced dynamics of the elementary segment of the axially moving sandwich belt with length dx and notations of the kinetics parameters pointed out in Figure 8.b* for both component belts in double belt system, similar as in the paper [75], we can write the following system of the transversal forced vibrations of the component belts in the axially moving double belt system:

$$\begin{aligned}
\rho A dx \frac{D^2 w_1(x,t)}{Dt^2} &= -\sigma A \sin\alpha_1 + \sigma A \sin(\alpha_1 + d\alpha_1) - \\
&\quad - b\frac{Dw_1(x,t)}{Dt}dx + c[w_2(x,t) - w_1(x,t)]dx - q_1(x,t)dx \\
\rho A dx \frac{D^2 w_2(x,t)}{Dt^2} &= -\sigma A \sin\alpha_2 + \sigma A \sin(\alpha_2 + d\alpha_2) - \\
&\quad - b\frac{Dw_2(x,t)}{Dt}dx - c[w_2(x,t) - w_1(x,t)]dx + q_2(x,t)dx
\end{aligned} \tag{1}$$

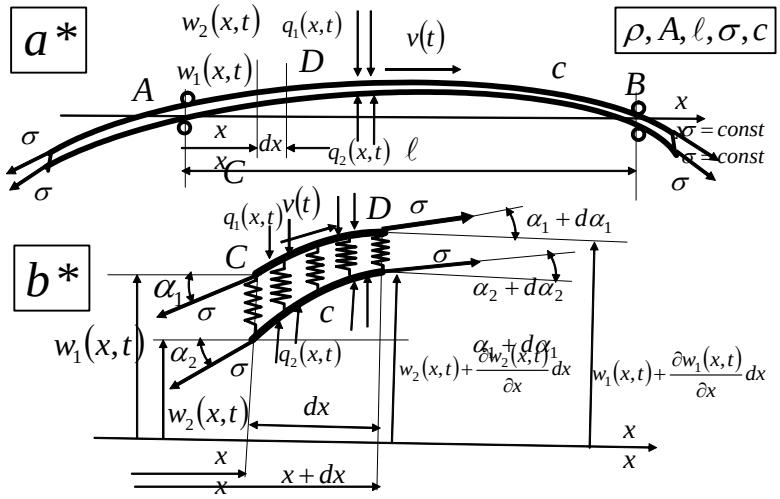


Figure 8. Transversal forced vibrations of the axially moving sandwich belts
a* Kinetics parameters of the transversal forced vibrations of the axially moving sandwich belts

b* Elementary segment of the axially moving sandwich belts with length dx and notations of the kinetics parameters for the case of the forced regime

Having in mind that the transversal belts' displacements are small it is right to take into account the approximations as in Refs. [72] and [75], and also, introducing the following denotation

$$c_0=\sqrt{\frac{\sigma}{\rho}}\ ,\quad \kappa=\sqrt{\frac{c}{\rho A}}\ ,\qquad 2\delta=\frac{b}{\rho A},\qquad \tilde{q}_i(x,t)=\frac{q_i(x,t)}{\rho A} \tag{2}$$

and the following partial differential operator: $\mathbf{L}_{x,t}[\bullet]$

$$\mathbf{L}_{x,t}[\bullet]=\frac{\partial^2}{\partial t^2}-\left(c_0^2-v_0^2\right)\frac{\partial^2}{\partial x^2}+2v_0\frac{\partial^2}{\partial x\partial t}+2\delta v_0\frac{\partial}{\partial x}+2\delta\frac{\partial}{\partial t}+\kappa^2 \tag{3}$$

and for the case $v=v_0=const$, previous partial differential equations (1) it is easy to rewrite in the following forms:

$$\begin{aligned}&\mathbf{L}_{x,t}[w_1(x,t)]-\kappa^2 w_2(x,t)+\tilde{q}_1(x,t)=0\\&\mathbf{L}_{x,t}[w_2(x,t)]-\kappa^2 w_1(x,t)-\tilde{q}_2(x,t)=0\end{aligned} \tag{4}$$

These partial differential equations are coupled by last terms.

4.2. 2. Solution of the basic decoupled partial differential equations

By using new independent coordinates in the following form:

$$\xi=x$$

$$\eta=\frac{v_0}{c_0^2-v_0^2}x+t \tag{5}$$

the partial differential operator (3) obtain the following

$$\tilde{\mathbf{L}}_{\xi,\eta}[\bullet]=\frac{c_0^2}{c_0^2-v_0^2}\tilde{\mathbf{L}}_{\eta}[\bullet]-\left(c_0^2-v_0^2\right)\tilde{\mathbf{L}}_{\xi}[\bullet] \tag{6}$$

and corresponding decomposition into two independent operators in the following forms:

$$\tilde{\mathbf{L}}_{\eta}[\bullet]=\left[\frac{\partial^2}{\partial\eta^2}+2\delta\frac{\partial}{\partial\eta}\right] \tag{7}$$

$$\tilde{\mathbf{L}}_{\xi}[\bullet]=\left[\frac{\partial^2}{\partial\xi^2}-\frac{2\delta v_0}{\left(c_0^2-v_0^2\right)}\frac{\partial}{\partial\xi}-\frac{\kappa^2}{\left(c_0^2-v_0^2\right)}\right] \tag{8}$$

then coupled partial differential equations (4) of the moving sandwich belts obtain the following form:

$$\begin{aligned}&\frac{c_0^2}{c_0^2-v_0^2}\tilde{\mathbf{L}}_{\eta}[w_1(\xi,\eta)]-\left(c_0^2-v_0^2\right)\tilde{\mathbf{L}}_{\xi}[w_1(\xi,\eta)]-\kappa^2 w_2(\xi,\eta)+\tilde{q}(\xi,\eta)_1=0\\&\frac{c_0^2}{c_0^2-v_0^2}\tilde{\mathbf{L}}_{\eta}[w_2(\xi,\eta)]-\left(c_0^2-v_0^2\right)\tilde{\mathbf{L}}_{\xi}[w_2(\xi,\eta)]-\kappa^2 w_1(\xi,\eta)-\tilde{q}_2(\xi,\eta)=0\end{aligned} \tag{9}$$

Basic decoupled partial differential equations are:

$$\widetilde{\mathbf{L}}_{\xi,\eta}[w_i(\xi,\eta)]=\frac{c_0^2}{c_0^2-v_0^2}\widetilde{\mathbf{L}}_{\eta}[w_i(\xi,\eta)]-\left(c_0^2-v_0^2\right)\widetilde{\mathbf{L}}_{\xi}[w_i(\xi,\eta)]=0\,,\ i=1,2 \tag{10}$$

Solution of the partial differential equation type from previous system (10) can be looked for Bernoulli's method of particular integrals in the form of multiplication of two functions (see book [53] by Rašković or Refs. [72] and [75]), from which the first $\mathbf{X}_{(i)}(\xi)$, $i=1,2$ depends only on space coordinate ξ and the second $\mathbf{Y}_{(i)}(\eta)$, $i=1,2$ is function of η:

$$w_{(i)}(\xi,\eta)=\mathbf{X}_{(i)}(\xi)\mathbf{Y}_{(i)}(\eta),\ \ i=1,2 \tag{11}$$

For beginning, the assumed solution (11) is introduced in previous system equation (10) and we obtain two decoupled ordinary differential equations in the following forms:

$$\widetilde{\mathbf{L}}_{\eta}\left[\mathbf{Y}_{(i)}(\eta)\right]+k^2\frac{c_0^2-v_0^2}{c_0^2}\mathbf{Y}_{(i)}(\eta)=0 \tag{12}$$

$$\widetilde{\mathbf{L}}_{\xi}\left[\mathbf{X}_{(i)}(\xi)\right]+\frac{k^2}{\left(c_0^2-v_0^2\right)}\mathbf{X}_{(i)}(\xi)=0 \tag{13}$$

and after denotations:

$$\widetilde{\omega}^2=k^2\frac{c_0^2-v_0^2}{c_0^2}\ ,\qquad \widetilde{\delta}=\frac{\delta v_0}{\left(c_0^2-v_0^2\right)},\qquad \widetilde{\lambda}^2=\frac{k^2-\kappa^2}{\left(c_0^2-v_0^2\right)}, \tag{14}$$

we obtain:

$$\frac{d^2\mathbf{Y}_{(i)}(\eta)}{d\eta^2}+2\delta\frac{d\mathbf{Y}_{(i)}(\eta)}{d\eta}+\widetilde{\omega}^2\mathbf{Y}_{(i)}(\eta)=0 \tag{15}$$

$$\frac{d^2\mathbf{X}_{(i)}(\xi)}{d\xi^2}-2\widetilde{\delta}\frac{d\mathbf{X}_{(i)}(\xi)}{d\xi}+\widetilde{\lambda}^2\mathbf{X}_{(i)}(\xi)=0 \tag{16}$$

Particular solution of the transversal displacement of the decoupled belts on the elastic Vincler type foundation, described by the partial differential equation (10) is in the form (11) must to satisfy the boundary conditions: displacements in the rolling bearings must be equal to zero:

$$w_{(i)}(x,t)\big|_{x=0}=0\,,\ w_{(i)}(\xi,\eta)\big|_{\substack{\xi=0\\ \eta=t}}=0\,,\ \mathbf{X}_{(i)}(0)=0\,,\ C_1=0\,,\ \ i=1,2\,,$$

$$w_{(i)}(x,t)\big|_{x=\ell}=0\,,\ w_{(i)}(\xi,\eta)\big|_{\substack{\xi=\ell\\ \eta=\ell\frac{v_0^2}{c_0^2-v_0^2}+t}}=0\,,\mathbf{X}_{(i)}(\ell)=0\,,\ C_2e^{\widetilde{\delta}\ell}\sin p\ell=0 \tag{17}$$

and then characteristic equation have the following roots: $p_s=\dfrac{s\pi}{\ell}$, $s=1,2,3,4,.....$

Than, we obtain series of particular solutions for each of the characteristic eigen numbers. Eigen amplitude functions are particular solutions of the ordinary differential equation (15) in the following forms (see Refs. [72] and [75]):

$$\mathbf{X}_{(i)s}(\xi)=e^{\tilde{\delta}\xi}\sin p_s\xi=e^{\tilde{\delta}\xi}\sin\frac{s\pi}{\ell}\xi$$

$$\text{for}\qquad p_{(1,2)s}=\mp\sqrt{\tilde{\lambda}_s^2-\tilde{\delta}^2}=\frac{s\pi}{\ell}\qquad\tilde{\lambda}>\tilde{\delta}\tag{18}$$

$$i=1,2\qquad s=1,2,3,4,.....$$

where

$$\tilde{\lambda}_s^2=\frac{k_s^2-\kappa^2}{\left(c_0^2-v_0^2\right)}>\tilde{\delta}=\frac{\delta v_0}{\left(c_0^2-v_0^2\right)}\qquad k_s^2=\tilde{\lambda}_s^2\left(c_0^2-v_0^2\right)+\kappa^2$$

$$k_s^2=\frac{\delta^2v_0^2}{\left(c_0^2-v_0^2\right)}+\kappa^2+\left(\frac{s\pi}{\ell}\right)^2\left(c_0^2-v_0^2\right)\qquad s=1,2,3,4,.....$$

(19)

$$\tilde{\omega}_s^2=k_s^2\frac{c_0^2-v_0^2}{c_0^2}=\frac{\delta^2v_0^2}{c_0^2}+\kappa^2\frac{c_0^2-v_0^2}{c_0^2}+\left(\frac{s\pi}{\ell}\right)^2\frac{\left(c_0^2-v_0^2\right)^2}{c_0^2}>\delta^2$$

Corresponding space - time η-functions are particular solutions of the ordinary differential equation (16) in the following forms (for detail see Ref. [75]):

:
$$\mathbf{Y}_{(i)s}(\eta)=e^{-\delta\eta}\left(A_sCh\tilde{q}_s\eta+B_sSh\tilde{q}_s\eta\right)$$

$$\text{for}\quad\tilde{q}_{(1,2)s}=\mp\sqrt{\delta^2-\tilde{\omega}_s^2}\qquad\tilde{\omega}_s<\delta$$

$$\mathbf{Y}_{(i)s}(\eta)=e^{-\delta\eta}\left(A_s\cos q_s\eta+B_s\sin q_s\eta\right)$$

$$\text{for}\quad q_{(1,2)s}=\mp\sqrt{\tilde{\omega}_s-\delta^2}\qquad\tilde{\omega}_s>\delta$$

$$q_{(1,2)s}=\mp\sqrt{\left(\kappa^2-\delta^2\right)\frac{c_0^2-v_0^2}{c_0^2}+\left(\frac{s\pi}{\ell}\right)^2\frac{\left(c_0^2-v_0^2\right)^2}{c_0^2}}\tag{20}$$

4.2.3. Approximation of the solution of the governing coupled partial differential equations

For solving coupled partial differential equations of transversal forced vibrations of the sandwich double belt system in the form (4) or (9) we take into calculus same eigen amplitude functions $\mathbf{X}_{(i)s}(\xi)$ for both belts in the form (18) and different unknown functions: $\mathbf{Y}_{(1)s}(\eta)$ and $\mathbf{Y}_{(2)s}(\eta)$. Than the solution suppose in the following expansions:

$$w_{(1)}(\xi,\eta)=\sum_{s=1}^{s=\infty}\mathbf{X}_{(1)\,s}(\xi)\mathbf{Y}_{(1)\,s}(\eta)=\sum_{s=1}^{s=\infty}e^{\tilde{\delta}\,\xi}\tilde{\mathbf{X}}_{(1)\,s}(\xi)\mathbf{Y}_{(1)\,s}(\eta)$$

$$w_{(2)}(\xi,\eta)=\sum_{s=1}^{s=\infty}\mathbf{X}_{(1)\,s}(\xi)\mathbf{Y}_{(2)\,s}(\eta)=\sum_{s=1}^{s=\infty}e^{\tilde{\delta}\,\xi}\tilde{\mathbf{X}}_{(1)\,s}(\xi)\mathbf{Y}_{(2)\,s}(\eta) \tag{21}$$

These expansion we put into the equations of the system (9), taking into account (15) and (16) previous system of the equations obtain the simplest form, and after multiplying first differential equation by $e^{-2\tilde{\delta}\,\xi}\mathbf{X}_{(i)r}(\xi)d\xi$ and second by $e^{-2\tilde{\delta}\,\xi}\mathbf{X}_{(i)s}(\xi)d\xi$ and integrating along belt's length ℓ between double belt system bearings and taking into account modified conditions of the orthogonality of eigen amplitude functions $\mathbf{X}_{(i)s}(\xi)$ for both belts in the form (18), as well as that some terms of the sum disappeared for different: $\tilde{\lambda}_s\neq\tilde{\lambda}_r$ for $s\neq r$, and in the corresponding result, we obtain s-th family system of the two coupled ordinary differential equations with respect to the unknown functions $\mathbf{Y}_{(1)s}(\eta)$ and $\mathbf{Y}_{(2)s}(\eta)$ in the following form:

$$\left\{\tilde{\mathbf{L}}_{\eta}\left[\mathbf{Y}_{(1)\,s}(\eta)\right]+\tilde{\lambda}_s^2\frac{\left(c_0^2-v_0^2\right)^2}{c_0^2}\mathbf{Y}_{(1)s}(\eta)-\kappa^2\frac{\left(c_0^2-v_0^2\right)}{c_0^2}\left[\mathbf{Y}_{(2)s}(\eta)-\mathbf{Y}_{(1)s}(\eta)\right]\right\}=\tilde{Q}_{1s}(\eta)$$

$$\left\{\tilde{\mathbf{L}}_{\eta}\left[\mathbf{Y}_{(2)\,s}(\eta)\right]+\tilde{\lambda}_s^2\frac{\left(c_0^2-v_0^2\right)^2}{c_0^2}\mathbf{Y}_{(2)s}(\eta)+\kappa^2\frac{\left(c_0^2-v_0^2\right)}{c_0^2}\left[\mathbf{Y}_{(2)s}(\eta)-\mathbf{Y}_{(1)s}(\eta)\right]\right\}=\tilde{Q}_{2s}(\eta) \tag{22}$$

where

$$\tilde{Q}_{(1)s}(\eta)=-\frac{\left(c_0^2-v_0^2\right)}{c_0^2}\frac{\int_0^{\ell}\tilde{q}_1(\xi,\eta)\mathbf{X}_{(1)s}(\xi)d\xi}{\int_0^{\ell}\left[\mathbf{X}_{(1)s}(\xi)\right]^2d\xi}$$

$$\tilde{Q}_{(2)s}(\eta)=\frac{\left(c_0^2-v_0^2\right)}{c_0^2}\frac{\int_0^{\ell}\tilde{q}_2(\xi,\eta)\mathbf{X}_{(1)s}(\xi)d\xi}{\int_0^{\ell}\left[\mathbf{X}_{(1)s}(\xi)\right]^2d\xi} \tag{23}$$

The solution of the system of second order no homogeneous ordinary differential equations (22) for the s-mode in the form of the expansion along eigen amplitude functions $X_{(1)s}(x)=X_{(1)s}(\xi)=X_{(2)s}(\xi)$, in the form (18), can be looked in the form: of the solutions for basic homogeneous system (see Ref. [75]) and we will apply the Lagrange's method of the variations of the constants of the eigen unknown function $\mathbf{Y}_{(1)s}(\eta)$ and $\mathbf{Y}_{(2)s}(\eta)$ in the form (20), introducing for integral constant the following unknown functions $C_{(1)s}(\eta),D_{(1)s}(\eta),M_{(1)s}(\eta),N_{(1)s}(\eta)$ of η for a s-mode, $s=1,2,3,4,....\infty$. We propose that $C_{(1)s}(\eta),D_{(1)s}(\eta),M_{(1)s}(\eta),N_{(1)s}(\eta)$ are functions of η and we can write:

$$\mathbf{Y}_{(1)s}(\eta)=e^{-\delta\eta}\left\{\left[C_{(1)s}(\eta)\cos\tilde{p}_s\eta+D_{(1)s}(\eta)\sin\tilde{p}_s\eta\right]+\left[M_{(1)s}(\eta)\cos\tilde{\tilde{p}}_s\eta+N_{(1)s}(\eta)\sin\tilde{\tilde{p}}_s\eta\right]\right\} \quad (24)$$

$$\mathbf{Y}_{(2)s}(\eta)=e^{-\delta\eta}\left\{\left[C_{(1)s}(\eta)\cos\tilde{p}_s\eta+D_{(1)s}(\eta)\sin\tilde{p}_s\eta\right]-\left[M_{(1)s}(\eta)\cos\tilde{\tilde{p}}_s\eta+N_{(1)s}(\eta)\sin\tilde{\tilde{p}}_s\eta\right]\right\} \quad (25)$$

where (for detail see Refs. [[72] and [75])

$$\tilde{p}_s=\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}_s}\quad,\quad s=1,2,3,4,..... \quad (26)$$

$$\tilde{\tilde{p}}_s=\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}},\qquad s=1,2,3,4,..... \quad (26^*)$$

In order to obtain first and second derivative with respect to η of the proposed forms of functions $\mathbf{Y}_{(1)s}(\eta)$ and $\mathbf{Y}_{(2)s}(\eta)$, we suppose that first derivatives of the functions $Y_{(i)s}(\eta)$, $i=1,2$, $s=1,2,3,4,....\infty$ with respect to the η are equal to the corresponding when coefficients $C_{(1)s}(\eta),D_{(1)s}(\eta),M_{(1)s}(\eta),N_{(1)s}(\eta)$ are constant and then we obtain the two equations-conditions. After introducing first and second derivatives of the proposed functions $Y_{(i)s}(\eta)$, $i=1,2$, $s=1,2,3,4,....\infty$ with respect to η into the system of no homogeneous second order ordinary differential equations (22) for the s-mode in the following form (18) and together with previous conditions for first derivatives, we obtain the system of the no homogeneous algebra equations along unknown first derivative of the unknown coefficients $C_{(1)s}(\eta),D_{(1)s}(\eta),M_{(1)s}(\eta),N_{(1)s}(\eta)$ with respect to η. After solving previous obtained system of the equations we obtain the first derivative of the unknown coefficients $C_{(1)s}(\eta),D_{(1)s}(\eta),M_{(1)s}(\eta),N_{(1)s}(\eta)$ with respect to η and after integrating for unknown coefficients-functions $C_{(1)s}(\eta),D_{(1)s}(\eta),M_{(1)s}(\eta),N_{(1)s}(\eta)$ we have the following expressions:

$$\begin{aligned}
C_{(1)s}(\eta)&=C_{(1)0s}-\frac{1}{2\tilde{p}_s}\int_0^\eta e^{\delta\eta}\left[\tilde{Q}_{(1)s}(\eta)+\tilde{Q}_{(2)s}(\eta)\right]\sin\tilde{p}_s\eta d\eta\\
D_{(1)s}(\eta)&=D_{(1)0s}+\frac{1}{2\tilde{p}_s}\int_0^\eta e^{\delta\eta}\left[\tilde{Q}_{(1)s}(\eta)+\tilde{Q}_{(2)s}(\eta)\right]\cos\tilde{p}_s\eta d\eta\\
M_{(1)s}(\eta)&=M_{(1)0s}-\frac{1}{2\tilde{\tilde{p}}_s}\int_0^\eta e^{\delta\eta}\left[\tilde{Q}_{(1)s}(\eta)-\tilde{Q}_{(2)s}(\eta)\right]\sin\tilde{\tilde{p}}_s\eta d\eta\\
N_{(1)s}(\eta)&=N_{(1)0s}+\frac{1}{2\tilde{\tilde{p}}_s}\int_0^\eta e^{\delta\eta}\left[\tilde{Q}_{(1)s}(\eta)-\tilde{Q}_{(2)s}(\eta)\right]\cos\tilde{\tilde{p}}_s\eta d\eta
\end{aligned} \quad (27)$$

where $\eta = \dfrac{v_0}{c_0^2 - v_0^2} x + t$. Now, previous unknown eigen functions $Y_{(i)s}(\eta)$, $i = 1,2$, $s = 1,2,3,4,....\infty$ depending on η for s - forced mode and eigen amplitude functions $\mathbf{X}_{(1)s}(\xi) = \mathbf{X}_{(2)s}(\xi) = e^{\tilde{\delta}\xi} \sin\dfrac{s\pi}{\ell}\xi$ are in the following form:

$$\mathbf{Y}_{(1)s}(\eta) = e^{-\delta\eta}\left[C_{(1)0s}\cos\tilde{p}_s\eta + D_{(1)0s}\sin\tilde{p}_s\eta\right] + \frac{1}{2\tilde{p}_s}\int_0^\eta e^{-\delta(\eta-\tau)}\left[\tilde{Q}_{(1)s}(\tau) + \tilde{Q}_{(2)s}(\tau)\right]\sin\tilde{p}_s(\eta-\tau)\tau d\tau$$
$$+ e^{-\delta\eta}\left\langle\left[M_{(1)0s}\cos\tilde{\tilde{p}}_s\eta + N_{(1)0s}\sin\tilde{\tilde{p}}_s\eta\right] + \frac{1}{2\tilde{\tilde{p}}_s}\int_0^\eta e^{-\delta(\eta-\tau)}\left[\tilde{Q}_{(1)s}(\tau) - \tilde{Q}_{(2)s}(\tau)\right]\sin\tilde{\tilde{p}}_s(\eta-\tau)d\tau\right\rangle \tag{28}$$

$$\mathbf{Y}_{(2)s}(\eta) = e^{-\delta\eta}\left[C_{(1)0s}\cos\tilde{p}_s\eta + D_{(1)0s}\sin\tilde{p}_s\eta\right] - e^{-\delta\eta}\left[M_{(1)0s}\cos\tilde{\tilde{p}}_s\eta + N_{(1)0s}\sin\tilde{\tilde{p}}_s\eta\right] -$$
$$+ \frac{1}{2\tilde{p}_s}\int_0^\eta e^{-\delta(\eta-\tau)}\left[\tilde{Q}_{(1)s}(\tau) + \tilde{Q}_{(2)s}(\tau)\right]\sin\tilde{p}_s(\eta-\tau)d\tau - \frac{1}{2\tilde{\tilde{p}}_s}\int_0^\eta e^{-\delta(\eta-\tau)}\left[\tilde{Q}_{(1)s}(\tau) - \tilde{Q}_{(2)s}(\tau)\right]\sin\tilde{\tilde{p}}_s(\eta-\tau)d\tau$$

where constants $C_{(1)0s}$, $D_{(1)0s}$, $M_{(1)0s}$ and $N_{(1)0s}$ are unknown constant defined by four initial conditions: belts' point elongations and velocities at the initial moment.

.The s-family of the particular solutions are in the following forms:

$$w_{(i)s}(x,t) = \mathbf{X}_{(i)s}(x)\mathbf{Y}_{(i)s}(x,t) \tag{29}$$

$$\begin{aligned} w_{(1)s}(\xi,\eta) = \mathbf{X}_{(1)s}(\xi)\mathbf{Y}_{(1)s}(\eta) &= e^{\tilde{\delta}\xi-\delta\eta}\sin\frac{s\pi}{\ell}\xi\left[C_{(1)0s}\cos\tilde{p}_s\eta + D_{(1)0s}\sin\tilde{p}_s\eta\right] + \\ &+ e^{\tilde{\delta}\xi-\delta\eta}\sin\frac{s\pi}{\ell}\xi\left[M_{(1)0s}\cos\tilde{\tilde{p}}_s\eta + N_{(1)0s}\sin\tilde{\tilde{p}}_s\eta\right] + \\ &+ \frac{1}{2\tilde{p}_s}e^{\tilde{\delta}\xi-\delta\eta}\sin\frac{s\pi}{\ell}\xi\int_0^\eta e^{\delta\tau}\left[\tilde{Q}_{(1)s}(\tau) + \tilde{Q}_{(2)s}(\tau)\right]\sin\tilde{p}_s(\eta-\tau)\tau d\tau \\ &+ \frac{1}{2\tilde{\tilde{p}}_s}e^{\tilde{\delta}\xi-\delta\eta}\sin\frac{s\pi}{\ell}\xi\int_0^\eta e^{\delta\tau}\left[\tilde{Q}_{(1)s}(\tau) - \tilde{Q}_{(2)s}(\tau)\right]\sin\tilde{\tilde{p}}_s(\eta-\tau)d\tau \end{aligned} \tag{30}$$

$$
\begin{aligned}
w_{(2)s}(\xi,\eta)=\mathbf{X}_{(\mathbf{1})s}(\xi)\mathbf{Y}_{(2)s}(\eta)=e^{\tilde{\delta}\,\xi-\delta\eta}\sin\frac{s\pi}{\ell}\xi\left[C_{(1)0s}\cos\tilde{p}_s\eta+D_{(1)0s}\sin\tilde{p}_s\eta\right]-\\
-e^{\tilde{\delta}\,\xi-\delta\eta}\sin\frac{s\pi}{\ell}\xi\left[M_{(1)0s}\cos\tilde{\tilde{p}}_s\eta+N_{(1)0s}\sin\tilde{\tilde{p}}_s\eta\right]-\\
+\frac{1}{2\tilde{p}_s}e^{\tilde{\delta}\,\xi-\delta\eta}\sin\frac{s\pi}{\ell}\xi\int_0^{\eta}e^{\delta\tau}\left[\tilde{Q}_{(1)s}(\tau)+\tilde{Q}_{(2)s}(\tau)\right]\sin\tilde{p}_s(\eta-\tau)d\tau-\\
-\frac{1}{2\tilde{\tilde{p}}_s}e^{\tilde{\delta}\,\xi-\delta\eta}\sin\frac{s\pi}{\ell}\xi\int_0^{\eta}e^{\delta\tau}\left[\tilde{Q}_{(1)s}(\tau)-\tilde{Q}_{(2)s}(\tau)\right]\sin\tilde{\tilde{p}}_s(\eta-\tau)d\tau
\end{aligned}
\tag{31}
$$

The s-family of the particular solutions for pure forced vibrations of a double belt system excited by external excitation distributed function depending only of time, are in the following forms:

$$
\begin{aligned}
w_{(1)}^{Forced}(x,t)=\sum_{s=1}^{\infty}\frac{e^{-\delta t}\sin\frac{s\pi}{\ell}x}{2\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}}}+\\
\int_0^{\eta=\frac{v_0}{c_0^2-v_0^2}x+t}e^{\delta\tau}\left[\tilde{Q}_{(1)s}(\tau)+\tilde{Q}_{(2)s}(\tau)\right]\sin\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}}_s(\eta-\tau)d\tau-\\
+\sum_{s=1}^{\infty}\frac{e^{-\delta t}\sin\frac{s\pi}{\ell}x}{2\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}}}\\
\int_0^{\eta=\frac{v_0}{c_0^2-v_0^2}x+t}e^{\delta\tau}\left[\tilde{Q}_{(1)s}(\tau)-\tilde{Q}_{(2)s}(\tau)\right]\sin\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}}(\eta-\tau)d\tau
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
w_{(2)}^{Forced}(x,t)=\sum_{s=1}^{\infty}\frac{e^{-\delta t}\sin\frac{s\pi}{\ell}x}{2\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}}}\\
\int_0^{\eta=\frac{v_0}{c_0^2-v_0^2}x+t}e^{\delta\tau}\left[\tilde{Q}_{(1)s}(\tau)+\tilde{Q}_{(2)s}(\tau)\right]\sin\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}}_s(\eta-\tau)d\tau-\\
-\sum_{s=1}^{\infty}\frac{e^{-\delta t}\sin\frac{s\pi}{\ell}x}{2\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}}}\\
\int_0^{\eta=\frac{v_0}{c_0^2-v_0^2}x+t}e^{\delta\tau}\left[\tilde{Q}_{(1)s}(\tau)-\tilde{Q}_{(2)s}(\tau)\right]\sin\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}}(\eta-\tau)d\tau
\end{aligned}
\tag{33}
$$

IV.2. 4. Appendix

Previous solutions are obtained on the basis previously obtained solutions of the coupled partial differential equations describing free transversal vibrations of the axially moving double belt system.

The s-family of the particular solutions for decoupled belts and for free transversal vibrations are:

$$w_{(i)s}(x,t)=\mathbf{X}_{(i)s}(x)\mathbf{Y}_{(i)s}(x,t) \tag{A.1}$$

$$w_{(i)s}(x,t)=e^{-\delta\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)+\tilde{\delta}x}$$

$$\sin\frac{s\pi}{\ell}x\left[R_s\cos\left\{\sqrt{\left(\frac{s\pi}{\ell}\right)^2\frac{\left(c_0^2-v_0^2\right)^2}{c_0^2}+\left(\kappa^2-\delta^2\right)\frac{c_0^2-v_0^2}{c_0^2}}\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)+\beta_s\right\}\right]$$

The generalized solution for decoupled belts transversal free vibrations is expressed by expansion

$$w_{(i)}(x,t)=\sum_{s=1}^{s=\infty}w_{(i)s}(x,t)=\sum_{s=1}^{s=\infty}\mathbf{X}_{(i)s}(x)\mathbf{Y}_{(i)s}(x,t) \tag{A.2}$$

$$w_{(i)}(x,t)=e^{-\delta\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)+\tilde{\delta}x}$$

$$\sum_{s=1}^{s=\infty}\sin\frac{s\pi}{\ell}x\left[R_s\cos\left\{\sqrt{\left(\frac{s\pi}{\ell}\right)^2\frac{\left(c_0^2-v_0^2\right)^2}{c_0^2}+\left(\kappa^2-\delta^2\right)\frac{c_0^2-v_0^2}{c_0^2}}\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)+\beta_s\right\}\right]$$

Solution of the coupled ordinary differential equations for free oscillations

$$\left\{\frac{d^2\mathbf{Y}_{(1)s}(\eta)}{d\eta^2}+2\delta\frac{d\mathbf{Y}_{(1)s}(\eta)}{d\eta}+\tilde{\omega}_s^2\mathbf{Y}_{(1)s}(\eta)-\tilde{\omega}_0^2\mathbf{Y}_{(2)s}(\eta)\right\}=0 \tag{A.3}$$

$$\left\{\frac{d^2\mathbf{Y}_{(2)s}(\eta)}{d\eta^2}+2\delta\frac{d\mathbf{Y}_{(2)s}(\eta)}{d\eta}+\tilde{\omega}_s^2\mathbf{Y}_{(2)s}(\eta)-\tilde{\omega}_0^2\mathbf{Y}_{(1)s}(\eta)\right\}=0$$

(A.4)

suppose in the following form:

$$\mathbf{Y}_{(i)s}(\eta)=D_{(i)s}e^{\dot{\imath}s\eta}\quad,\quad \mathbf{i}=\sqrt{-1}\,,\qquad i=1,2 \tag{A.4}$$

Characteristic equation of the formalized dynamical system have the following four sets of the characteristic eigen numbers:

$$\lambda_{(1,2,3,4)s}=-\delta\mp\sqrt{\delta^2-\tilde{\omega}_s^2\mp\tilde{\omega}_0^2}\,,\quad \mathbf{i}=\sqrt{-1}$$

$$\lambda_{(1,2)s} = -\delta \mp \mathbf{i}\sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left\{\left(c_0^2 - v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2 - \delta^2\right\}} = -\delta \mp \mathbf{i}\,\tilde{p}_s \quad ,$$

$$s = 1,2,3,4,..... \tag{A.5}$$

$$\lambda_{(3,4)s} = -\delta \mp \mathbf{i}\sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left\{\left(c_0^2 - v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2 - \delta^2 + 2\kappa^2\right\}} = -\delta \mp \mathbf{i}\,\tilde{\tilde{p}}_s \,,$$

$$s = 1,2,3,4,..... \tag{A.6}$$

The ratio between amplitudes of the own space-time η - functions $\mathbf{Y}_{(\mathbf{1})s}(\eta)$ and $\mathbf{Y}_{(2)s}(\eta)$, $\mathbf{Y}_{(i)s}(\eta) = D_{(i)s}e^{\lambda_s \eta}$ is not difficult to obtain in the form $\frac{D_{(1)s(r)}}{D_{(2)s(r)}} = \mp 1$, $s = 1,2,3,4,..... \; r = 1,2,3,4..$ Then, we can conclude that considered system of the coupled ordinary differential equations have four different and complex sets of roots and eigen characteristic numbers for defining unknown own space-time η - functions $\mathbf{Y}_{(\mathbf{1})s}(\eta)$ and $\mathbf{Y}_{(2)s}(\eta)$: having four sets of characteristic numbers we can conclude that corresponding four sets of the particular solutions for composing the functions $\mathbf{Y}_{(\mathbf{1})s}(\eta)$ and $\mathbf{Y}_{(2)s}(\eta)$ exists in the following form:

$$\mathbf{Y}_{(\mathbf{1})s(1,2,3,4)}(\eta) = \begin{cases} e^{-\delta\eta}\cos\tilde{p}_s\eta \\ e^{-\delta\eta}\sin\tilde{p}_s\eta \\ e^{-\delta\eta}\cos\tilde{\tilde{p}}_s\eta \\ e^{-\delta\eta}\sin\tilde{\tilde{p}}_s\eta \end{cases} \tag{A.7}$$

where is $\eta = \frac{v_0}{c_0^2 - v_0^2}x + t$, and $s = 1,2,3,4,.....$, or in developed form:

$$\mathbf{Y}_{(\mathbf{1})s(1,2,3,4)}(\eta) = \begin{cases} e^{-\delta\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right)}\cos\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right)\sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left\{\left(c_0^2 - v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2 - \delta^2\right\}} \\ e^{-\delta\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right)}\sin\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right)\sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left\{\left(c_0^2 - v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2 - \delta^2\right\}} \\ e^{-\delta\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right)}\cos\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right)\sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left\{\left(c_0^2 - v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2 - \delta^2 + 2\kappa^2\right\}} \\ e^{-\delta\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right)}\sin\left(\frac{v_0}{c_0^2 - v_0^2}x + t\right)\sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left\{\left(c_0^2 - v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2 - \delta^2 + 2\kappa^2\right\}} \end{cases} \tag{A.8}$$

Finally the unknown space-time η - functions $\mathbf{Y}_{(\mathbf{1})s}(\eta)$ and $\mathbf{Y}_{(2)s}(\eta)$ for free double belt system transversal vibrations we obtain in the following forms:

$$\mathbf{Y}_{(1)s}(\eta)=e^{-\delta\eta}\left[C_{(1)s}\cos\tilde{p}_s\eta+D_{(1)s}\sin\tilde{p}_s\eta\right]+e^{-\delta\eta}\left[M_{(1)s}\cos\tilde{\tilde{p}}_s\eta+N_{(1)s}\sin\tilde{\tilde{p}}_s\eta\right] \quad (A.9)$$

$$\mathbf{Y}_{(2)s}(\eta)=e^{-\delta\eta}\left[C_{(1)s}\cos\tilde{p}_s\eta+D_{(1)s}\sin\tilde{p}_s\eta\right]-e^{-\delta\eta}\left[M_{(1)s}\cos\tilde{\tilde{p}}_s\eta+N_{(1)s}\sin\tilde{\tilde{p}}_s\eta\right] \quad (A.10)$$

The s-family of the particular solutions for free vibrations are in the following forms:

$$w_{(i)s}(x,t)=\mathbf{X}_{(i)s}(x)\mathbf{Y}_{(i)s}(x,t),\ s=1,2,3,4,..... \quad (A.11)$$

$$w_{(1)s}(x,t)=e^{-\delta t}\sin\frac{s\pi}{\ell}x\left\{R_s\cos\left\langle\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}}+\beta_s\right\rangle\right\}+$$
$$+e^{-\delta t}\sin\frac{s\pi}{\ell}x\left\{U_s\cos\left\langle\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}}+\gamma_s\right\rangle\right\} \quad (A.12)$$

$$w_{(2)s}(x,t)=e^{-\delta t}\sin\frac{s\pi}{\ell}x\left\{R_s\cos\left\langle\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}}+\beta_s\right\rangle\right\}-$$
$$-e^{-\delta t}\sin\frac{s\pi}{\ell}x\left\{U_s\cos\left\langle\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}}+\gamma_s\right\rangle\right\} \quad (A.13)$$

where R_s, U_s, β_s and γ_s are unknown constants defined by initial conditions.

Then, finally, the generalized solutions of the based coupled partial differential equations are expressed by expansion in the following forms (free vibrations):

$$w_{(i)}(x,t)=\sum_{s=1}^{s=\infty}w_{(i)s}(x,t)=\sum_{s=1}^{s=\infty}\mathbf{X}_{(i)s}(x)\mathbf{Y}_{(i)s}(x,t) \quad (A.14)$$

for coupled

$$w_{(1)}(x,t)=e^{-\delta t}\sum_{s=1}^{\infty}\sin\frac{s\pi}{\ell}x\left\{R_s\cos\left\langle\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}}+\beta_s\right\rangle\right\}+$$
$$+e^{-\delta t}\sum_{s=1}^{\infty}\sin\frac{s\pi}{\ell}x\left\{U_s\cos\left\langle\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}}+\gamma_s\right\rangle\right\} \quad (A.15)$$

$$w_{(2)}(x,t)=e^{-\delta t}\sum_{s=1}^{\infty}\sin\frac{s\pi}{\ell}x\left\{R_s\cos\left\langle\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2-\delta^2\right\}}+\beta_s\right\rangle\right\}-$$
$$-e^{-\delta t}\sum_{s=1}^{\infty}\sin\frac{s\pi}{\ell}x\left\{U_s\cos\left\langle\left(\frac{v_0}{c_0^2-v_0^2}x+t\right)\sqrt{\frac{c_0^2-v_0^2}{c_0^2}\left\{\left(c_0^2-v_0^2\right)\left(\frac{s\pi}{\ell}\right)^2-\delta^2+2\kappa^2\right\}}+\gamma_s\right\rangle\right\} \quad (A.16)$$

where R_s, U_s, β_s and γ_s are unknown constants defined by initial conditions, two by initial transversal displacements of component belts and by two transversal velocities of the component belts (for detail see References [72] and [75]).

IV.2.5. Concluding Remarks

Subject of mathematical description and analytical study, presented in this part of the paper, is a theoretical, pure classical model of hybrid elastodynamic model very useful for university teaching of elastodynamics as a fundamental part of the engineering science (mechanical, civil and physics), as well as, a good introduction of the students and engineers of the multifrequency wave phenomena in real mechanical systems with moving material.

If we compare the expressions for coupled and uncoupled belts, we can conclude that for uncoupled belts' transverse free vibrations contain one frequency damped vibrations in one eigen amplitude shape, and for coupled vibrations contain two frequency damped vibrations in each one amplitude shape, and that these two-frequency dumped vibrations are uncoupled with relation of the other shape own vibrations. This is visible directly from corresponding expressions (A.12), (A.13) or (A.14) and (A.16) presented in Appendix.

For analysis forced regimes, we can use terms expressed by (30), (31) and (32) from which, we can conclude that forced vibrations in each mode should be contain three frequencies which are thwo frequencies of the free own double belt system vibrations, $\tilde{p}_s = \sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left[\left(\frac{s\pi}{\ell}\right)^2\left(c_0^2 - v_0^2\right) - \delta^2\right]}$ and $\tilde{\tilde{p}}_s = \sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left[\left(\frac{s\pi}{\ell}\right)^2\left(c_0^2 - v_0^2\right) + 2\kappa^2 - \delta^2\right]}$, and one frequency of external forced excitation, frequencies Ω_i. Free vibrations regimes are two frequency, and forced are three, or multifrequency, depending of number of frequencies of applied external transverse excitations.

From last expressions for particular or generalized solutions (30), (31) and (32) as expressions of transverse displacements of double belt system, we can conclude that we can separate eigen amplitude functions $\mathbf{X}_{(i)s}(\xi)$ along the space ξ-length-time $\eta(x,t)$ coordinate system as well two eigen phase functions $\tilde{\beta}_s(x)$ and $\tilde{\tilde{\beta}}_s(x)$ expressed by:

$$\tilde{\beta}_s(x) = \frac{v_0^2}{c_0^2 - v_0^2} x \sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left[\left(\frac{s\pi}{\ell}\right)^2\left(c_0^2 - v_0^2\right) - \delta^2\right]}, \qquad s = 1,2,3,4,..... \tag{33}$$

$$\tilde{\tilde{\beta}}_s(x) = \frac{v_0^2}{c_0^2 - v_0^2} x \sqrt{\frac{c_0^2 - v_0^2}{c_0^2}\left[\left(\frac{s\pi}{\ell}\right)^2\left(c_0^2 - v_0^2\right) + 2\kappa^2 - \delta^2\right]}, \quad s = 1,2,3,4,..... \tag{34}$$

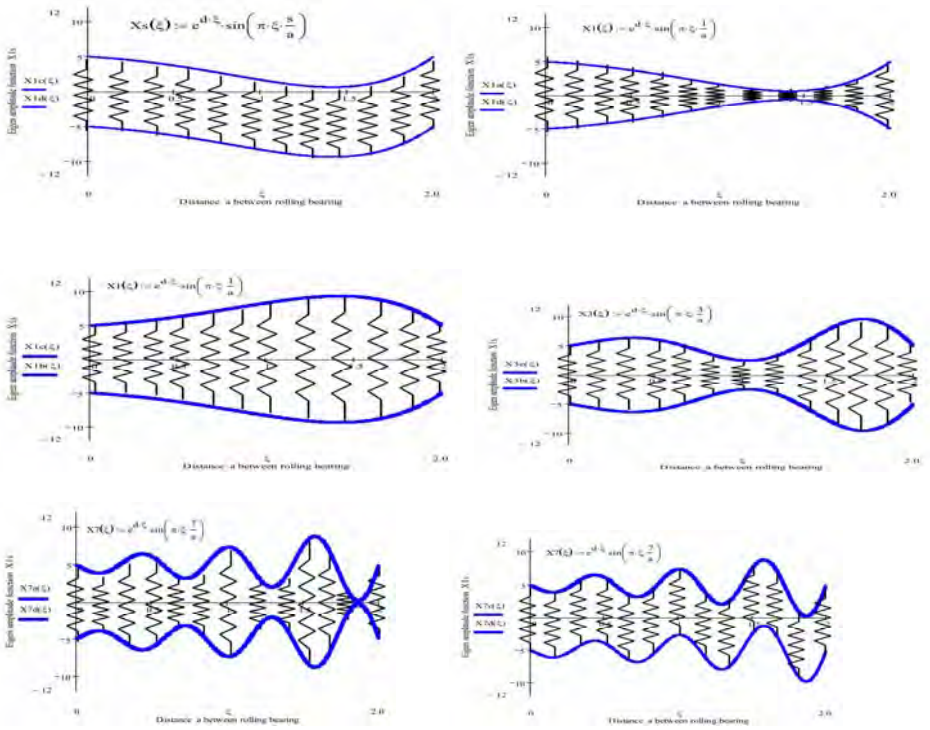


Figure 9. Sixth cases pf the possible vibrations firms of the double belt system with elastic layer for different eigen amplitude functions $\mathbf{X}_{(i)s}(\xi)=e^{\tilde{\delta}\xi}\sin\frac{s\pi}{\ell}\xi$ for the solution in the $\xi,\ \eta=\frac{v_0}{c_0^2-v_0{}^2}x+t$ coordinates system,

If we compare the expressions for solutions with respect to the other way analysis, of the solutions for coupled and uncoupled belts, we can conclude that for uncoupled belts' transverse displacements of forced vibrations contain one frequency damped vibrations and corresponding frequency forced regime in one eigen amplitude shape, with one eigen phase functions $\tilde{\beta}_s(x)$ (see expression from Ref. [75]) and for coupled double belt system vibrations contain two frequency damped free vibrations and corresponding frequency forced regime, as well as corresponding combinations in each one amplitude shape with two eigen phase functions $\tilde{\beta}_s(x)$ and $\tilde{\tilde{\beta}}_s(x)$ expressed by (33) and (34). Also, in other way, we can compare amplitude forms of the dynamics of coupled and uncoupled belts and conclude that dynamics of the uncoupled belts containing two types of eigen amplitude functions: $\mathbf{X}^{(C)}_{(i)s}(x)$ ***sin SIN*** AND $\mathbf{X}^{(S)}_{(i)s}(x)$ ***sin cos*** corresponding to one frequency free vibration mode and that double belt system dynamics contains

also two types eigen amplitude functions, but each of the both two frequencies: $\tilde{\mathbf{X}}^{(C)}_{(i)s}(x)$ ***sin cos*** and $\tilde{\mathbf{X}}^{(S)}_{(i)s}(x)$ ***sin sin*** for first frequency of the mode and $\tilde{\tilde{\mathbf{X}}}^{(C)}_{(i)s}(x)$ ***sin cos*** and $\tilde{\tilde{\mathbf{X}}}^{(S)}_{(i)s}(x)$ ***sin sin*** for second of the same mode, contained in expressions (30), (31), (32) and (A.12), (A.13), (A.14) (A.15).

In Figure 9. sixth cases of the possible vibrations firms of the double belt system with elastic layer for different eigen amplitude functions $\mathbf{X}_{(i)s}(\xi) = e^{\tilde{\delta}\xi}\sin\frac{s\pi}{\ell}\xi$ for the solution in the $\xi,\ \eta = \frac{v_0}{c_0^2 - v_0^{\ 2}}x + t$ coordinates system are presented.

We haven't information if this theoretical model of a sandwich double belt system was applied in real systems, but in our opinion it is possible to use this hybrid model of sandwich belt system in the different kind of conveyer in which is necessary that upper (or lower) belt haven't vibrations under transversal periodic excitation. It is possible in the condition of the dynamic absorption, when only lower (or upper) belt is in the forced regime of vibrations. This sandwich double belt system can be project as a dynamical absorber, when upper belt in the system is excited by external periodic excitation no vibrations, and only lower belt have forced vibrations.

Series of the papers [69-92] contain results of analysis based on analytical expressions describing dynamics of hybrid systems with complex structure. These system contains coupled plates, beams or belts.

4.3. A review of the study of the transfer energy between sub-systems in the complex structure systems.

4.3.1. Transfer energy in spring pendulum system

For introducing to the problem of the energy transfer or transient in the hybrid non-linear systems, it is useful to take, for simple analysis, into consideration the change energy between parts of the energy carrying on the generalized coordinates ϕ and ρ in the very known system, known under name *spring pendulum system,* with two degree of freedom. For the analysis of the energy in the spring pendulum we can write the kinetic and potential energies in the forms (see Refs. [72], [44], [56] and [83] by Hedrih (Stevanović)):

$$E_k = \frac{1}{2}m\left[\dot{\rho}^2 + (\rho + \ell)^2\dot{\phi}^2\right]$$

and

$$E_p = \frac{1}{2}c\rho^2 + mg(\rho + \ell)(1 - \cos\phi) \tag{1}$$

where: m is mass of the pendulum, ℓ length of pendulum string-neglected mass spring in the static equilibrium state of the pendulum, and c spring axial rigidity and ϕ and ρ are respectfully, angle and extension part of length of the string-spring of the

pendulum with comparison of the sprig length in static equilibrium state of the pendulum, taken as the generalized coordinates of the system. For the linearized case for kinetic energy, after neglecting small member - part of kinetic energy on the generalized coordinate ϕ, we can taking into account following expression:

* Expression $E_{k2}=\frac{1}{2}m(\rho+\ell)^2\dot{\phi}^2$ changes into approximation

$$E_{k2}\approx\frac{1}{2}m\left(\ell\dot{\phi}\right)^2. \tag{2}$$

Only for small oscillations – perturbations from equilibrium position it is possible to use approximation of the expression for kinetic and potential energy in the form:

$$E_k\approx\frac{1}{2}m\left[\dot{\rho}^2+\left(\ell\dot{\phi}\right)^2\right] \quad \text{and} \quad E_p\approx\frac{1}{2}c\rho^2+\frac{1}{2}mg\ell\phi^2 \tag{3}$$

For that linerized case the generalized coordinates are normal coordinates of the small oscillations of the spring pendulum around equilibrium position $\rho=0,\phi=0$ and coordinates are decoupled. In this linearized case of the spring pendulum model, the energy carried on the these normal coordinates are uncoupled and transfer or transient of the total energy don't appeared between proper parts of the separate normal coordinate and on the separate processes defined by normal coordinates are conservative systems each with one degree of the freedom. In this case each of the coordinate there are conversion of the energies from kinetic to potential, but sum of the both of one normal coordinates is constant.

$$E_{k\rho}\approx\frac{1}{2}m\dot{\rho}^2 \quad \text{and} \quad E_{p\rho}\approx\frac{1}{2}c\rho^2 \tag{4}$$

$$E_{k\phi}\approx\frac{1}{2}m\left(\ell\dot{\phi}\right)^2 \quad \text{and} \quad E_{p\phi}\approx\frac{1}{2}mg\ell\phi^2 \tag{5}$$

This is visible from system of the differential equations in the linearized form:

$$\ddot{\rho}+\omega_2^2\rho=0 \text{ where } \omega_2^2=\frac{c}{m}$$

$$\ddot{\phi}+\omega_1^2\phi=0 \text{ where } \omega_1^2=\frac{g}{\ell}. \tag{6}$$

but for the non-linear case the interaction between coordinates is present and then energy transient appears.

$$E_k=\frac{1}{2}m\left[\dot{\rho}^2+\ell^2\dot{\phi}^2+\rho^2\dot{\phi}^2+2\rho\ell\dot{\phi}^2\right] \quad \text{and}$$

$$E_p=\frac{1}{2}c\rho^2+mg\ell(1-\cos\phi)+mg\rho(1-\cos\phi) \tag{7}$$

We can separate the following parts:

I* Kinetic and potential energies carrying on the coordinate ρ are:

$$E_{k\rho}=\frac{1}{2}m\dot{\rho}^2 \quad \text{and} \quad E_{p\rho}=\frac{1}{2}c\rho^2+mg\rho \tag{8}$$

By analysing these previous expressions, we can see that with these expressions for decoupled oscillator with coordinate ρ, we have pure linear oscillator or harmonic oscillator with coordinate ρ and frequency $\omega_2^2 = \frac{c}{m}$, and separated process is isochronous.

II* Kinetic and potential energies carrying on the coordinate ϕ are

$$E_{k\phi} = \frac{1}{2} m\ell^2 \dot{\phi}^2 \quad \text{and} \quad E_{p\phi} = mg\ell(1-\cos\phi) \tag{9}$$

By analysing these previous expressions we can see that with these expression for decoupled oscillator with coordinate ϕ, we have pure non-linear oscillator with coordinate ϕ, and separated process is no isochronous. For linearized case this oscillator have eigen frequency $\omega_1^2 = \frac{g}{\ell}$.

III* Then formally, we can conclude that in the spring pendulum, we have coupled two oscillators, one pure linear with one degree of freedom, and second non-linear, also with one degree of freedom. In the hybrid system these oscillators are coupled and mechanical energy of the coupling contain two parts: one kinetic energy and second potential energy. Then, in the coupling, hybrid connections with static and dynamic kinetic properties are introduced.

Kinetic and potential energies of the coordinate ϕ and ρ interaction in the non-linear hybrid model are:

$$E_{k(\phi,\rho)} = \frac{1}{2} m[\rho + 2\ell]\rho\dot{\phi}^2 \quad \text{and}$$

$$E_{p(\phi,\rho)} = -mg\rho\cos\phi \tag{10}$$

For non-linear case ordinary differential equations are in the following form:

$$\ddot{\rho} + \omega_2^2 \rho = -g(1-\cos\phi) \tag{11}$$

$$\ddot{\phi} + \omega_1^2 \phi = \omega_1^2(\phi - \sin\phi) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \frac{1}{\ell^2}\rho(\rho+2\ell)\ddot{\phi} \tag{12}$$

or in non-linear approximation forms for small oscillations around zero coordinates $\rho = 0, \phi = 0$ or around stable equilibrium position of the spring pendulum are

$$\ddot{\rho} + \omega_2^2 \rho \approx -g\left(\frac{\phi^2}{2} - \frac{\phi^4}{24} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + ...\right) \tag{13}$$

$$\ddot{\phi} + \omega_1^2 \phi \approx -\omega_1^2\left(\frac{\phi^3}{3!} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - ....\right) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \frac{1}{\ell^2}\rho(\rho+2\ell)\ddot{\phi} \tag{14}$$

If we introduce phase coordinate, then we can write:

$$v = \dot{\rho}$$

$$\dot{v} = -\omega_2^2 \rho - g(1-\cos\phi)$$

$$u = \dot{\phi} \tag{15}$$

$$\dot{u} = -\omega_1^2\phi + \omega_1^2(\phi - \sin\phi) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho + \ell) - \frac{1}{\ell^2}\rho(\rho + 2\ell)\ddot{\phi}$$

or in the approximation

$$v = \dot{\rho}$$

$$\dot{v} \approx -\omega_2^2\rho - g\left(\frac{\phi^2}{2} - \frac{\phi^4}{24} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + ...\right)$$

$$u = \dot{\phi}$$

$$\dot{u} \approx -\omega_1^2\phi - \omega_1^2\left(\frac{\phi^3}{3!} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - ....\right) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho + \ell) - \frac{1}{\ell^2}\rho(\rho + 2\ell)\dot{u} \qquad (16)$$

From system equations (11)-(12), as well from their approximations (13)-(14), we can see that their right hand parts are non-linear and are functions of generalized coordinates, as well as of the generalized coordinates first and second derivatives. Also we can see that generalized coordinates ϕ and ρ are around their zero values, when $\rho = 0, \phi = 0$ at the stable equilibrium position of the spring pendulum, and that also are main coordinates of the linearized model. It is reason that the asymptotic averaged method is applicable for obtaining first asymptotic approximation of the particular solutions and it is possible to use for energy analysis of the transfer energy between energies carried by generalized coordinates ϕ and ρ in this non-linear system with two degree of freedom, but formally, we can take into account that we have two oscillators, one non linear and one linear each with one degree of freedom as two sub-systems coupled in the hybrid system with two degree of freedom, by hybrid connection realized by statically and dynamical connections. This interconnection have two parts of energy interaction between sub-systems expressed by kinetic and potential energies in the forms expressed by (10).
Taking into consideration some conclusion from considered system of the spring pendulum, we can conclude also that it is important to consider more simple case of the coupling between linear and non-linear systems with one degree of freedom with different types of the coupling realized by simple static or dynamic elements, for to investigate hybrid phenomena in the coupled sub-systems.

4. 3.2. Forced vibratos of spring pendulum

Let consider the energy transfer between parts of the energy carrying on the generalized coordinates ϕ and ρ in the *spring pendulum system* with two degrees of freedom excited by external excitations. For that analysis of the energy in the spring pendulum in the forced regime excited by external one frequency excitation – generalized forces $M_\phi(t) = M_0\cos(\Omega_\phi t + \vartheta_\phi)$ and $F_\rho(t) = F_0\cos(\Omega_\rho t + \vartheta_\rho)$, we can write the kinetic and potential energies in the forms (1). By taking into account all comments and asymptotic approximation as in the introductory part of this paper, as well as corresponding expressions (2) – (5), system of the differential equations of the linearized system is in the following form (see Refs. [72], [44], [56] and [83] by Hedrih (Stevanović)):

$$\ddot{\rho} + \omega_2^2 \rho = h_{0\rho} \cos\left(\Omega_\rho t + \vartheta_\rho\right) \tag{17}$$

where $\omega_2^2 = \dfrac{c}{m}$, $h_{0\rho} = \dfrac{F_0}{m}$

$$\ddot{\phi} + \omega_1^2 \phi = h_{0\phi} \cos\left(\Omega_\phi t + \vartheta_\phi\right) \tag{18}$$

where $\omega_1^2 = \dfrac{g}{\ell}$, $h_{0\phi} = \dfrac{M_0}{m\ell^2}$.

Solutions of the linearized equations (17) an (18) are:

$$\rho(t) = R_2 \cos\left(\omega_2 t + \alpha_{02}\right) + \frac{h_{0\rho}}{\omega_2^2 - \Omega_\rho^2} \cos\left(\Omega_\rho t + \vartheta_\rho\right) \tag{19}$$

$$\phi(t) = R_1 \cos\left(\omega_1 t + \alpha_{01}\right) + \frac{h_{0\phi}}{\omega_1^2 - \Omega_\phi^2} \cos\left(\Omega_\phi t + \vartheta_\phi\right) \tag{20}$$

For that linearized case both chosen coordinates are main coordinates of the linearized model, and from solutions (19) – (20), we can see that free and also, forced vibrations are uncoupled, and not interaction between free, and also forced modes of the vibrations. Then, we have two uncoupled oscillators with different eigen circular frequencies $\omega_1^2 = \dfrac{g}{\ell}$ and $\omega_2^2 = \dfrac{c}{m}$ and different forced external excitation frequencies Ω_ϕ and Ω_ρ and with possibilities of appearance two main uncoupled resonant regimes, when $\Omega_{\phi,resonant}^2 = \omega_1^2 = \dfrac{g}{\ell}$ and $\Omega_{\rho,resonant}^2 = \omega_2^2 = \dfrac{c}{m}$.

In this case for linearized models and in the resonant cases, expressions for solutions are in the following forms:

$$\rho(t)\Big|_{\Omega_{\rho,resonant}=\omega_2} = \rho_0 \cos\omega_2 t + \frac{\dot{\rho}_0}{\omega_2} \sin\omega_2 t + \frac{h_{0\rho}}{2\omega_2}\left[\omega_2 t \sin\left(\omega_2 t + \vartheta_\rho\right) - \sin\omega_2 t \sin\vartheta_\rho\right] \tag{21}$$

$$\phi(t)\Big|_{\Omega_{\phi,resonant}=\omega_1} = \phi_0 \cos\omega_1 t + \frac{\dot{\phi}_0}{\omega_1} \sin\omega_1 t + \frac{h_{0\phi}}{2\omega_1}\left[\omega_1 t \sin\left(\omega_1 t + \vartheta_\phi\right) - \sin\omega_1 t \sin\vartheta_\phi\right] \tag{22}$$

But, for the non-linear case the interaction between coordinates is present and then energy transient appears.

Expressions for kinetic and potential energies are in the same forma as presented and analyzed in first part V.1.1 for free vibrations and named by (1)-(5) and (7)-(10). Then, the expressions for coordinates are different and must be taken in the forms (19)-(20) and (21)-(22).

By analyze corresponding expressions, we can see that with these expression for decoupled oscillator with coordinate ρ, we have pure linear oscillator or harmonic oscillator with coordinate ρ and frequency $\omega_2^2 = \dfrac{c}{m}$, and separated process is isochronous. By analyzed these corresponding expressions, we can see that with these expressions for decoupled oscillators with coordinate ϕ, we have pure non-linear

oscillator with coordinate ϕ, and separated process is no isochronous. For linearyzed case this oscillator have eigen frequency $\omega_1^2 = \frac{g}{\ell}$.

For forced non-linear case differential equations of the system non-linear oscillation are in the following form:

$$\ddot{\rho} + \omega_2^2 \rho = -g(1-\cos\phi) + h_{0\rho}\cos(\Omega_\rho t + \vartheta_\rho) \tag{23}$$

$$\ddot{\phi} + \omega_1^2 \phi = \omega_1^2(\phi - \sin\phi) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \frac{1}{\ell^2}\rho(\rho+2\ell)\ddot{\phi} + h_{0\phi}\cos(\Omega_\phi t + \vartheta_\phi) \tag{24}$$

or in non-linear approximation forms for small oscillations around zero coordinates $\rho = 0, \phi = 0$ or around stable equilibrium position of the spring pendulum

$$\ddot{\rho} + \omega_2^2 \rho \approx -g\left(\frac{\phi^2}{2} - \frac{\phi^4}{24} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + ...\right) + h_{0\rho}\cos(\Omega_\rho t + \vartheta_\rho) \tag{25}$$

$$\ddot{\phi} + \omega_1^2 \phi \approx -\omega_1^2\left(\frac{\phi^3}{3!} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - ....\right) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \frac{1}{\ell^2}\rho(\rho+2\ell)\ddot{\phi} + h_{0\phi}\cos(\Omega_\phi t + \vartheta_\phi) \tag{26}$$

If we introduce phase coordinate, then we can write:

$$v = \dot{\rho}$$

$$\dot{v} = -\omega_2^2\rho - g(1-\cos\phi) + h_{0\rho}\cos(\Omega_\rho t + \vartheta_\rho)$$

$$u = \dot{\phi}$$

$$\dot{u} = -\omega_1^2\phi + \omega_1^2(\phi - \sin\phi) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \frac{1}{\ell^2}\rho(\rho+2\ell)\dot{u} + h_{0\phi}\cos(\Omega_\phi t + \vartheta_\phi) \tag{25}$$

or in the approximation

$$v = \dot{\rho}$$

$$\dot{v} \approx -\omega_2^2\rho - g\left(\frac{\phi^2}{2} - \frac{\phi^4}{24} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + ...\right) + h_{0\rho}\cos(\Omega_\rho t + \vartheta_\rho)$$

$$u = \dot{\phi} \tag{26}$$

$$\dot{u} \approx -\omega_1^2\phi - \omega_1^2\left(\frac{\phi^3}{3!} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - ....\right) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \frac{1}{\ell^2}\rho(\rho+2\ell)\dot{u} + h_{0\phi}\cos(\Omega_\phi t + \vartheta_\phi)$$

From system of the differential equations (23)-(24), as well as from their approximations (25)-(26), we can see that their right hand parts are non-linear and are functions of generalized coordinates, as well as of the generalized coordinates first and second derivatives with respect to time and function of time. Also, we can see that generalized coordinates ϕ and ρ around their zero values, when $\rho = 0, \phi = 0$ at the stable equilibrium position of the spring pendulum are also main coordinates of the linearized model. It is reason that the asymptotic averaged method is applicable for obtaining first asymptotic approximation of the solutions

Then, it is possible that first asymptotic approximations of the solutions of the system of non-linear differential equations (23)-(24), take into account in the following asymptotic approximations for the small spring pendulum forced elongations in the form:

$$\begin{aligned}\rho &= a_\rho(t)\cos\left(\omega_1 t + \varphi_\rho(t)\right)\\ \phi &= a_\phi(t)\cos\left(\omega_2 t + \varphi_\phi(t)\right)\end{aligned} \tag{27}$$

where amplitudes $a_\rho(t)$ and $a_\varphi(t)$ and phases $\varphi_\rho(t)$ and $\varphi_\varphi(t)$ are defined by system of first order non-linear differential equations in first asymptotic approximation in the following form:

$$\begin{aligned}
\dot{a}_\rho(t) &= \frac{h_{0\rho}}{\left(\omega_2 + \Omega_\rho\right)}\sin\left(\varphi_\rho(t) - \vartheta_\rho\right)\\
\dot{\varphi}_\rho(t) &= \omega_2 - \Omega_\rho - \frac{h_{0\rho}}{a_\rho(t)\left(\omega_2 + \Omega_\rho\right)}\cos\left(\varphi_\rho(t) - \vartheta_\rho\right)\\
\dot{a}_\phi(t) &\approx -\frac{h_{0\phi}}{2\left(\omega_1 + \Omega_\phi\right)}\sin\left(\varphi_\phi(t) - \vartheta_\phi\right) + \frac{h_{0\phi}}{3\left(\omega_1 + \Omega_\phi\right)}\frac{a_\rho^2(t)}{\ell^2}\sin\left(\varphi_\phi(t) - \vartheta_\phi\right)\\
\dot{\varphi}_\phi(t) &\approx \omega_1 - \Omega_\phi + \frac{\omega_1}{12}\left[1 - \frac{a_\rho^2(t)}{2\ell^2}\right] - \frac{h_{0\phi}}{2a_\phi(t)\left(\omega_1 + \Omega_\phi\right)}\cos\left(\varphi_\phi(t) - \vartheta_\phi\right) +\\
&\quad + \frac{h_{0\phi}}{3a_\phi(t)\left(\omega_1 + \Omega_\phi\right)}\frac{a_\rho^2(t)}{\ell^2}\cos\left(\varphi_\phi(t) - \vartheta_\phi\right)
\end{aligned} \tag{28}$$

where $\Omega_\phi \approx \omega_1$ and $\Omega_\rho \approx \omega_2$ are external excitation frequencies in the resonant rages corresponding eigen frequencies of corresponding linearized system. Previous system of four non-linear and first order differential equation in the first asymptotic approximation are obtained by asymptotic Krilov-Bogoliyubov-Mitropolyskiy method and for small amplitudes of external excitations and in the resonant rages of the both frequencies.

4. 3.3. Concluding remarks

Taking into consideration some conclusion from considered system of the spring pendulum, we can conclude, also, that it is important to consider more simple case of the coupling between linear and non-linear systems each with one degree of freedom with different types of the coupling realized by simple static or dynamic elements (see Refs. [72], [44], [56] and [83] by Hedrih (Stevanović)) for to investigate hybrid phenomena in the non-linear system forced dynamics.

Also, it is possible to use for energy analysis of the transfer energy between energies carried by generalized coordinates ϕ and ρ in this non-linear system forced dynamics with two degrees of freedom, but formally, we can take into account that, we have two oscillators, one non-linear and one linear each with one degree of freedom as two sub-systems coupled in the hybrid system with two degree of freedom, by hybrid connection realized by static and dynamic coupling. This interconnection have two part

of energy interaction between sub-systems expressed by kinetic and potential energy in the form (10).

Taking into consideration some conclusion for considered system of the spring pendulum forced oscillations, we can conclude also that it is important to consider more simple case of the coupling between linear and non-linear systems each with one degree of freedom with different types of the coupling realized by simple static or dynamic elements, for to investigate hybrid phenomena in the system forced dynamics.

4.4. Analysis of the trigger of coupled singularities in nonlinear dynamic of no ideal system

4.4.1. Free vibrations of the heavy mass particle along rotate rough curvilinear line with Coulomb friction

For beginning let to consider free vibrations of the heavy mass particle along rotate rough curvilinear line with Coulomb's type friction, see Figure 8.a*. For the case that curvilinear line is in the vertical rotate plane Oxz around vertical Oz axis, we can take that equation of the curve-linear line is: $z = f(x)$, or $f_1(x,z) = z - f(x) = 0$ and with the following properties $f(-x) = f(x)$ and that coordinate pole is in the zero point $f(0) = 0$ in which line have minimum (see Figure 8.a*). Also we take that curvilinear line rotate around vertical Oz axis with constant angular velocity $\vec{\Omega} = \Omega\vec{k}$ (see Ref. [103] by Hedrih (Stevanović)).

Heavy mass particle, mass m, moving along rough curvilinear line with Coulomb's type sliding friction coefficient μ, is loaded by proper weight mg, as a active conservative force and by four no ideal constraint reactions, one F_N - normal ideal constrain reaction, second F_{BN} in binormal direction and two additional, $F_{\mu 1}$ first tangential component of the no ideal constraint reaction induced by friction and proportional to the normal component reaction F_N, $F_{\mu 1} = -\mu F_N \, sign \, \vec{v}_{rel}$, and $F_{\mu 2}$ second tangential component of the no ideal constraint reaction induced by friction caused by pressures in the binormal direction and proportional to the binormal component of the inertia force F_{BN}, $F_{\mu 2} = -\mu F_{BN} \, sign \, \vec{v}_{rel}$, caused by curvilinear line rotation around vertical Oz axis with constant angular velocity $\vec{\Omega} = \Omega\vec{k}$,

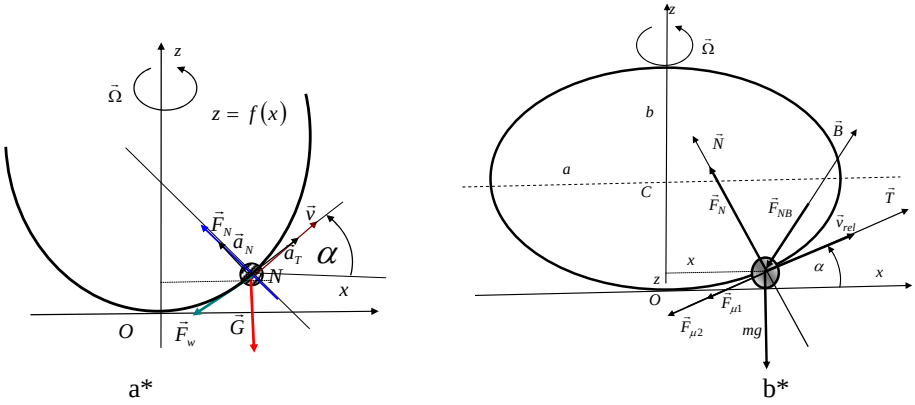


Figure 8. Heavy material particle motion along rough curvilinear line with Coulomb friction

Force of the inertia of mass particle realtive motion along the curvilinear line which rotate around vertical Oz axis with constant angular velocity $\vec{\Omega}=\Omega\vec{k}$, have two components. One component is force of the inertia of the circle rotation around vertical axis in the form $\vec{F}_{jp}=m\Omega^2 x\vec{u}$, and second is Coriolis inertia force of the system and we can write: $\vec{F}_{NB}=-\vec{F}_{jC}=2m\left[\vec{\Omega},\vec{v}_{rel}\right]=2m\Omega v_{rel}\cos\alpha\vec{B}$. Corresponding force of Coulomb's type friction is in the form: $\vec{F}_{\mu 2}=-\mu\left|\vec{F}_{NB}\right|\dfrac{\vec{v}_{rel}}{\left|\vec{v}_{rel}\right|}=-2\mu m\Omega v_{rel}\cos\alpha\dfrac{\vec{v}_{rel}}{\left|\vec{v}_{rel}\right|}=-2\mu m\Omega\vec{v}_{rel}\cos\alpha$.

By use principle of dynamical equilibrium we obtain expression for the intensity of normal and binormal components of curvilinear constraint reactions corresponding differential double non-linear equation of the heavy mass particle motion along rotate arbitrary curvilinear rough line, with angular velocity of rotation $\vec{\Omega}$, and defined by function $z=f(x)$, for the case that the coefficient of the Coulomb's type sliding friction is μ, s in the following form:

$$\frac{d}{dt}\left(\dot{x}\sqrt{1+z'^2}\right)\pm\mu\dot{x}^2\frac{z''}{\sqrt{1+z'^2}}+\Omega^2\frac{x}{\sqrt{1+z'^2}}\left(1\mp\mu z'\right)+$$
$$+\frac{g}{\sqrt{1+z'^2}}\left(z'\pm\mu\right)\pm 2\mu\Omega\dot{x}=0\quad\begin{cases}for & v_{rel}>0\\ for & v_{rel}<0\end{cases}\qquad(1)$$

For the case of heavy mass particle motion along no ideal arbitrary rough curvilinear line without rotation, differential equation is in the form:::

$$\frac{d}{dt}\left(\dot{x}\sqrt{1+z'^2}\right)+g\frac{z'}{\sqrt{1+z'^2}}\pm\mu\frac{1}{\sqrt{1+z'^2}}\left(\dot{x}^2z''+g\right)=0\quad\begin{cases}for & v_{rel}>0\\ for & v_{rel}<0\end{cases}\qquad(2)$$

Let consider special case of the rough curvilinear line with friction along normal surface contact (without last term $\pm 2\mu\Omega\dot{x}$ in (1)) and let introduce new variable in the

following form: $u=\dot{x}^2$, then previous differential double equation (1) of the mass particle motion along rough line is possible to transform in the following form:

$$\frac{du}{dx}+2u\frac{z''(z'\pm\mu)}{(1+z'^2)}=-2\Omega^2\frac{x}{(1+z'^2)}(1\mp\mu z')-\frac{2g}{(1+z'^2)}(z'\pm\mu)\quad\begin{cases}for & v_{rel}>0\\ for & v_{rel}<0\end{cases} \tag{3}$$

Previous differential double equation (3) of the material particle motion along rough curvilinear line according new helping coordinate u is ordinary double differential equation first order with changeable coefficients and type in following form: $\frac{du}{dx}\pm P(x)u=Q(x)$, with following solution:

$$[\dot{x}(x)]^2=$$
$$=e^{-2\int\frac{z''(z'\pm\mu)}{(1+z'^2)}dx}\left[-2\int\left[\Omega^2\frac{x}{(1+z'^2)}(1\mp\mu z')+\frac{g}{(1+z'^2)}(z'\pm\mu)\right]e2^{\int\frac{z''(z'\pm\mu)}{(1+z'^2)}dx}dx+C\right] \tag{4}$$

From the previous first integral, the following equation of the phase trajectories in the phase plane $(x,\dot{x})$ we obtain:

$$v_{rel}^2(x)=$$
$$=(1+z'^2)e^{-2\int\frac{z''(z'\pm\mu)}{(1+z'^2)}dx}\left[-2\int\left[\Omega^2\frac{x}{(1+z'^2)}(1\mp\mu z')+\frac{g}{(1+z'^2)}(z'\pm\mu)\right]e2^{\int\frac{z''(z'\pm\mu)}{(1+z'^2)}dx}dx+C\right] \tag{5}$$

where C integral constant depending of initial conditions, angular coordinate and angular velocity at initial moment, or starting terminate mass particle positions for next phase trajectory branch.

For reason to compare properties of kinetic parameters of main considered system dynamics and corresponding fictive (neglecting terms with acquire of velocity $\dot{x}^2$) for comparison we transform corresponding differential double equations in the form of the system of first order differential double equations and for obtaining singularities for main system and fictive systems use conditions that right hand side all equations must be equal to zero (null). Then we obtain the following conditions:

*For main system dynamics:

$$\frac{dx}{dt}=v=0$$
$$\frac{dv}{dt}=-\dot{x}^2\frac{z'z''}{(1+z'^2)}\mp\mu\dot{x}^2\frac{z''}{(1+z'^2)}-\Omega^2\frac{x(1\mp\mu z')}{(1+z'^2)}-\frac{g(z'\pm\mu)}{(1+z'^2)}=0 \tag{6}$$

*For corresponding fictive systems

$$\frac{dx}{dt}=v=0$$
$$\frac{dv}{dt}=-\Omega^2\frac{x(1\mp\mu z')}{(1+z'^2)}-\frac{g(z'\pm\mu)}{(1+z'^2)}=0 \tag{7}$$

and

$$\frac{dx}{dt} = v = 0$$

$$\frac{dv}{dt} = -\frac{\left(\Omega^2 x + gz'\right)}{\left(1+z'^2\right)} = 0 \tag{8}$$

We can see that for listed main system and for fictive system, conditions for obtaining singularities are same. Depending of the curvilinear line form $z = f(x)$, we obtained two nonlinear algebra equations in the following forms:

$$-\Omega^2 \frac{x\left(1 \mp \mu z'\right)}{\left(1+z'^2\right)} - \frac{g\left(z' \pm \mu\right)}{\left(1+z'^2\right)} = 0 \qquad \text{and} \qquad \frac{\left(\Omega^2 x + gz'\right)}{\left(1+z'^2\right)} = 0 \tag{9}$$

from which we can obtain, one or more roots.

If corresponding algebra double equation (9) have one root for $\mu = 0$ then words are about one equilibrium position with „one side left" and „one side right" bifurcation of the equilibrium position and one fictive trigger of coupled singularities caused by Coulomb's type friction between mass particle and rough curvilinear line.

If corresponding algebra double equation (9) have odd number of roots for $\mu = 0$ then words are about trigger of coupled singularities in a dynamics of a basic non-linear system correspond to the system with friction. In this case corresponding algebra double equation (9) for $\mu \neq 0$ have corresponding odd number of roots for each of the sets of the sign $\pm$, but all these roots are selected in two subsets, first one an "one side right" singularities and other "one side left" singularities correspond to the „one side left" and „one side right" relative equilibrium positions. Then, each roots of the corresponding algebra double equation (9) for $\mu = 0$, have two corresponding roots obtained from corresponding algebra double equation (9) for $\mu \neq 0$ and then there are present new fictive triggers of coupled one side singularities. Then we have trigger of the coupled triggers of coupled one side left, one central and one side right singularities, which are present in the system with Coulomb's type friction and with a corresponding nonlinear system with ideal constraints and with minimum a trigger of coupled singularities in its nonlinear dynamics.

Example 1. For the case that line is a circle shaped by $(z-R)^2 + x^2 = R^2$, $z = R - \sqrt{R^2 - x^2}$, for $\mu = 0$ and $\mu \neq 0$, from (9) there are two corresponding algebra equations, one of which for $\mu \neq 0$ is algebra double equations:

$$\Omega^2 x + g\frac{x}{\sqrt{R^2 - x^2}} = 0 \qquad \text{and}$$

$$\Omega^2 x\left(1 \mp \frac{\mu x}{\sqrt{R^2 - x^2}}\right) + g\left(\frac{x}{\sqrt{R^2 - x^2}} \pm \mu\right) = 0 \tag{10}$$

From first algebra equation for $\mu = 0$ of previous system (10) is visible that $x = 0$ is a root correspond to the equilibrium position, but there are also pair of the roots:

$x_{1,3}=\pm\sqrt{R^2-\frac{g^2}{\Omega^4}}$ for $\left|\frac{g}{R\Omega^2}\right|\le 1$. In this case for $\mu=0$ in system dynamics minimum a trigger of coupled three singularities exists. Also, we can conclude that second algebra equation for $\mu\neq 0$ have minimum two roots. First approximation of the minimum vales of first two roots are $x_{1,3}\approx\pm 2\mu\frac{g}{\Omega^2}$, which correspond to the "one side right" and "one side left" equilibrium positions and with $x=0$ build a trigger of coupled two one side singularities appeared as a result of bifurcation by introducing Coulomb's type friction. By qualitative analyzing of the second algebra double equation from system (10) in the form:

$$x^4\left(1+\mu^2\right)-x^2\left(R^2-\left(1+4\mu^2\right)\frac{g^2}{\Omega^4}\right)-x\left(\pm 2\mu\frac{gR^2}{\Omega^2}\right)-4\mu^2\frac{g^2R^2}{\Omega^4}=0,$$

we conclude that, also, one trigger of coupled three triggers of coupled one side singularities appear.

Example 2. For the case that line is an ellipse shaped by $\left(\frac{z-R}{a}\right)^2+\frac{x^2}{b^2}=1$, $z=R\pm a\sqrt{1-\frac{x^2}{b^2}}$, for $\mu=0$ and $\mu\neq 0$, from (9) there are two corresponding algebra equations, one of which for $\mu\neq 0$ is algebra double equations (see Figure 8.b*):

$$\Omega^2 x\pm ag\frac{\frac{x}{b^2}}{\sqrt{1-\frac{x^2}{b^2}}}=0 \text{ and } \Omega^2 x\left(1-\mu a\frac{\frac{x}{b^2}}{\sqrt{1-\frac{x^2}{b^2}}}\right)+g\left(\pm a\frac{\frac{x}{b^2}}{\sqrt{1-\frac{x^2}{b^2}}}\pm\mu\right)=0 \quad (11)$$

From first algebra equation for $\mu=0$ of previous system (11) is visible that $x=0$ is a root correspond to the equilibrium position, but there are also pair of two roots: $x_{1,3}=\pm b\sqrt{1-\left(\frac{ag}{b^2\Omega^2}\right)^2}$ for $\frac{ag}{b^2\Omega^2}<1$. In this case for $\mu=0$ in system dynamics, minimum a trigger of coupled three singularities exists. Also, we can conclude that second algebra equation for $\mu\neq 0$ have minimum two roots. First approximation of the minimum vales of first two roots are $x_{1,3}\approx\pm 2\mu\frac{g}{\Omega^2}$, which correspond to the "one side right" and "one side left" equilibrium positions and with $x=0$ build a trigger of coupled two one side singularities appeared as a result of bifurcation by introducing Coulomb's type friction. By qualitative analyzing of the second algebra double equation from system (11) in the form: $\left(x\pm\mu\frac{g}{\Omega^2}\right)^2\left(R^2-x^2\right)=x^2\left(\frac{g}{\Omega^2}\mp\mu x\right)^2$, we conclude that appear also one trigger of coupled three triggers of coupled one side singularities.

4.4. 2. Theorem of trigger of coupled singularities

Previous considered differential double equations of the heavy mass particle along rough curvilinear line with Coulomb's type friction is possible to express in the following generalized form of differential double equation with double signs (see Ref. [103] by Hedrih (Stevanović)).:

$$\ddot{x} \pm b_{\mu}\dot{x}^2 + g\left[k, F\left(x, \mp x_{\mu}\right)\right] f\left(x, \pm x_{\mu}\right) = 0 \quad for \quad \dot{x} \gtrless 0 \tag{12}$$

where b_{μ} coefficient depending of Coulombs type coefficient of friction, and x_{μ} parameter in coordinate dimension depending of Coulombs type coefficient of friction and with a corresponding governing differential equation for ideal system dynamics for $x_{\mu} = 0$ in the following form:

$$\ddot{x} + g\left[k, F\left(x\right)\right] f\left(x\right) = 0 \tag{13}$$

4.4. 3. Theorem on the existence of a trigger of the coupled singularities and the separatrix in the form of number eight in the conservative system.

By using nonlinear dynamic analysis of systems with described nonlinear phenomenon of the trigger of coupled singularities and corresponding families of phase portraits and potential energies (see References [84-103]) as well as the corresponding experimental investigations of such non-linear dynamics in mechanical engineering systems with coupled rotation motions (see Refs. [96] and [87]) it was easy to define and to prove a series of the theorem of the existence of a trigger of coupled singularities in non-linear dynamical conservative and no conservative systems with periodical structure.

Theorem: *In the system whose dynamics can be described with the use of non-linear differential equation in the form* (see Refs. [88] and [89]):

$$\ddot{x} + g\left[k, F\left(x\right)\right] f\left(x\right) = 0 \tag{14}$$

and whose potential energy is in the form:

$$\mathbf{E} = m\int_0^x g\left[k, F\left(x\right)\right] f\left(x\right) dx = \mathbf{G}\left[k, F\left(x\right)\right] \tag{15}$$

in which the functions $f(x)$ *and* $g(x)$ *are:*

$$F\left(x\right) = \int_0^x f\left(x\right) dx \qquad \text{and} \qquad G\left(k, x\right) = \int_0^x g\left(k, x\right) dx \tag{16}$$

and satisfy the following conditions:

$$\begin{aligned} &f\left(-x\right) = -f\left(x\right) && g\left(k, x + nT_0\right) = g\left(k, x\right) \\ &f\left(x + nT_0\right) = f\left(x\right) && g\left(k, -x\right) = g\left(k, x\right) \\ &f\left(0\right) = 0 && g\left[k, F\left(x_r\right)\right] = 0, \ for\ k \in \left(k_1, k_2\right) \cup \left(k_2, k_3\right) \ldots \end{aligned} \tag{17}$$

$f(x_s)=0$, $x_s=sT_0$, $s=1,2,3,4,...$ $x_r=\pm x_0\pm rT_0$,

$r=0,1,2,3,4,...$ $|x_0|<\dfrac{T_0}{2}$

$g[k,F(x)]\neq 0$, *for* $k\notin(k_1,k_2)\cup(k_2,k_3)....$

and both functions $f(x)$ *and* $g(x)$ *have one maximum or minimum in the interval between two zero roots:*

a for parameters values* $k\notin(k_{1\neq},k_2)\cup(k_2,k_3)....$, *outside of the intervals* $(k_{1\neq},k_2)\cup(k_2,k_3)....$, ***the trigger of singularities*** *in the local area* ***does not exist.***

b for parameters values* $k\in(k_{1\neq},k_2)\cup(k_2,k_3)....$, *inside of the intervals* $(k_{1\neq},k_2)\cup(k_2,k_3)....$, ***the series of triggers of coupled singularities in the local domains exist.***

We can see that for the case a* the second derivative of the potential energy can be positive or negative: :

$$\left.\frac{d^2\mathbf{E_p}}{dx^2}\right|_{x=x_s}=\begin{cases}\text{for}\quad \dfrac{df(0)}{dx}>0\quad \mathbf{E_{pmin}}\quad x_0\\ \text{stable equilibrium p.}\\ \\ \text{for}\quad \dfrac{df(x_s)}{dx}>0\quad \mathbf{E_{pmin}}\quad x_s\\ s=2p,\ p=1,2,3,4,..\qquad \text{stable equilibrium p.}\\ \\ \text{for}\quad \dfrac{df(x_s)}{dx}<0\quad \mathbf{E_{p\,max}}\quad x_s\quad s=2p-1,\\ p=1,2,3,4,..\\ \text{unstable equilibrium p.}\end{cases} \tag{18}$$

and equilibrium positions can be stable and unstable with corresponding singular points alternatively change, periodically, with period T_0 .from stable center to unstable saddle point, and corresponding phase portrait is without trigger of coupled singularities and without separatrix in the form of number eight.

Also we can see that for the case b* the second derivative of the potential energy can be positive or negative

$$\left.\frac{d^2\mathbf{E_p}}{dx^2}\right|_{x=x_s}=m\left\{g[k,F(x)]\frac{df(x)}{dx}\right\}\Bigg|_{x=x_s}<0\quad \mathbf{E_{p\,max}}$$

x_s .*unstable equilibrium position*

$$\left.\frac{d^2\mathbf{E_p}}{dx^2}\right|_{x=x_r}=m\left\{\frac{dg[k,F(x)]}{dF(x)}[f(x)]^2\right\}\Bigg|_{x=x_r}>0\quad \mathbf{E_{p\,min}} \tag{19}$$

x_r .*stable equilibrium positions*

and equilibrium positions can be stable and unstable with corresponding singular points alternatively change, periodically, with period T_0.from stable center to unstable saddle points, and corresponding phase portrait is with triggers of coupled singularities and with series of the separatrix in the form of numbers eights. Then, the triggers of coupled singularities exist in the phase portrait in the intervals defined by: $x \in \left(-\frac{T_0}{2} + sT, \frac{T_0}{2} + sT \right)$ $s = 0,1,2,3,4,...$.

Integral energy of the system is in the form:

$$\dot{x}^2 + 2G\left[k, F\left(x\right)\right] = \dot{x}_0^2 + 2G\left[k, F\left(x\left(t_0\right)\right)\right] = const \tag{20}$$

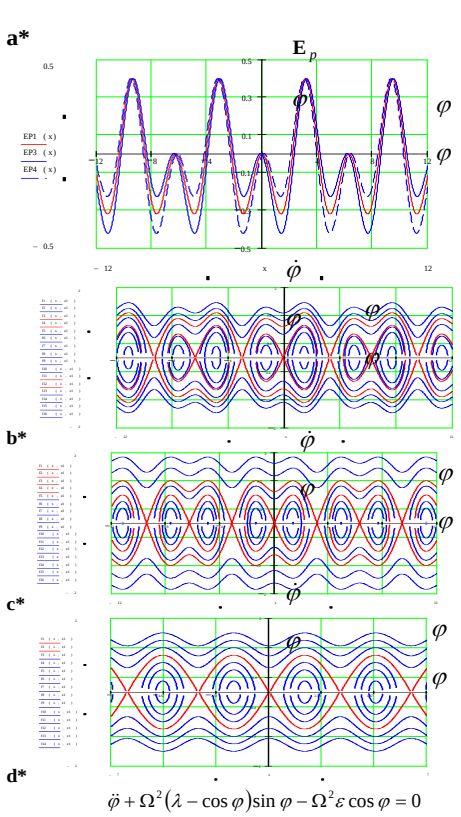
Equation of homoclinic orbit in the form number "eight" trough homoclinic point (0,0) is:

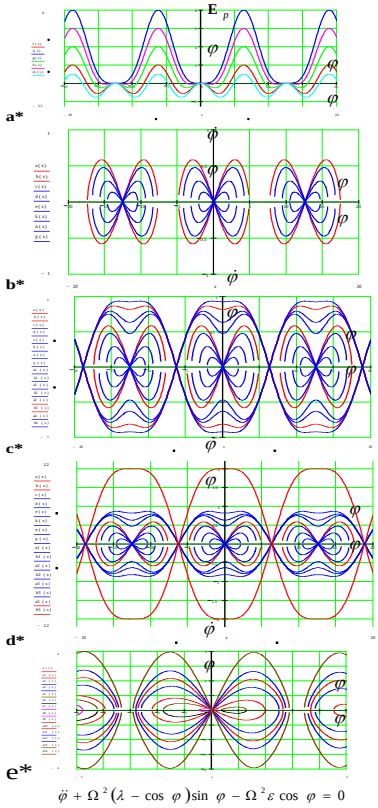
$$\dot{x}^2 + 2G\left[k, F\left(x\right)\right] = 2G\left[k, F\left(0\right)\right] = h_{0hc} = const \tag{21}$$

for $g\left[k, F\left(x_r\right)\right] = 0,\ for\ k \in \left(k_1, k_2\right) \cup \left(k_2, k_3\right)....$ in which the functions $F\left(x\right)$ and $G\left(k, x\right)$ are in the form (16) and satisfy the conditions (17).

In Figure 9 B* and B* equivalent of potential energy $\mathbf{E}_p\left(\varphi\right)$ graph of basic ideal mechanical system, which corresponds to no ideal Coulomb's type friction, is presented. In Figures 9 B*b*, B*c* and B*d* the sets of the homoclinic phase trajectory layering, for $\alpha_0 = 0$ and different values of the $k = \frac{1}{\lambda} = \left| \frac{g}{\ell\Omega^2} \right| \leq 1$ and axis eccentricity are presented. Homoclinic orbits in the form of number eight appear and disappear with changing parameter $k = \frac{1}{\lambda} = \left| \frac{g}{\ell\Omega^2} \right| \lesseqgtr 1$. Two sets of the of the singular points: $\varphi_s = s\pi$, $s = 1,2,3,4,...$ and $\varphi_s = \arccos\frac{g}{\ell\Omega^2} \pm 2s\pi$, $s = 1,2,3,4,...$ fort $k = \frac{1}{\lambda} = \left| \frac{g}{\ell\Omega^2} \right| \leq 1$ exists together with homoclinic orbits – separatrix in the form of number eight.

In Figure 9. A*a* equivalent of potential energy $\mathbf{E}_p\left(\varphi\right)$ graph of basic ideal mechanical system, which corresponds to no ideal Coulomb's type friction, is presented as a function of coordinate the φ. In Figures 9. (A*b*), (A*c*) and (A*d*) series of the phase trajectory portraits, for $\alpha_0 = 0$, and different values of the $k = \frac{1}{\lambda} = \left| \frac{g}{\ell\Omega^2} \right| \lesseqgtr 1$ and eccentricity of axis of circle rotation are presented. Two sets (A*b*) and (A*c*) of the of the singular points in phase portraits are visible: $\varphi_s = s\pi$, $s = 1,2,3,4,...$ and $\varphi_s = \arccos\frac{g}{\ell\Omega^2} \pm 2s\pi$, $s = 1,2,3,4,...$ for $k = \frac{1}{\lambda} = \left| \frac{g}{\ell\Omega^2} \right| \leq 1$.One set (A*d*) of singular points in phase trajectory portrait is visible: $\varphi_s = s\pi$, $s = 1,2,3,4,...$ for $k = \frac{1}{\lambda} = \left| \frac{g}{\ell\Omega^2} \right| \geq 1$.





A* B*

Figure 9. A* Equivalent of potential energy $\mathbf{E}_p(\varphi)$ graph of basic ideal mechanical system **(A*a*)** corresponding to no ideal Coulomb's type friction and phase trajectory portrait **(A*b*)**, (A*c*) and (A*d*) for $\alpha_0=0$ and different values of the $k=\frac{1}{\lambda}=\left|\frac{g}{\ell\Omega^2}\right|\lesseqgtr 1$ and axis eccentricity. Two sets (**A***b*) and (**A*c***) of the of the singular points: $\varphi_s=s\pi$, $s=1,2,3,4,...$ and $\varphi_s=\arccos\frac{g}{\ell\Omega^2}\pm 2s\pi$, $s=1,2,3,4,...$ fort $k=\frac{1}{\lambda}=\left|\frac{g}{\ell\Omega^2}\right|\le 1$.

One set (A*d*) of singular points $\varphi_s=s\pi$, $s=1,2,3,4,...$ for $k=\frac{1}{\lambda}=\left|\frac{g}{\ell\Omega^2}\right|\ge 1$.

Figure 9. B* Equivalent of potential energy $\mathbf{E}_p(\varphi)$ graph of basic ideal mechanical system (B* a*) correspond to no ideal wich Coulomb's type friction and homoclinic phase trajectory layering (B* b*), (B* c*) and (B* d*) for $\alpha_0=0$ and different values of the $k=\frac{1}{\lambda}=\left|\frac{g}{\ell\Omega^2}\right|\le 1$ and axis eccentricity.

4.4. 4. Triggers of coupled singularities in non-linear dynamics of coupled double rotor systems with Coulomb's type friction

In this part, we start with a new model of the non-linear dynamics of two coupled rigid rotors with mass particle debalances and no ideal surfaces between rotor shafts and cylindrical bearing where appear Coulomb's type friction (for detail see Reference [87]).

In Figure 10. a* the structure of the coupled double rotor system with Coulomb's type friction into contact surfaces between rotor shafts and cylindrical bearings is presented. In Figure 10. b* decomposition of this system with plan of the Coulomb's type friction forces is presented.

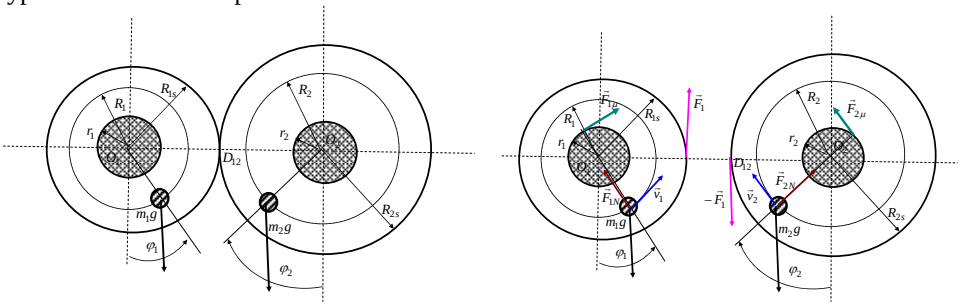


Figure 10. Coupled double rotor system (a*)with Coulomb's type friction into contact surfaces between discs and shafts; Decomposition (b*) of the system with plan of the Coulombs type friction forces

Governing nonlinear differential double equation of the coupled double rotor system dynamics with Coulomb's type friction into contact surfaces between rotor shafts and cylindrical bearings take the following form:

$$mR^3\left[\frac{1}{k}+\lambda p^2\right]\ddot{\varphi}\pm\mu mrR^2\left[\cos k\varphi-k\lambda p^2\cos\varphi\right]\ddot{\varphi}+mgR\sin\varphi\left(\frac{R}{k}\pm\mu r\cos k\varphi\right)-$$
$$-\lambda pmgR_2\sin k\varphi\left(R\mp\mu r\cos\varphi\right)\mp \tag{22}$$
$$\mp\mu rmg\left[\cos\varphi+\frac{R}{g}\dot{\varphi}^2\right]\left(\frac{R}{k}\pm\mu r\cos k\varphi\right)\mp\mu\lambda rmg\left[\cos k\varphi+\frac{pR}{g}k^2\dot{\varphi}^2\right]\left(R\mp\mu r\cos\varphi\right)=0$$

For $k=2$ and for ideal constraints (by neglecting friction) previous differential equation obtain the following form:

$$\ddot{\varphi}+\frac{g}{R\left(1+2\lambda p^2\right)}\left(1-4\lambda p\cos\varphi\right)\sin\varphi=0 \tag{23}$$

Obtained differential equations (23) is in the class (13) and then on the basis of the listed theorem of trigger of coupled singularities in chapter VII.2.1.. we can conclude that non-linear dynamics in basic system when condition $\frac{1}{4\lambda p}\le 1$ is satisfied appear a trigged of coupled singularities and in the phase trajectory portrait appear a homoclinic

orbit in the form of number eight. Sets of singularities are: first $\varphi_s = s\pi$ and second $\varphi_s = \arccos\frac{1}{4\lambda p}$ for $4\lambda p \geq 1$, presented in Figure 9..

4.4. 5. Concluding remarks

Systems with coupled multi-step rotors are important for engineering applications, then it is important to investigate ideal as well as no ideal system nonlinear dynamics. Also, no stability in the working processes of like that system dynamics caused higher level of noise and vibrations. Present Coulomb's type frictions in these kind of system dynamics caused new instability and more higher level of noise and vibrations. This is reason that is important to investigate non-linear phenomena in dynamics of other corresponding ideal as well as no ideal system dynamics. Also, system with vibro-impacts are important for engineering practice. Vibro-impacts are strong non-linearity with discontinuities in the system kinetic parameters and alternations of the forced and velocities directions in comparison before and after impacts (see Reference [102] an [104] by Hedrih (Stevanović), Raičević and Jović).

4.5. A review of the study of non-linear and stochastic vibrations through scientific research projects and doctoral dissertation and magistar thesis defended at Mechanical engineering faculty University of Niš in period 1972-2011 in area of Mechanics

IV.5.1. The study of the transfer of energy between sub-systems coupled in hybrid system (see Refs. [106-109], [55-56] by Heedrih (Stevanović) (1975, 1995a,b, 1887a,b, 2007a,b, 2008a,b), [57-60] by Hedrih (Stevanović) and Simonović (2009a,b,c) and [45] and [69]Hedrih (Stevanović) and Hedrih A. (2009a,b)) is very important for different applications. Two papers by author (see Refs. [72] and [76] by Heedrih (Stevanović) (2005, 2006 and 2008) presents analytical analysis of the transfer of energy between plates for free and forced transversal vibrations of an elastically connected double-plate system. Energy analysis of vibro-impact system dynamics with curvilinear trajectories and no ideal constraints was done by Jović in 2009 and in 2011 in his two theses (see References [128] and [12]), for Magister of science as well for doctor's of sciences degrees. Potential energy and stress state in material with crack was study by Jovanović and presented in his Doctor's Degree Thesis in 2009 (see Ref. [126]). Energy analysis of the non-linear oscillatory motions of elastic and deformable bodies was done by Hedrih (Stevanović) her doctor's degree thesis in 1975 (see Ref. [109]). Energy analysis longitudinal oscillations of rods with changeable cross sections was original research results in 1995 presented by Filipovski in his magister of sciences degree thesis (see Ref. [119]). For all previous results see References from list in Appendix I – References VII – [105-130] and Appendix II – References VIII – [131-140] .

4.5.2. When, at an international conference ICNO in Kiev in 1969, my professor of mechanics and mathematics, D. P. Rašković (1910-1985) (see Refs. [32], [33], [34], [53] and [54] Rašković (1965,1985) presented me to academician Yuri Alekseevich

Mitropolskiy (1917-2008) (see Refs. [61-68] by Mitropolskiy (1955, 1964, 1968, 1976 and 2003) and when I started really to understand the differences between linear and non-linear phenomena in dynamics of mechanical real systems, I knew I was on the right path of research which enchanted me ever more by understanding new phenomena and their variety in non-linear dynamics of realistic engineering and other dynamical systems. (First my knowledge about properties of non-linearity and the non-linear function I obtained in gymnasium from my excellent professor of mathematics Draginja Nikolić and during my research Matura work on the subject of Non-linear elementary functions and their graphics as a final high school examination.)

For beginning of this chapter, a review survey of original results of the author and of researchers from Faculty of Mechanical Engineering University of Niš (see References from lists in Appendix I – References VII – [105-130] and Appendix II – References VIII – [131-140]), inspired and/or obtained by the asymptotic method of Krilov-Bogolyubov-Mitropolyskiy, and as a direct influence of professor Rašković scientific instruction and also by published Mitropolskiy's papers and monographs, as well as publications by Kiev Mathematical institute scientists in area of non-linear and stochastic dynamics . These results have been published in scientific journals, and were presented on the scientific conferences and in the bachelor degree works (see [107] by Stevanović, (1967)), Magister of sciences theses (see [108] by Stevanović (1972), [110] by Kozić (1982), [112] by Pavlović (1982), [114] by Mitić (1985), [118] by Pavlov (993), [119] by Filipovski (1995), [121] by Janevski* (2004), [122] by Simonović (2008)) and doctoral dissertations (see [109] by (Stevanović) Hedrih (1975), [111] by Kozić (1990), [113] by Pavloić (1990), [115] by Mitić (1994), [127] by Knežević (2000), [124] by Perić (2005), [126] by Jovanović (2009), [127] by Janevski* (2010), [128] by Jović** (2011), [129] by Simonović (2011) and [130] by Veljović (2011)) supervised by Mitropolskiy (in period from 1972 to 1975) or by Rašković (in period from 1964 to 1974) and by Hedrih in period from 1976 to 2011 year as well. For all previous results see References from list in Appendix I – References VII – [105-130] and Appendix II – References VIII – [131-140].

In area of stochastic stability a scientific supports by series of consultation to researchers was given by Kiev stochastic research group at Institute of Matmematics NANU , S.T. Ariaratnam (Canada) and A. Tylikowski (Polad) and also by their papers.

The original results contain asymptotic analysis of the non-linear oscillatory motions of elastic bodies: beams, plates, shells and shafts (see References by (Stevanović) Hedrih (1972, 1981, 1978, 1983, 1984, 1985, 1995); Hedrih and Rašković (1974); Hedrih, Kozić, Pavlović, Mitić and Filipovski (1983, 1984, 1985, 1986, 1993, 1996, 1995)). Also, late a series of new research results are obtained by Janevski in 2003 [121] and by Simonovic in 2008 [122, 129] an in 2011 and also by Veljovic in 2011 [130] . The multi-frequency oscillatory motion of elastic bodies was studied. Corresponding system of partial differential equations of system dynamics, as well as system of first approximation of ordinary differential equations for corresponding numbers of amplitudes and phases of multi-frequency regimes of elastic bodies non-linear oscillations were composed. The characteristic properties of non-linear systems passing through coupled multi-frequency resonant state and mutual influences between excited modes were discovered.

In the same cited papers amplitude-frequency and phase frequency curves for stationary and no stationary coupled multi-frequency resonant kinetic states, based on the numerical experiment on the system of ordinary differential equations in first approximation are presented. Resonant jumps are pointed out in the both series of graphical presentation: amplitude-frequency and phase frequency curves for the case of the resonant interactions between modes in the same frequency resonant intervals.

Using ideas of averaging and asymptotic methods Krilov-Bogoliyubov-Mitropolyskiy in the Doctoral dissertation [109] and in References (see Refs. Hedrih (Stevanović) (1975, 1972, 1981, 1978, 1983, 1984, 1985, 1995)) author gives the first asymptotic approximations of the solutions for one-, two- , three- and four-frequency vibrations of non-linear elastic beams, shaft and thin elastic plates, as well as of the thin elastic shells with positive constant Gauss's curvatures and finite deformations, and system of the ordinary differential equations in first asymptotic approximation for corresponding numbers of amplitudes and phases for stationary and no stationary vibration regimes.

Some results of an investigation of multi-frequency vibrations in single-frequency regime in non-linear systems with many degrees of freedom and with slow-changing parameters are presented by Stevanović and Rašković article (1974). Application of the Krilov-Bogolyubov-Mitropolskiy asymptotic method for study of elastic bodies non-linear oscillations and energetic analysis of the elastic bodies oscillatory motions give new results in theses [108] by Stevanović in 1975. One-frequency transversal oscillations of thin rectangular plate with non-linear constitutive material stress-strain relations and non-linear transversal vibrations of a plate with special analysis of influence of weak non-linear boundary conditions are contents of the articles by Hedrih (1979, 1981).

First approximation of an asymptotic particular solution of the non-linear equations of a thin elastic shell with positive Gauss' curvature in two-frequency regime is pointed out in the article by Hedrih (1983). Two-frequency oscillations of the thin elastic shells with finite deformations and interactions between harmonics have been studied by Hedrih and Mitić (1983) and multi frequency forced vibrations of thin elastic shells with a positive Gauss's curvature and finite displacements by Hedrih (1984). Also, on the mutual influence between modes in non-linear systems with small parameter applied to the multi-frequencies plate oscillations are studied by Hedrih, Kozić, Pavlović and Mitić (1984).

Multi-frequency forced vibrations of thin elastic shells with a positive Gauss' curvature and finite deformations and initial deformations influence of the shell middle surface to the phase-frequency characteristics of the non-linear stationary forced shell's vibrations and numerical analysis of the four-frequency vibrations of thin elastic shells with Gauss' positive curvature and finite deformations are content of reference by Hedrih and Mitić (1985). Also, initial displacement deformation influence of the thin elastic shell middle surface to the resonant jumps appearance was investigated by same authors Hedrih and Mitić (1987). By means of the graphical presentations from the cited References, analysis was made and some conclusions about non-linear phenomenon in multi-frequency vibrations regimes were pointed out. Some of these conclusions we quote here: Non-linearities are the reason for the appearance of interaction between modes in multi-frequency regimes; In the coupled resonant state one or several resonant

jumps appear on the amplitude-frequency and phase frequency curves; these resonant jumps are from smaller to greater amplitudes and vice versa.

Unique trigger of coupled singularities (Hedrih (2003)) with one unstable homoclinic saddle type point, and with two singular stable center type points appear in one frequency stationary resonant kinetic state. It is visible on the phase-frequency as well as on the amplitude-frequency graphs for stationary resonant state.

In the case of the multi-frequency coupled resonant state and in the appearance of the more resonant coupled modes in resonant range of corresponding frequencies, unique trigger of coupled singularities, and multiplied triggers of coupled singularities (see Refs. by Hedrih, 2004, 2005) appear. Maximum number of triggers of coupled singularities is adequate to number of coupled modes and resonant frequencies of external excitations. Multiplied triggers contain multiple unstable saddle homoclinic points in the mapped phase plane as the number of resonant frequencies of external excitations. For example, if a four-frequency coupled resonant process in u-v plane is in question, four homoclinic saddle type points appear. The appearance of these unstable homoclinic saddle points requires further study, since it induces instability in a stationary non-linear multi-frequency kinetic process.

By use a double circular plate system, presented in the References by Hedrih (Stevanović) and Simonović (2005,2006 and 2007), the multi-frequency analysis of the non-linear dynamics with different approaches and by use different kinetic parameters of multi-frequency regimes is pointed out. Series of the amplitude-frequency and phase-frequency graphs as well as eigen-time functions–frequency graphs are obtained for stationary resonant states and analysed according present singularities and triggers of coupled singularities, as well as resonant jumps.

An analogy between non-linear phenomena in particular multi-frequency stationary resonant regimes of multi circular plate system non-linear dynamics, multi-beam system non-linear dynamics and corresponding regimes in chain system non-linear dynamics is identified (see References by Hedrih (Stevanović) listed in the reference list from period 1972-2010).

Using differential equations systems of the first approximation of multi-frequency regime of stationary and no stationary resonant kinetic states, we analysed the energy of excited modes and transfer of energy from one to other modes. On the basis of this analysis, the question of excitation of lower frequency modes by higher frequency mode in the non-linear multi-frequency vibration regimes was opened.

4.5.3. In the Reference by Hedrih (Stevanović) and Hedrih (2009), the expressions for the kinetic and potential energy as well as energy interaction between chains in the double DNA chain helix are obtained and analyzed for a linearized model. Corresponding expressions of the kinetic and potential energies of these uncoupled main chains are also defined for the eigen main chains of the double DNA chain helix. By obtained expressions, we concluded that there is no energy interaction between eigen main chains of the double DNA chain helix system. Time expressions of the main coordinates of the two eigen main chains are expressed by time, and eigen circular frequencies are obtained. Also, generalized coordinates of the double DNA chain helix are expressed by time correspond to the sets of the eigen circular frequencies. These data contribute to better understanding of biomechanical events of DNA transcription that occur parallel with biochemical processes. Considered as a linear mechanical

system, DNA molecule as a double chain helix has its eigen circular frequencies and that is its characteristic. Mathematically it is possible to decouple it into two chains with their set with corresponding eigen circular frequencies which are different. This may correspond to different chemical structure (the order of base pairs) of the complementary chains of DNA. We are free to propose that every specific set of base pair order has its eigen circular frequencies and its corresponding oscillatory energy and it changes when DNA chains are coupled in the system of double chain helix. Oscillations of base pairs and corresponding oscillatory energy for specific set of base pairs may contribute to conformational chances of DNA double helix, and its unzipping and folding.

4. 5.4. General concluding remarks

For limited length of paper, now we made only some comments concerning the following

* Lissajous' curves, orthogonal asynchronous and synchronous oscillations, asynchronization and synchronization of subsystem in hybrid system dynamics.

Series of **Lissajous curves as well as new series of the generalized Lissajous curves** obtained by software MathCad as a results of the coupled orthogonal multi-frequency oscillations are suitable for to build a method of asynchronization/synchronization for applications to the discrete continuum for synchronization some parts of discrete continuum. By this method based on attractors of asynchronization/synchronization of the component oscillations of the subsystems of hybrid system is possible and suitable for to obtain conditions of the integrity of the dynamical system. Generalized **Lissajous curves** can be used as attractors of asynchronization/synchronization of the component subsystem oscillations which are coupled as that these oscillations are orthogonal. By changing some parameters of the coupled oscillators synchronization and by use current software tools as it is MathCad (or MathLab or Mathematica), the visualization of the transformation of the generalized **Lissajous curve,** up to its degeneration into part of straight line, can be obtain as results of the orthogonal coupling of oscillatory multi-frequency signals. If this degeneration is not possible, then these oscillators it is not possible to synchronized and corresponding parameter is not parameter of synchronization. If as results of the change of some parameters of the coupled oscillators synchronization is transformation of the generalized **Lissajous curve** into one unique line then it is possible to obtain system parameters of the attractor of partially synchronization or asynchronization of the coupled oscillators.

Also, there are some models of the discrete continuum in plane or in the space, which mass particles moves oscillatory as result of coupled two in plane, or three in space, orthogonal multi-frequency oscillations Trajectories of this mass particles are generalized **Lissajous curves.** Applications of the knowledge about generalized **Lissajous curves** is important for constructions of some processing machines with working processes based on the motion of the coupled orthogonal multi frequency vibrations.

Ito's stochastic differential equations and applications to stochastic oscillations of mechanical systems with hereditary properties are also actual mathematical task for open possibilities in engineering practice, as well as in other area of science.

Mathematical analogy and phenomenological mapping by use mathematical models in applications to disparate physical models dynamics open very large interactions between different area of science and easier transfer knowledge from one area of science to other.

Also, one of main education task of Serbian mathematicians and other university professor to fined minimum volume of the classical and new current mathematical knowledge necessary to be in the programs of Ph.D. study enough for mathematical background of new Ph.D. specialist for their next two decade research and possibility to accept new and future mathematical discovery and be competent to applied these new mathematical knowledge in research and practice, as well as to define new mathematical tasks appear from his research and to directed to mathematicians for future research.

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136. Sub-Projects 5.1. Thema: Stress and Strain State of the Deformable Bodies and 5.2. Theme: Vector Interpretation of Body Kinetic Parameters, as a part of Project: Actual Problems on Mechanic (1990-1995), Project Leader prof. dr Mane Šašić Ministry of Sciences, Technology and Development of Republic Serbia. Some research results included in three Magistar of sciences theses of Ljubiša Perić, Dragan Jovanović and Snežana Mitić. Mathematical Institute SANU.
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142. Project 1409 Stability and nonlinear oscillations of visco-elastic and composite continuous systems, (2002 – 2005). Support: Ministry of Sciences and Environmental Protection of Republic of Serbia. Some research results included in a Magistar of sciences these of Goran Janevski. Mechanical Engineering Faculty University of Niš. (Project Leaders R. Pavloviić and P. Kozić).
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DOPRINOSI KLASIČNOJ I ANALITIČKOJ MEHANICI: PREGLED AUTOROVIH REZULTATA

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Ključne reči: Pregledni, vektorska metoda, vector momenta mase, vector devijacionog momenta mase, rotator, spregnute rotacije, mimoilazne ose, bazni vektori

tangentnog prostora vektora položaja, ugaona brzina baznih vektora tangentnog prostora, brzina ekstenzije baznog vektora, reonomne veze, reonomne coordinate, pokretljivost, asimptotska aproksimacija rešenja, asimptotska metoda usrednjenja Krilov-Bogoljubov-Mitropolzski, metoda varijacije konstanata, nasledni sistem, reološko i relaksaciono jezgro, standardni nasledni element, integrod-diferencijalna jednačina, izvod necelog reda, kovarijantne koordinate, kontravarijantne koordinate, fizičke koordinate, metoda diskretnog kontinuuma, prostorna frakcionog reda struktura, glavne sopstvene površinske mreže, glavni sopstveni lanci, oscilator frakcionog reda, karakteristični brojevi sistema frakcionog reda, prenos signala, višefrekventni, materijalne tačke, kruto telo, reduktor, deformabilno telo, sistem više tela, transverzalni, longitudinalni, spregnute ploče, spregnute trake, spregnute grede, stohastička stabilnost.

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