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# BOUNDARY LAYER FLOW AND HEAT TRANSFER IN A MICROPOLAR FLUID PAST A PERMEABLE FLAT PLATE

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# Boundary layer flow and heat transfer in a micropolar fluid past a permeable flat plate

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**Abstract**

An analysis is performed to study the shear stress, the couple-stress and heat transfer characteristics of a laminar mixed convection boundary layer flow of a micropolar fluid past an isothermal permeable plate. The governing nonsimilar boundary layer equations are analyzed using the (i) series solution for small $\xi$, (ii) asymptotic solution for large $\xi$ and (iii) primitive-variable formulation and the stream function formulation are being used for all $\xi$. The effects of the material parameters, such as, the vortex viscosity parameter, $K$, and the transpiration parameter, $s$, on the shear stress, the couple-stress and heat transfer have been investigated. The agreement between the solutions obtained from the stream-function formulation and the primitive-variable formulation is found to be excellent.

**Keywords**: Boundary layer, heat transfer, micropolar fluid, permeable flat plate

**Nomenclature**

| | | | |
|---|---|---|---|
| $f$ | dimensionless stream function; | $m$ | couple-stress |
| $g$ | dimensionless microrotation; | $n$ | a real number |
| $H$ | characteristic length | $\bar{N}$ | angular velocity |
| $j$ | microinertia per unit mass | $N$ | dimensionless angular velocity |
| $K$ | vortex viscosity parameter | $Pr$ | Prandtl number |
| | | $q$ | surface heat flux |

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| | |
|---|---|
| $Re$ | Reynolds number |
| $\bar{T}$ | temperature |
| $s$ | transpiration parameter |
| $\bar{T}$ | Temperature of the fluid in the boundary layer region |
| $\bar{T}_w$ | Temperature of the fluid at the surface |
| $\bar{T}_\infty$ | Temperature of the ambient fluid |
| $s$ | transpiration parameter |
| $U_0$ | free stream velocity |
| $\bar{u}, \bar{v}$ | velocity components in $\bar{x}, \bar{y}$ direction |
| $H$ | characteristic length dimensionless velocities in the $x$- and $y$-direction respectively |
| $V_0$ | surface mass flux |
| $x, y$ | nondimensional streamwise and cross-stream Cartesian coordinates |
| $Y$ | similarity variable |

*Greek symbols*

| | |
|---|---|
| $\kappa$ | thermal conductivity of the fluid |
| $\theta$ | dimensionless temperature in the boundary layer |
| $\psi$ | stream function |
| $\mu$ | dynamic viscosity |
| $\alpha$ | thermal diffusivity |
| $\nu$ | viscosity coefficient |
| $\rho$ | density of the fluid |
| $\tau$ | surface shear stress |
| $\gamma$ | gyroviscosity coefficient |

# 1 Introduction

The concept of micropolar fluids introduced by Eringen [1] deals with a class of fluids, which exhibit certain microscopic effects arising from the local structure and micromotions of the fluid elements. These fluids contain dilute suspensions of rigid micromolecules with individual motions, which support stress and body moments and are influenced by spin-inertia. The theory of micropolar fluid and its extension to thermomicropolar fluids [2] may form suitable non- Newtonian fluid models which can be used to analyze the behavior of exotic lubricants [3]-[4], colloidal suspensions or polymeric fluids [5], liquid crystals [6]-[7], and animal blood [8]. Kolpashchikov et al. [9] have derived a method to measure micropolar parameters experimentally. A through review of this subject and application of micropolar fluid mechanics has been provided by Ariman et al. [10]-[11]. On the otherhand, Rees and Bassom [12] investigated the Blasius boundary-layer flow of a micropolar fluid over a flat plate. In this investigation detailed numerical results have also been presented as an asymptotic analysis for large distances from the leading edge.

Studies of heat convection in micropolar fluids have been focused on flat plate [13]-[17] and on a wavy surface [18]. Hossain and Chowdhury

[19] investigated the effect of material parameters on the mixed convection flow of thermomicropolar fluid from a vertical as well as a horizontal heat surface, taking into consideration that the spin-gradient viscosity is non-uniform. Later, Hossain et al. [20] investigated the problem for a viscous incompressible thermomicropolar fluid with uniform spin gradient over a flat plate with a small inclination to the horizontal.

The importance of suction and blowing in controlling the boundary layer thickness and the rate of heat transfer has motivated many researchers to investigate its effects on forced and free convection flows. Eichhorn [21] considered power law variations in the plate temperature and transpiration velocity and gave similarity solutions of the problem. Sparrow and Cess [22] discussed the case of constant plate temperature and transpiration velocity and obtained series expansions for temperature and velocity distributions in powers of $x^{1/2}$, where $x$ is the distance in the stream-wise direction measured from the leading edge. Later, Merkin [23]-[24] and Perikh et al. [25] presented numerical solutions for free convection heat transfer with blowing along an isothermal vertical flat plate. Hartnett and Eckert [26] and Sparrow and Starr [27] reported the characteristics of heat transfer and skin-friction for pure forced convection with blowing; the former dealt with a non-similar case. Locally non-similar solutions for convection flow with arbitrary transpiration velocity were obtained by Kao [28]-[29], applying Görtler-Meksin transformations. Free convection flow along a vertical plate with arbitrary blowing and wall temperature has also been investigated by Vedhanayagam et al. [30]. With this understanding Yucel [31] investigated mixed convection micropolar fluid flow over horizontal plate with uniform surface mass flux blowing and suction through the surface. Recently, Attia [32] investigated the steady laminar flow with heat generation of an incompressible micropolar fluid impinging on a porous flat plate considering a uniform suction or blowing is applied normal to the plate, which is maintained at a constant temperature. Most recently, the MHD boundary-layer flow of a micropolar fluid past a wedge with variable wall temperature has been discussed by Ishak et al. [33]. In addition to this mixed convection flow of a micropolar fluid from an isothermal vertical plate has been investigated by Jena and Mathur [34]

The problem considered here is the boundary layer flow and heat transfer from a permeable flat surface with uniform surface temperature and uniform surface mass flux. So far the authors concern, this has not been discussed in the literature. The transformed boundary layer equations are

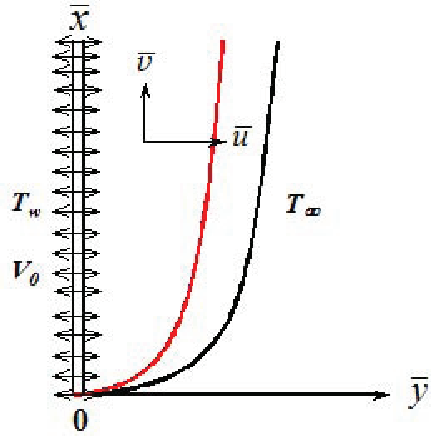



Figure 1: The flow configuration and the coordinate system

solved numerically near to and far from the leading edge, using extended series solutions and asymptotic series solutions. Solutions for intermediate locations are obtained using the primitive-variable formulation as well as by the stream-function formulation. In this investigation, we have considered only the suction case and the effects of the material parameters such as the vortex viscosity parameter, $K$, and the transpiration parameter, $s$, on the shear stress, the couple-stress and heat transfer are presented graphically. The results illustrate the different behavior that occurs when these parameters are varied.

## 2 Mathematical Formulation

A two-dimensional steady, laminar boundary layer flow of a micropolar fluid along a permeable vertical flat plate is considered. The temperature of the ambient fluid is assumed to be uniform at $T$ and that of the surface at $T_w$. The coordinate system and the flow configuration are shown in Fig. 1. Under the usual boundary layer approximation, following Ahmadi [11] and Rees and Bossom [12], the equations of conservation of mass, momentum and energy that govern the flow are given as below:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\mu+\kappa}{\rho}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\kappa}{\rho}\frac{\partial \bar{N}}{\partial \bar{y}}, \tag{2}$$

$$\frac{\partial \bar{N}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{N}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{N}}{\partial \bar{y}} = \frac{\gamma}{\rho j}\frac{\partial^2 \bar{N}}{\partial \bar{y}^2} - \frac{2\kappa}{\rho j}\left(\bar{N} + \frac{\partial \bar{u}}{\partial \bar{y}}\right), \tag{3}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha \nabla^2 \bar{T}. \tag{4}$$

Here, $\bar{x}, \bar{y}$ are the coordinates parallel with and perpendicular to the flat surface, $\bar{u} - \bar{v}$ are the velocity components, $\bar{p}$ the pressure, $\bar{N}$ the component of the gyration vector normal to the $\bar{x} - \bar{y}$ plane, and $j$ is the microinertia density. Furthermore, $\rho$ is the fluid density, $\mu$ the dynamic viscosity, $\kappa$ vortex viscosity and $\gamma$ is the spin-gradient viscosity given by $\gamma = (\mu + \kappa/2)j$ (see [11]). We follow the work of many recent authors by assuming that $j$ is a constant and, therefore, it is set equal to a reference value, $j_0$; consequently the equation for the microinertia density (3) is trivially satisfied.

The boundary conditions to be satisfied by equations (1)-(4) are

$$\bar{u} = 0, \quad \bar{v} = V_0, \quad \bar{N} = -n\frac{\partial \bar{u}}{\partial \bar{y}}, \quad \bar{T} = \bar{T}_w \quad \bar{y} = 0,$$

$$\bar{u} = U_0, \qquad \bar{T} = \bar{T}_\infty \quad \bar{y} \to \infty, \tag{5}$$

where $V_0$ represents the suction velocity of the fluid through the surface of the plate. In this study we shall consider only the suction case (rather than blowing) and therefore $V_0$ is taken as positive throughout. Furthermore, $n$ is a constant, $0 \le n \le 1$. The case $n = 0$ corresponds to the strong concentration of micro-elements. This indicates $\bar{N} = 0$ near the wall suggests that the concentration of the particles is strong enough so that the micro-elements near the walls are unable to rotate because of this concentration. The case, $n = \frac{1}{2}$, on the other hand, indicates the vanishing of anti-symmetric part of the stress tensor and denotes weak concentration. The case $n = 1$ may be used for the modeling of turbulent boundary layer flows ([16]).

Now we introduce the following dimensionless dependent and independent variables:

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y}}{L}Re^{1/2}, \quad u = \frac{\bar{u}}{U_0}, \quad v = \frac{\bar{v}}{L}Re^{1/2}$$

$$\bar{N} = \frac{U_0}{L}Re^{1/2}N, \quad \theta = \frac{\bar{T} - \bar{T}_\infty}{\bar{T}_w - \bar{T}_\infty}, \quad s = \frac{V_0 L}{\nu}Re^{-1/2} \tag{6}$$

in the above set of equations, where $x, y$ are the non dimensional coordinate axis, $u, v$ the non dimensional velocity components, $L$ is reference length, $N$ is the non dimensional angular momentum, and $\theta$ is the non dimensional temperature.

Thus equations (1)-(4) take the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (1+K)\frac{\partial^2 u}{\partial y^2} + K\frac{\partial N}{\partial y} \tag{8}$$

$$u\frac{\partial N}{\partial x} + v\frac{\partial N}{\partial y} = \left(1+\frac{K}{2}\right)\frac{\partial^2 N}{\partial y^2} - K\left(2N + \frac{\partial u}{\partial y}\right) \tag{9}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{Pr}\nabla^2\theta \tag{10}$$

In equations (6) $Re(= U_0L/\nu)$ is the Reynolds number, $K(= \kappa/\mu)$, appeared in equations (8) and (9), is termed as the vortex viscosity parameter and in equation (10) $\mathrm{Pr}(= \nu/\alpha)$ is the Prandtl number. Here we also use $j_0 = L^2$ in equation (9).

The boundary conditions now become

$$u = 0, \quad v = s, \quad N = -n\frac{\partial u}{\partial y}, \quad \theta = 1 \quad \text{at} \quad y = 0,$$

$$u = 1, \quad \bar{T} = 0 \quad \text{as} \quad y \to \infty. \tag{11}$$

From application point of view, we need to find the values of shear stress, $\tau$ , the couple-stress,$m$ , and rate of heat transfer, $q$ , at the surface of the plate, that may be obtained by the relations given below:

$$\bar{\tau} = (\mu+\kappa)\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0}, \quad \bar{m} = \gamma\left(\frac{\partial \bar{N}}{\partial \bar{y}}\right)_{\bar{y}=0} \bar{q} = -k\left(\frac{\partial T}{\partial \bar{y}}\right)_{\bar{y}=0} \tag{12}$$

Using (6) on the relations (12), we obtain

$$\tau = [1+(1-n)K]\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad m = \left(1+\frac{K}{2}\right)\left(\frac{\partial N}{\partial y}\right)_{y=0} q = -\left(\frac{\partial \theta}{\partial y}\right)_{y=0} \tag{13}$$

where $\tau, m$ and $n$ are dimensionless shear-stress, the couple-stress and the rate of heat transfer, respectively, which are defined by

$$\tau = \frac{\bar{\tau}LRe^{-1/2}}{\mu U_0}, \quad m = \frac{\bar{m}}{\rho U_0^2 L}, \quad q = \frac{\bar{q}LRe^{-1/2}}{k(T_w - T_\infty}$$

# 3 Solution methodology

To Investigate the present problem we have employed two formulations, namely, the primitive-variable formulation and the stream-function formulation, method of solution of which are presented in the following sections.

## 3.1 Primitive-variable transformation

To get the set of equations (7)-(10) in convenient form for integration, we define the following one parameter group of transformation for the dependent and the independent variables:

$$u = U,\ v = x^{-1/2}(V + s\xi),\ N = x^{-1/2}G,\ \theta = \Theta,\ \xi = x^{1/2},\ Y = x^{-1/2}y \tag{14}$$

Thus the equations (7)-(10) are transformed to

$$\frac{1}{2}\xi\frac{\partial U}{\partial \xi} - \frac{1}{2}Y\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial Y} = 0, \tag{15}$$

$$\frac{1}{2}\xi U\frac{\partial U}{\partial \xi} + \left(V + s\xi - \frac{1}{2}YU\right)\frac{\partial U}{\partial Y} = (1+K)\frac{\partial^2 U}{\partial Y^2} + K\frac{\partial G}{\partial Y}, \tag{16}$$

$$-\frac{1}{2}UG+\frac{1}{2}\xi XU\frac{\partial G}{\partial \xi}+\left(V + s\xi - \frac{1}{2}YU\right)\frac{\partial G}{\partial Y} = (1+K/2)\frac{\partial^2 G}{\partial Y^2}-K\xi^2\left(2G + \frac{\partial U}{\partial Y}\right), \tag{17}$$

$$\frac{1}{2}\xi U\frac{\partial \Theta}{\partial Y} + \left(V + s\xi - \frac{1}{2}YU\right)\frac{\partial \Theta}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 \Theta}{\partial Y^2}. \tag{18}$$

Appropriate boundary conditions are

$$U = 0, \quad V = 0, \quad G = -n\frac{\partial U}{\partial Y}, \quad \Theta = 1 \quad \text{at} \quad Y = 0,$$

$$U = 1, \quad G = 0, \quad \Theta = 0 \quad \text{as} \quad Y \to \infty. \tag{19}$$

Once we know the values of $U, V, G$ and $\Theta$ and their derivatives, we are at position to find the values of shear stress,$\tau$ the couple-stress,$m$ and rate of heat transfer,$q$ from the following relations obtained from (13):

$$\tau = [1 + (1-n)K]\,\xi^{-1}\left(\frac{\partial U}{\partial Y}\right)_{Y=0},$$

$$m = \left(1 + \frac{K}{2}\right)\xi^{-2}\left(\frac{\partial G}{\partial Y}\right)_{Y=0}, \tag{20}$$

$$q = -\xi^{-1}\left(\frac{\partial \Theta}{\partial Y}\right)_{Y=0}.$$

## 3.2 Stream function formulation

To get the set of equations (7)-10) in convenient form for integration, we define the following one parameter group of transformation for the dependent and the independent variables:

$$\psi = x^{1/2}(f(\xi,Y)+s\xi),\ N = x^{-1/2}g(\xi,Y),\ \theta = h(\xi,Y)$$
$$\xi = x^{1/2},\ Y = x^{-1/2}y \tag{21}$$

Here $Y$ is the pseudo-similarity variable and $\psi$ is the stream function that satisfies equation (7) and is defined by

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial y}. \tag{22}$$

Equations (19), (21) and (22) thus reduce to

$$(1+K)f''' + \frac{1}{2}ff'' + Kg' + s\xi f'' = \frac{1}{2}\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right), \tag{23}$$

$$(1+K/2)g'' + \frac{1}{2}(fg'+gf') - K\xi^2(2g+f'') + s\xi g' = \frac{1}{2}\xi\left(f'\frac{\partial g}{\partial \xi} - g'\frac{\partial f}{\partial \xi}\right), \tag{24}$$

$$\frac{1}{Pr}h'' + \frac{1}{2}fh' + s\xi h' = \frac{1}{2}\xi\left(f'\frac{\partial h}{\partial \xi} - h'\frac{\partial f}{\partial \xi}\right). \tag{25}$$

Boundary conditions take the form

$$f(\xi,0) = f'(\xi,0) = 0,\quad g(\xi,0) = -nf''(\xi,0),\quad h(\xi,0) = 1,$$
$$f'(\xi,0) = 1,\quad g(\xi,0) = 0,\quad h(\xi,0) = 0. \tag{26}$$

In these equations, primes denote differentiation of the functions with respect to $Y$. Here solution of the set of equations (23)-(25) is obtained by implicit finite difference method together with the Keller-box elimination technique (also known as Keller box method), introduced by Keller and Cebeci [[35] and described in more detail in Cebeci and Bradshaw [36]. In this case, the expressions for the shear stress, the couple-stress and rate of heat transfer given in (19) become

$$\tau = [1 + (1-n)]\,\xi^{-1} f''(\xi,0),$$

$$m = \left(1 + \frac{K}{2}\right)\xi^{-2}g'(\xi,0), \tag{27}$$
$$q = -\xi^{-1}hg'(\xi,0).$$

### 3.3 Asymptotic solutions

**Solutions for small $\xi$**

Near the leading edge, or equivalently for small $\xi$, we expand the functions $f, g$ and $h$ in powers of $\xi$ as given below:

$$f(\xi,Y) = \sum_{i=0}^{\infty}\xi^i f_i(Y), \quad g(\xi,Y) = \sum_{i=0}^{\infty}\xi^i g_i(Y), \quad h(\xi,Y) = \sum_{i=0}^{\infty}\xi^i h_i(Y) \tag{28}$$

Substituting these into equations (23)-(25) and then equating the terms of like powers of $\xi$ to zero, we get the following pairs of ordinary differential equations for the functions $f_i$, $g_i$ and $h_i$:

$$(1+K)f_0''' + \frac{1}{2}f_0 f_0'' + Kg_0' = 0,$$
$$(1+K/2)g_0'' + \frac{1}{2}(f_0 g_0' + g_0 f_0') = 0,$$
$$\frac{1}{Pr}h_0'' + \frac{1}{2}fh_0' = 0 \tag{29}$$
$$f_0(0) = f_0'(0) = 0, \quad g_0(0) = -nf_0''(0), \quad h_0(0) = 1,$$
$$f_0'(\infty) = 1, \quad g_0(\infty) = 0, \quad h_0(\infty) = 0. \tag{30}$$

The higher order equations, for $i > 1$, are as follows:

$$(1+K)f_i''' + sf_{i-1}'' + Kg_i' = \frac{1}{2}\sum_{r=0}^{l}\left(rf_r'f_{i-r}' - (1+r)f_r'f_{i-r}''\right),$$

$$(1+K/2)g_i'' + sg_{i-1}' - K(2g_{i-2} + f_{i-2}'') = \frac{1}{2}\sum_{r=0}^{l}\left\{(r-1)g_r f_{i-r}' - (1+r)f_r g_{i-r}'\right\},$$

$$(1+K/2)h_i'' + sh_{i-1}' = \frac{1}{2}\sum_{r=0}^{l}\left\{rh_{i-r}f_r' - (1+r)f_r h_{i-r}'\right\}. \tag{31}$$

$$f_i(0) = f_i'(0) = 0, \quad g_i(0) = -nf_i''(0), \quad h_i(0) = 0,$$
$$f_i'(\infty) = 0, \quad g_i(\infty) = 0, \quad h_i(\infty) = 0. \tag{32}$$

for $i = 1, 2, \ldots$.

In the above equations the functions $f_0$, $g_0$ and $h_0$ are the well-known free convection similarity solutions for flow around a constant temperature semi-infinite vertical plate and the functions $f_i$, $g_i$ and $h_i$ $(i = 1, 2, 3, \ldots)$ are effectively first and higher order corrections to the flow due to the effect of the transpiration of the fluid through the surface of the plate, Further the equations for each $i \geq 1$) are linear, but coupled, and may be found by pair-wise sequential solution. These pair of equations has been integrated using an implicit Runge-Kutta-Butcher (Butcher [37]) initial value solver together with the iteration scheme of Nachtsheim and Swigert [38]. In the present investigation, solutions of 10 sets of equations have been obtained.

The solution of the above equations enables the calculation of the various flow parameters near the leading edge, such as the values of shear stress, $\tau$, the couple-stress, $m$, and rate of heat transfer, $q$. Using the relation given in (27), the quantities $\tau$, $m$ and $q$ can now be calculated, respectively, from the following expressions:

$$\tau = [1 + (1-n)K]\, \xi^{-1} \sum_{i=0}^{\infty} \xi^i f_i''(0)$$
$$m = \left(1 + \frac{K}{2}\right) \xi^{-2}\tau = \left(1 + \frac{K}{2}\right) \xi^{-1} \sum_{i=0}^{\infty} \xi^i g_i'(0) \tag{33}$$
$$q = -\xi^{-1}\tau = \left(1 + \frac{K}{2}\right) \xi^{-1} \sum_{i=0}^{\infty} \xi^i h_i'(0)$$

## Solutions for large $\xi$

In this section attention has been given to the solution of equations (23)-(25) when $\xi$ is large. The order of magnitude analysis of various terms in (23)-(25) shows that the largest terms are $f'''$ and $\xi f''$ in (23), $g''$ and $\xi f'$ in (24), and $h''$ and $\xi h'$ in (25). In the respective equations both the terms have to be balanced in magnitude and the only way to do this is to assume that $Y$ be small and hence derivatives are large. Given that $h = O(1)$ as $\xi \to \infty$, it is essential to find appropriate scaling for $f$ and $Y$. On balancing

$f'''$ and $\xi f''$ in (23), it is found that $Y = O(\xi^{-1})$ and $f = O(\xi^{-1})$ as $\xi \to \infty$. Therefore the following transformations are introduced:

$$f = \xi^{-1} F(\xi, \eta), \;\; g = \xi G(\xi, \eta), \;\; h = H(\xi, \eta), \;\; \eta = \xi Y. \tag{34}$$

Equations (23)-(25) together with the transformations given in (34) then become

$$(1 + K) F''' + K G' + s F'' = \frac{1}{2} \xi^{-1} \left( F' \frac{\partial F'}{\partial \xi} - F'' \frac{\partial F}{\partial \xi} \right), \tag{35}$$

$$(1 + K/2) G'' - K(2G + F'') + s G' = \frac{1}{2} \xi^{-1} \left( F' \frac{\partial G}{\partial \xi} - G' \frac{\partial F}{\partial \xi} \right), \tag{36}$$

$$\frac{1}{Pr} H'' + s H' = \frac{1}{2} \xi^{-1} \left( F' \frac{\partial H}{\partial \xi} - H' \frac{\partial F}{\partial \xi} \right). \tag{37}$$

For sufficiently large $\xi$, we can write the above equations as follows

$$(1 + K) F''' + K G' + s F'' = 0, \tag{38}$$

$$(1 + K/2) G'' - K(2G + F'') + s G' = 0, \tag{39}$$

$$\frac{1}{Pr} H'' + s H' = 0. \tag{40}$$

Boundary conditions take the form

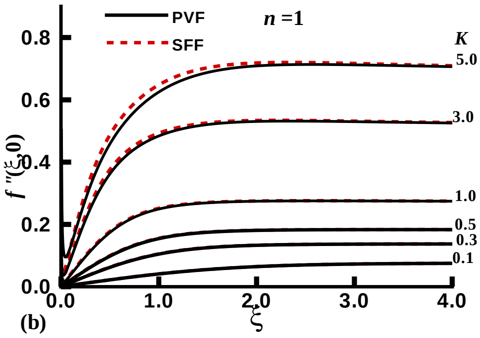
$$F(\xi, 0) = F'(\xi, 0) = 0, \quad G(\xi, 0) = -n F''(\xi, 0), \quad H(\xi, 0) = 1,$$

$$F'(\xi, \infty) = 1, \quad G(\xi, \infty) = 0, \quad H(\xi, \infty) = 0. \tag{41}$$

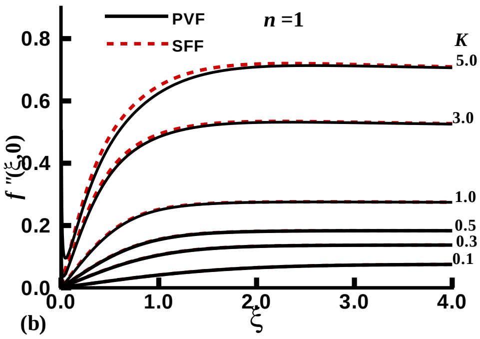
The various flow parameters such as the shear stress, the couple-stress and rate of heat transfer may be calculated from the following relations:

$$\begin{aligned} \tau &= \left[1 + (1 - n) K\right] F''(\xi, 0), \\ m &= \left(1 + \frac{K}{2}\right) G'(\xi, 0), \\ q &= -H'(\xi, 0). \end{aligned} \tag{42}$$

Numerical values of $\tau$, $m$ and $q$ thus obtained are compared with that obtained from primitive variable formulation through figures 2 and 3.




(a) $n = 0$



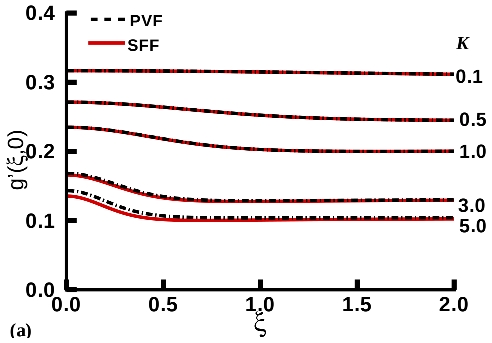

(b) $n = 1$

Figure 2: Development of wall shear stress $f''(\xi, 0)$ as a function of $\xi$ for various values of $K$.

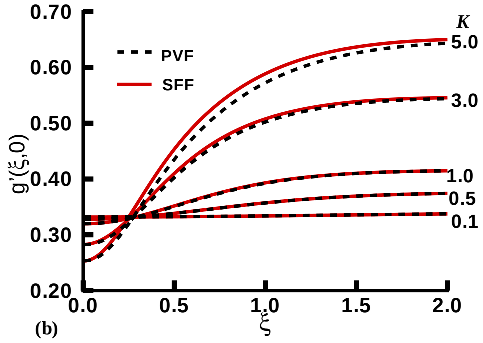
## 4 Results and discussion

Numerical computation were carried out mainly for fluid having a Prandtl number, $\Pr = 10.0$ while the value of the vortex viscosity parameter, $K =$ 0.0, 2.0 and 5.0 and the transpiration parameter, $s =$ 0.5, 1.5 and 3.0.

Representative numerical values of $X^{-1/2}g'(100, 0)$ obtained from the present integration of equation (23) are entered into the Table 1 for comparison with those of Rees and Bossom [12].




(a)




(b)

Figure 3: Development of change of the gyration component at the wall, $g'(\xi, 0)$ as a function of $\xi$

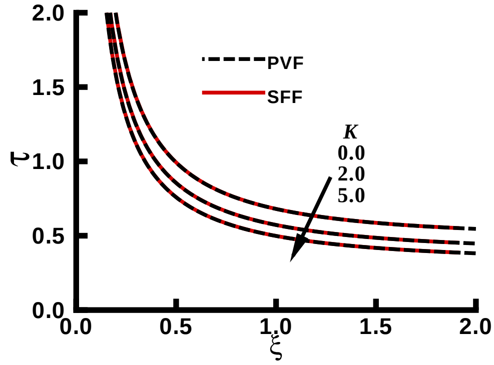
**Table 1**
Numerical values of $X^{-1/2}g'$ at $X = 100$ and $Y = 0$ for $n = 0$ and different values of $K$

| $K$ | Rees et al. [12] | present |
|---|---|---|
| 0.1 | -0.06895 | -0.06908 |
| 0.3 | -0.1050 | -0.10518 |
| 0.5 | -0.1211 | -0.12124 |
| 1.0 | -0.1354 | -0.13555 |
| 3.0 | -0.1285 | -0.12859 |
| 5.0 | -0.1145 | -0.11455 |

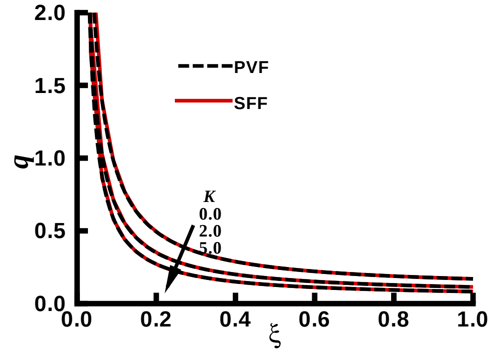
From this table it may be seen that the present solutions are in excellent agreement with Rees and Bossom [12]. Further, throughout figures 2 to 5 the solid lines represent solutions from primitive variable formulation (PVF) and the dotted are from stream-function formulation (SFF).

In Figs 2 and 3 we have shown a comparison between the stream-function formulation and the primitive-variable formulation of the variations of both the shear stress and of the rate of change of the gyration component at the solid boundary with considering $s = 0.0$, $Pr = 10.0$ and a range of values of $K$ with $n$ fixed. The agreement between these formulations is seen to be extremely good. We further observe that these figures are exactly similar to that of Rees et al. [12].

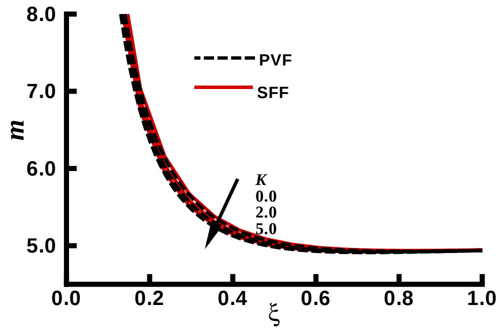
A comparison between the results obtained by the stream-function formulation and the primitive-variable formulation is shown in Figs 4 and 5. From the figures, it is evident that there is an excellent agreement between these two results, which is expected. Hence rest of the results presented and discussed here is based on primitive-variable formulation. In Fig.6, we depict the values of local shear stress, $\tau$, the local heat transfer, $q$, at the surface and the distribution of the local couple-stress, $m$, in the boundary layer for different values of the vortex viscosity parameter $K$ (= 0.0, 2.0 and 5.0) while the Prandtl number, Pr = 10, the transpiration parameter, $s = 1.0$, and the temperature gradient parameter, $n = 0.5$. In these figures the curves marked by solid, broken and dotted curves represent respectively, the results obtained by the finite difference method, the series solution method and the asymptotic method. From these figures it may be seen that an increase in the value of the vortex viscosity parameter $K$ leads to an increase in the value of the local shear stress, the local heat transfer and the local couple-stress. We further observe that for any selected value of the vortex viscosity parameter $K$, values of the shear stress, heat transfer and the couple-stress reach the respective asymptotic values smoothly. The heat transfer also reaches its asymptotic values at smaller $\xi$; whereas the shear stress and the couple-stress does so at comparatively larger value of $\xi$. We further observe that as the value of $K$ increases, the shear stress, the heat transfer and the couple-stress reach their asymptotic values faster. Finally, it may be concluded that the values of the shear stress, the heat transfer and the couple-stress obtained by the three methods are in excellent agreement with each other when the value of $K$ is large. The effect of the transpiration parameter, $s$, on the local shear stress, the heat transfer and the couple-stress are presented graphically in Fig.7.




(a)

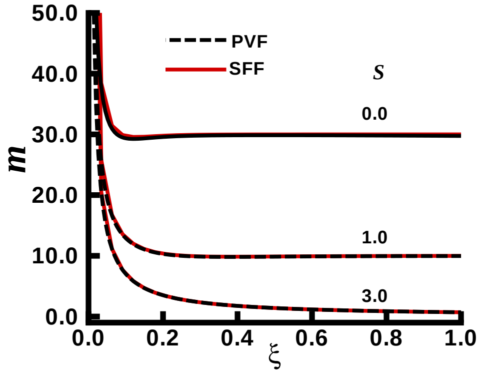



(b)

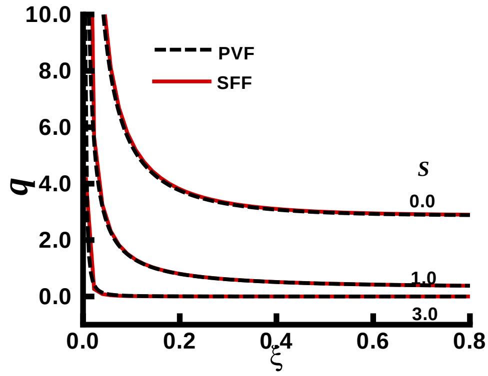



(c)

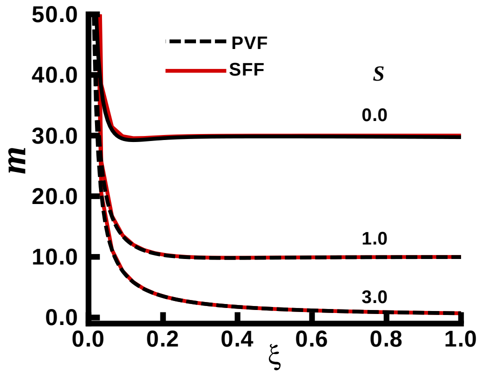
Figure 4: The surface shear stress, $\tau$ , the couple-stress, $m$, and the heat transfer, $q$, for $n = 0.5, K = 0.5, Pr = 10.0$ and for various values of $s$.




(a)

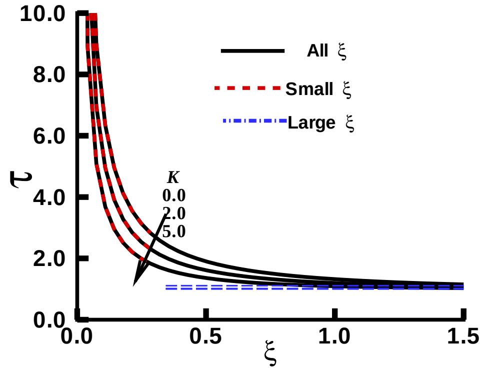



(b)

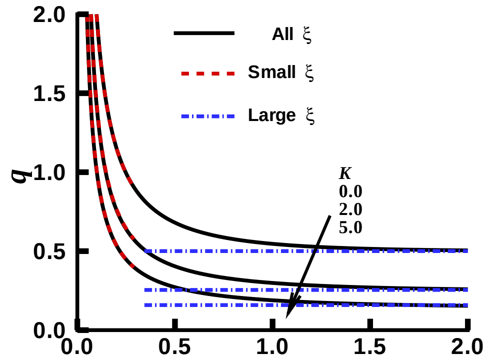



(c)

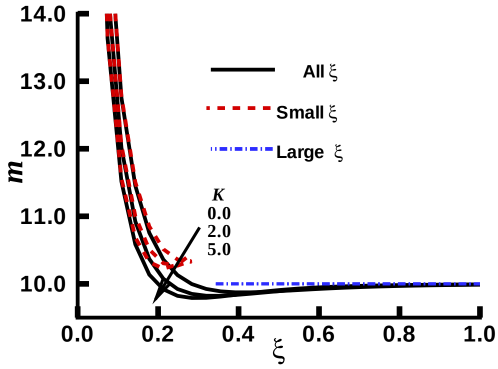
Figure 5: The surface shear stress, $\tau$ , the couple-stress, $m$, and the heat transfer, $q$, for $n = 0.5, K = 0.5, Pr = 10.0$ and for various values of $s$.




(a)

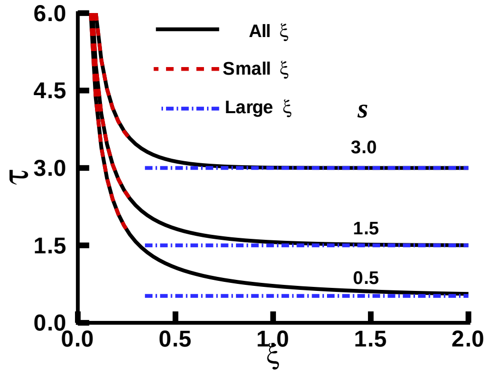



(b)

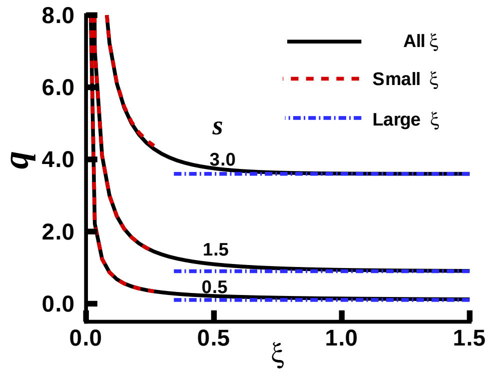



(c)

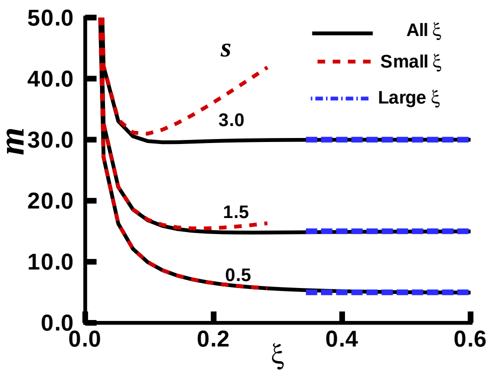
Figure 6: The surface shear stress, $\tau$ , the couple-stress, $m$, and the heat transfer, $q$, for $n = 0.5, s = 1.0, Pr = 10.0$ and for various values of $K$.




(a)




(b)




(c)

Figure 7: The surface shear stress, $\tau$ , the couple-stress, $m$, and the heat transfer, $q$, for $n = 0.5, K = 0.5, Pr = 10.0$ and for various values of $s$.

It is observed from these figures that an increase in the value of s increases the value of the shear stress, the heat transfer and the couples-tress. We also observe that for any selected value of $s$, values of the shear stress, the heat transfer and the couple-stress tend to their respective asymptotic values. As the value of $s$ increases, the shear stress, the heat transfer and the couple-stress reach their asymptotic values faster In this case we also found that the results from the three methods are in excellent agreement.

# 5 Conclusions

In the present study we have investigated the effects of the vortex viscosity parameter, $K$, and the transpiration parameter, $s$, on laminar mixed convection boundary layer flow of a micropolar fluid past a vertical permeable flat plate. The governing boundary layer equations have been simulated employing four distinct methods, namely: (1) the series solution for small $\xi$ ; (2) the asymptotic solution for large $\xi$; (3) the implicit finite difference method together with Keller-box scheme; and the primitive-variable formulation method for all $\xi$. Results are expressed in terms of the surface shear stress, the couple-stress and heat transfer rate. From the present investigation it may be concluded that:

(i) Agreement between the solutions of the stream-function formulation and the primitive-variable formulation found to be excellent.

(ii) An increase in the value of the vortex viscosity parameter, $K$, leads to an increase in the value of the surface shear stress, the heat transfer rate and the local couple-stress.

(iii) Value of the surface shear stress, the heat transfer rate or the couple-stress increases due to increase in the rate of increase in fluid injection parameter $s$.

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## Protok graničnog sloja preko propusne ravne ploče i prenos toplote u mikropolarnoj tečnosti

Izvršena je analiza u cilju proučavanja smičućeg napona, naponskog sprega i karakteristika prenosa toplote laminarne mešovite konvekcije graničnog sloja preko izotermalne propusne ploče protokom mikropolarnog fluida. Vladajuće neslične jednačine graničnog sloja su analizirane korišćenjem (i) rešenja pomoću reda za male $\xi$, (ii) asimptotskog rešenja za velike $\xi$ i (iii) pristupa sa primitivnim promenljivim, a formulacija strujne funkcije se koristi za sve $\xi$. Efekti materijalnih parametara, kao što su, parametar vrtložne viskoznosti, $K$, i parametar transpiracije, $s$, na smicajni napon, naponski spreg i prenos toplote su istraženi. Za slaganje izmedju rešenja dobijenih iz formulacije strujnom funkcijom i formulacije sa primitivnim promenljivim je utvrdjeno da je odlično.

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