ON LINEAR AND NONLINEAR
FRACTIONAL PDEs

Jamshad Ahmad   **Qazi Mahmood ul Hassan**
Syed Tauseef Mohyud-Din

On linear and nonlinear fractional PDEs

Jamshad Ahmad[*]   **Qazi Mahmood ul Hassan**[†]

Syed Tauseef Mohyud-Din [‡]

Abstract

In this study, Variational Iteration Method (VIM) has been applied to obtain the analytical solutions of fractional order nonlinear partial differential equations. The iteration procedure is based on a relatively new approach which is called Jumarie's fractional derivative. Several examples have been solved to elucidate effectiveness of the proposed method and the results are compared with the exact solution, revealing high accuracy and efficiency of the method.

Keywords: Fractional Fokker-Plank Equation, Fractional Fornberg-Whitham equation, Variational Iteration Method, Modified Riemann–Liouville derivative.

1   Introduction

In last two decades, fractional differential equations have gained much interest due to exact description of nonlinear phenomena in fluid flow, seismology, biology, chemistry, economic, probability and statistics, acoustics, material science, engineering and other areas of science. However, fractional calculus is three centuries old as the conventional calculus [1]. Derivatives and integrals of fractional arbitrary orders have found many applications in recent

---

[*]Department of Mathematics, Faculty of Sciences, HITEC University Taxila Cantt Pakistan, e-mail:jamshaid_uetian@yahoo.com

[†]Department of Mathematics, Faculty of Sciences, HITEC University Taxila Cantt Pakistan

[‡]Department of Mathematics, Faculty of Sciences, HITEC University Taxila Cantt Pakistan

study [2,3]. Several analytical methods have been proposed for solving fractional differential equation of complex physical nature, such as Adomian's Decomposition Method (ADM)[4], Differential Transform Method (DTM) [5], Homotopy Perturbation Method (HPM) [6].

Obidat and Momani applied the VIM to fractional differential equations in fluid mechanics [7]. Inc [8] applied the VIM to solve the space and time fractional Burger's equations. Safari et al. [9] used the VIM to obtain an analytic approximate solution of space-fractional KdV-Burgers- Kuramoto equations. Abidi and Omrani [10] used the Homotopy Analysis Method (HAM) for the Fornberg-Whitham equation and Hassan et al. [19] applied (HAM) for solving Space and Time-fractional KdV Equations. Gupta and singh [11] applied the Homotopy Perturbation Method (HPM) for the fractional Fornberg-Whitham equation. More recently, Lu [12] has used the Variational Iteration Method (VIM) to obtain an approximate solution of Fornberg-Whitham equation. In this paper, we extend the application of the FVIM with modified Riemann-Liouville derivative in order derive the analytical approximate solutions to nonlinear time-fractional Fokker-Plank and Fornberg-Whitham equations.

2   Basic definitions

We give some basic definitions, notations and properties of the fractional calculus theory which are used further in this paper:

Definition 2.1 Assume $f : R \rightarrow R,\ x \rightarrow f(x)$, denote a continuous (but not necessarily differentiable) function and let the partition $h > 0$ in the interval [0, 1]. Jumarie's derivative is defined through the fractional difference:

$$\Delta^{\alpha} = (FW - 1)^{\alpha} f(x) \tag{1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(x + (\alpha - k)),$$

where, $FW f(x) = f(x + h)$. Then the fractional derivative is defined as the following limit:

$$f^{(\alpha)} = \lim_{h \to 0} \frac{\Delta^{\alpha} [f(x) - f(0)]}{h^{\alpha}}. \tag{2}$$

This definition is close to the standard definition of derivative, and as a direct result, the $\alpha\,th$ derivative of a constant $0 < \alpha < 1$; is zero.

Definition 2.2 The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ for a function $f \in C_\mu \ \mu \geq -1$ is defined as

$$I_x^\alpha f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_0^x \left(x - \xi\right)^{\alpha - 1} f\left(\xi\right) d\xi, \ \alpha > 0, \ t > 0. \tag{3}$$

Definition 2.3 The Jumarie's modified Riemann–Liouville derivative is defined as

$$I_x^\alpha f\left(x\right) = \frac{1}{\Gamma\left(m - \alpha\right)} \frac{d^m}{d\,x^m} \int_0^x \left(x - \xi\right)^{m - \alpha} \left(f\left(\xi\right) - f\left(0\right)\right) d\xi, \tag{4}$$

where, $x \in [0, 1]$, $m - 1 < \alpha \leq m$, $m \geq 1$.

The proposed modified Riemann–Liouville derivative as shown in Eq. (4) is strictly equivalent to Eq. (2). Meanwhile, we would introduce some properties of the fractional modified Riemann–Liouville derivative

(a) Fractional Leibnitz product law

$$D_x^\alpha\left(u\,v\right) = u^{\left(\alpha\right)} v + u\,v^{\left(\alpha\right)}. \tag{5}$$

(b) Fractional Leibnitz formulation

$$I_x^\alpha\,D_x^\alpha f\left(x\right) = f\left(x\right) - f\left(0\right), \ 0 < \alpha \leq 1. \tag{6}$$

Therefore, the integration by part can be used during the fractional calculus

$$I_x^\alpha\,u^{\left(\alpha\right)}\,v = \left.\left(u\,v\right)\right|_a^b - I_b^\alpha\,u\,v^{\left(\alpha\right)}. \tag{7}$$

Definition 2.4 Fractional derivative of compounded functions is defined as

$$d^\alpha f\left(x\right) \cong \Gamma\left(1 + \alpha\right) df, \ 0 < \alpha < 1. \tag{8}$$

Definition 2.5 The integral with respect to $\left(d\,\xi\right)^\alpha$ is defined as the solution of fractional differential equation given by equation

$$d\,y \cong f\left(x\right)\left(d\,x\right)^\alpha, \ y\left(0\right) = 0, \ x \geq 0, \ 0 < \alpha \leq 1, \tag{9}$$

$$y \cong \int_0^x f\left(\xi\right)\left(d\,\xi\right)^\alpha = \alpha \int_0^x \left(x - \xi\right)^{\alpha - 1} f\left(\xi\right) d\xi, \ 0 < \alpha \leq 1. \tag{10}$$

For example $f(x) = x^\beta$ in Eq. (10), one obtains

$$\int_0^x \xi^\beta \, (d\xi)^\alpha = \frac{\Gamma(1+\alpha)\,\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} \, x^{\alpha+\beta}, \; 0 < \alpha \le 1 \qquad (11)$$

and for a continuous function f, we have

$$\int_0^x \frac{\partial^\alpha f(t)}{\partial t^\alpha} \, (d\xi)^\alpha = \Gamma(1+\alpha)\,(f(x) - f(0)) \, . \qquad (12)$$

Definition 2.6 Assume that the continuous function $f : R \to R$, $x \to f(x)$ has a fractional derivative of order $k\,\alpha$, for any positive integer k and any α $0 < \alpha \le 1$, then the following equality holds, which is

$$f(x+h) = \sum_{k=0}^\infty \frac{h^{\alpha k}}{\alpha k!} \, f^{(\alpha k)}(x), \; 0 < \alpha \le 1, \qquad (13)$$

On making the substitution $h \to x$ and $x \to 0$, we obtain the fractional Mc-Laurin series

$$f(x) = \sum_{k=0}^\infty \frac{x^{\alpha k}}{\alpha k!} \, f^{(\alpha k)}(0), \; 0 < \alpha \le 1. \qquad (13')$$

3 Analysis of Fractional Variational Iteration Method (FVIM)

In order to elucidate the solution procedure of the VIM, we consider the following fractional differential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} \, u(x, t) = K[x] \, u(x, t) + q(x, t), \; t > 0, \; x \in R, \qquad (14)$$

subject to the initial condition

$$u(x, 0) = f(x) \, .$$

Here $K[x]$ is the differential operator, $f(x)$ and $q(x, t)$ are continuous functions. According to VIM introduced by He [13], we can construct a correction functional for Eq. (14) as follows

$$u_{n+1}(x, t) = u_n(x, t) + I^\alpha \left[\lambda \left(\frac{\partial^\alpha u_n}{\partial t^\alpha} - K[x] \, u(x, t) - q(x, t) \right) \right],$$

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \lambda(\xi) \Big(-q(x,\xi)$$

$$+ \frac{\partial^\alpha u_n}{\partial \xi^\alpha}(x,\xi) - K[x]u(x,\xi) \Big) d\xi. \tag{15}$$

Combining Eq.(10) and (15), we obtained a proposed correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(\xi) \Big(\frac{\partial^\alpha u_n}{\partial \xi^\alpha}(x,\xi)$$

$$- K[x]u(x,\xi) - q(x,\xi) \Big) (d\xi)^\alpha. \tag{16}$$

It is obvious that the successive approximation u_j, $j \geq 0$ can be established by determining λ via variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta \tilde{u}_n = 0$. Therefore, we first determine Lagrange's multiplier that will be identified optimally via integration by parts. The successive approximation of the $u_{n+1}(x,t)$, $n \geq 0$ solution $u(x,t)$ will be readily obtained upon using the Lagrange's multiplier and by using any selective function u_0. The initial values are usually used for selecting the zeroth approximation u_0. With λ determined, several approximations u_j, $j \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n \to \infty} u_n(x,t). \tag{17}$$

4   Numerical examples

In this section, we apply the FVIM to find the solutions of nonlinear time-fractional Fokker-Plank and Fornberg-Whitham equations, and compared them with those obtained by other methods.

Example 1. Consider the following non-homogeneous space fractional equation

$$\frac{\partial^\alpha u}{\partial x^\alpha} - \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + (2 - 2t - 2x), \quad (t > 0, \ x \in R, \ 1 < \alpha \leq 2), \tag{18}$$

with the conditions

$$u(0,t) = t^2, \quad u(x,0) = x^2, \quad u_t(0,t) = 0.$$

The correction functional is read as

$$
\begin{aligned}
u_{n+1}(x, t) \;=\; & u_n(x, t) + \frac{1}{\Gamma(1+\alpha)} \int_0^x \lambda(\xi) \left(\frac{\partial^\alpha u_n(\xi, t)}{\partial \xi^\alpha} - \frac{\partial u_n(\xi, t)}{\partial \xi} \right. \\
& \left. - \frac{\partial u_n(\xi, t)}{\partial t} - (2 - 2t - 2\xi) \right) (d\xi)^\alpha,
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
u_{n+1}(x, t) \;=\; & u_n(x, t) + \frac{1}{\Gamma(1+\alpha)} \int_0^x \lambda(\xi) \left(\frac{\partial^\alpha u_n(\xi, t)}{\partial \xi^\alpha} - \frac{\partial \tilde{u}_n(\xi, t)}{\partial \xi} \right. \\
& \left. - \frac{\partial \tilde{u}_n(\xi, t)}{\partial t} - (2 - 2t - 2\xi) \right) (d\xi)^\alpha.
\end{aligned} \tag{20}
$$

The function $\tilde{u}_n$ is a restricted term which means that $\delta \tilde{u}_n = 0$, via the variational theory. Taking variation δ to both sides of Eq. (19) and applying optimal condition $\delta u_{n+1} = 0$, one obtains

$$
0 = \delta u_n + \frac{1}{\Gamma(1+\alpha)} \delta \int_0^x \lambda(\xi) \frac{\partial^\alpha u_n}{\partial \xi^\alpha} (d\xi)^\alpha, \tag{21}
$$

Using Eq. (8), applying integration by parts on second term of right hand and after equating, we have

$$
\frac{\partial^\alpha \lambda(\xi)}{\partial \xi^\alpha} = 0, \text{ and } 1 + \lambda(\xi) = 0. \tag{22}
$$

The generalized Lagrange multiplier can be identified by the above equations $\lambda(\xi) = -1$. Substituting Eq. (21) into the functional Eq. (19) yields the iteration formulation as follows

$$
u_{n+1} = u_n - \frac{1}{\Gamma(1+\alpha)} \int_0^x \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - \frac{\partial u_n}{\partial \xi} - \frac{\partial u_n}{\partial t} - (2 - 2t - 2\xi) \right) (d\xi)^\alpha. \tag{23}
$$

The initial approximation is read as

$$
u_0(x, t) = t^2 + \frac{2 - 2t}{\Gamma(1+\alpha)} x^\alpha - \frac{2}{\Gamma(2+\alpha)} x^{1+\alpha}.
$$

The other approximations are

$$u_1\left(x,\,t\right) = u_0 - \frac{1}{\Gamma\left(1+\alpha\right)} \int_0^x \left\{ \frac{\partial^\alpha u_0}{\partial \xi^\alpha} - \frac{\partial u_0}{\partial \xi} - \frac{\partial u_0}{\partial t} \right\} \left(d\xi\right)^\alpha$$

$$= \frac{2-2t}{\Gamma\left(2\alpha\right)} x^{2\alpha-1} - \frac{4}{\Gamma\left(1+2\alpha\right)} x^{2\alpha} + \frac{2t}{\Gamma\left(1+\alpha\right)} x^\alpha,$$

$$u_2\left(x,t\right) = u_{10} - \frac{1}{\Gamma\left(1+\alpha\right)} \int_0^x \left(\frac{\partial^\alpha u_0}{\partial \xi^\alpha} - \frac{\partial u_0}{\partial \xi} - \frac{\partial u_0}{\partial t} \right) \left(d\xi\right)^\alpha$$

$$= \frac{2-2t}{\Gamma\left(3\alpha-1\right)} x^{3\alpha-2} - \frac{6}{\Gamma\left(3\alpha\right)} x^{3\alpha} + \frac{2t}{\Gamma\left(2\alpha\right)} x^{2\alpha-1} + \frac{2}{\Gamma\left(2+\alpha\right)} x^{2\alpha},$$

$$\vdots$$

The solution is

$$u\left(x,t\right) = t^2 + \frac{2-2t}{\Gamma\left(1+\alpha\right)} x^\alpha - \frac{2}{\Gamma\left(2+\alpha\right)} x^{1+\alpha} + \frac{2-2t}{\Gamma\left(2\alpha\right)} x^{2\alpha-1}$$

$$- \frac{4}{\Gamma\left(1+2\alpha\right)} x^{2\alpha} + \frac{2t}{\Gamma\left(1+\alpha\right)} x^\alpha + \frac{2-2t}{\Gamma\left(3\alpha-1\right)} x^{3\alpha-2}$$

$$- \frac{6}{\Gamma\left(3\alpha\right)} x^{3\alpha} + \frac{2t}{\Gamma\left(2\alpha\right)} x^{2\alpha-1} + \frac{2}{\Gamma\left(2+\alpha\right)} x^{2\alpha} + \dots \qquad (24)$$

which is exactly the same as the one that was obtained in [14] using the Adomian's decomposition method. Due to self canceling "noise" terms appearance, by replacing $\alpha = 2$ in Eq. (24), yields the exact solution:

$$u\left(x,\,t\right) = x^2 + t^2$$

Example 2. Consider the following nonlinear time-fractional hyperbolic equation [15, 16]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right), \qquad \left(t>0,\ x\in R,\ 1<\alpha\leq 2\right), \qquad (25)$$

with the initial conditions

$$u\left(x,\,0\right) = x^2,\ u_t\left(x,\,0\right) = -2x^2.$$

The exact solution is $u(x, t) = \left(\frac{x}{t+1}\right)^2$.

The correction functional is read as

$$u_{n+1} = u_n + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda(\xi) \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha}\right. \tag{26}$$

$$\left. - u_n \frac{\partial^2 u_n}{\partial x^2} - \left(\frac{\partial u_n}{\partial x}\right)^2\right)(d\xi)^\alpha$$

Making the correction functional stationary, the general Lagrange multiplier can be identified as $\lambda(\xi) = -1$. Substituting value of the general Lagrange multiplier in the Eq. (26), we get the following iteration formula

$$u_{n+1} = u_n - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha}\right. \tag{27}$$

$$\left. - u_n \frac{\partial^2 u_n}{\partial x^2} - \left(\frac{\partial u_n}{\partial x}\right)^2\right)(d\xi)^\alpha.$$

The initial approximation is given as

$$u_0(x, t) = x^2(1 - 2t),$$

The other approximations are

$$u_1(x, t) = u_0 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha u_0}{\partial \xi^\alpha} - u_0 \frac{\partial^2 u_0}{\partial x^2} - \left(\frac{\partial u_0}{\partial x}\right)^2\right)(d\xi)^\alpha$$

$$= \frac{6x^2}{\Gamma(1+\alpha)}\left(t^\alpha - \frac{4\Gamma(1+\alpha)}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{8\Gamma(1+\alpha)}{\Gamma(3+\alpha)}t^{2+\alpha}\right),$$

$$\vdots$$

Consequently, we have the following solution of Eq. (25) in a series form

$$u(x, t) = x^2(1 - 2t) + \frac{6x^2}{\Gamma(1+\alpha)}\left(t^\alpha\right. \tag{28}$$

$$\left. - \frac{4\Gamma(1+\alpha)}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{8\Gamma(1+\alpha)}{\Gamma(3+\alpha)}t^{2+\alpha}\right) + \dots$$

Example 3. Consider the nonlinear time fractional Fokker - Plank equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left(-\frac{\partial}{\partial x}\left(\frac{4u}{x}-\frac{x}{3}\right)+\frac{\partial^2 u}{\partial x^2}\right)u\left(x,t\right), \quad (t>0, x \in R, 0 < \alpha \le 1),$$

(29)

subject to the initial condition

$$u\left(x,0\right) = x^2.$$

The correction functional is read as

$$u_{n+1} = u_n + \frac{1}{\Gamma\left(1+\alpha\right)}\int_0^t \lambda\left(\xi\right)\left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha}\right.$$

(30)

$$\left.+\frac{\partial}{\partial x}\left(\left(\frac{4u_n}{x}-\frac{x}{3}\right)u_n\right)-u_n\frac{\partial^2 u_n}{\partial x^2}\right)\left(d\xi\right)^\alpha,$$

where, $\lambda\left(\xi\right) = -1$ can be determined optimally via variational theory. The iteration formula is

$$u_{n+1} = u_n - \frac{1}{\Gamma\left(1+\alpha\right)}\int_0^t \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha}\right.$$

(31)

$$\left.+\frac{\partial}{\partial x}\left(\left(\frac{4u_n}{x}-\frac{x}{3}\right)u_n\right)-u_n\frac{\partial^2 u_n}{\partial x^2}\right)\left(d\xi\right)^\alpha,$$

The initial approximation is given as

$$u_0\left(x,t\right) = x^2$$

$$u_1 = u_0 - \frac{1}{\Gamma\left(1+\alpha\right)}\int_0^t \left(\frac{\partial^\alpha u_0}{\partial \xi^\alpha}+\frac{\partial}{\partial x}\left(\left(\frac{4u_0}{x}-\frac{x}{3}\right)u_0\right)-u_0\frac{\partial^2 u_0}{\partial x^2}\right)\left(d\xi\right)^\alpha$$

$$= \frac{x^2}{\Gamma\left(1+\alpha\right)}t^\alpha,$$

$$u_2 = u_1 - \frac{1}{\Gamma\left(1+\alpha\right)}\int_0^t \left(\frac{\partial^\alpha u_1}{\partial \xi^\alpha}+\frac{\partial}{\partial x}\left(\left(\frac{4u_1}{x}-\frac{x}{3}\right)u_1\right)-\frac{\partial^2}{\partial x^2}\left(u_1^2\right)\right)\left(d\xi\right)^\alpha$$

$$= \frac{x^2}{\Gamma\left(1+2\alpha\right)}t^{2\alpha},$$

$$u_3 = u_2 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha u_2}{\partial \xi^\alpha} + \frac{\partial}{\partial x} \left(\left(\frac{4u_2}{x} - \frac{x}{3} \right) u_2 \right) - u_2 \frac{\partial^2 u_2}{\partial x^2} \right) (d\xi)^\alpha$$

$$= \frac{x^2}{\Gamma(1+3\alpha)} t^{3\alpha},$$

$$\vdots$$

The series solution is given by

$$u(x,t) = x^2 \left(1 + \frac{1}{\Gamma(1+\alpha)} t^\alpha + \frac{1}{\Gamma(1+2\alpha)} t^{2\alpha} \right.$$
$$\left. + \frac{1}{\Gamma(1+3\alpha)} t^{3\alpha} \dots \right).$$
$$= \sum_{k=0}^\infty \frac{x^2}{\Gamma(1+k\alpha)} t^{k\alpha}$$
$$= x^2 E_\alpha(t^\alpha). \tag{32}$$

which is the exact solution of the nonlinear time fractional PDE obtained by Odibat and Momani [17].

Example 4. Consider the following nonlinear time-fractional Fornberg-Whitham equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} = u \frac{\partial^3 u}{\partial x^3} - u \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial x} \frac{\partial^3 u}{\partial x^3}, \quad (t > 0, x \in R, 0 < \alpha \le 1), \tag{33}$$

with initial condition as

$$u(x, 0) = \frac{4}{3} e^{\frac{1}{2} x}.$$

The exact travelling wave solution to the above initial value problem is given by [18]

$$u(x, t) = \frac{4}{3} e^{\frac{1}{2} x - \frac{2}{3} t}.$$

The correction functional is read as

$$u_{n+1} = u_n + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda(\xi) \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - \frac{\partial^3 u_n}{\partial x^2 \partial t} + \frac{\partial u_n}{\partial x} \right.$$
$$\left. - u_n \frac{\partial^3 u_n}{\partial x^3} + u_n \frac{\partial u_n}{\partial x} - 3 \frac{\partial u_n}{\partial x} \frac{\partial^3 u_n}{\partial x^3} \right) (d\xi)^\alpha, \tag{34}$$

Making the correction functional stationary, the general Lagrange multiplier can be identified as $\lambda(\xi) = -1$. Substituting value of the general Lagrange multiplier in the Eq. (34), we get the following iteration formula

$$
\begin{aligned}
u_{n+1} \;=\; u_n &- \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - \frac{\partial^3 u_n}{\partial x^2 \partial t} + \frac{\partial u_n}{\partial x} \right. \\
&- \left. u_n \frac{\partial^3 u_n}{\partial x^3} + u_n \frac{\partial u_n}{\partial x} - 3 \frac{\partial u_n}{\partial x} \frac{\partial^3 u_n}{\partial x^3} \right) (d\xi)^\alpha .
\end{aligned} \tag{35}
$$

We start with an initial approximation

$$
u_0(x, t) = \frac{4}{3} e^{\frac{1}{2} x}.
$$

Then, we can obtain the other components by means of

$$
\begin{aligned}
u_1 = u_0 &- \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha u_0}{\partial \xi^\alpha} - \frac{\partial^3 u_0}{\partial x^2 \partial t} \right. \\
&+ \frac{\partial u_0}{\partial x} - u_0 \frac{\partial^3 u_0}{\partial x^3} + u_0 \frac{\partial u_0}{\partial x} - 3 \frac{\partial u_0}{\partial x} \frac{\partial^3 u_0}{\partial x^3} \right) (d\xi)^\alpha \\
&= -\frac{2}{3\Gamma(1+\alpha)} e^{\frac{1}{2} x} t^\alpha,
\end{aligned}
$$

$$
\begin{aligned}
u_2 = u_1 &- \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^\alpha u_1}{\partial \xi^\alpha} - \frac{\partial^3 u_1}{\partial x^2 \partial t} \right. \\
&+ \frac{\partial u_1}{\partial x} - u_1 \frac{\partial^3 u_1}{\partial x^3} + u_1 \frac{\partial u_1}{\partial x} - 3 \frac{\partial u_1}{\partial x} \frac{\partial^3 u_1}{\partial x^3} \right) (d\xi)^\alpha \\
&= \frac{1}{6} e^{\frac{1}{2} x} \left(\frac{2}{\Gamma(1+\alpha)} t^{2\alpha} - \frac{1}{\Gamma(2\alpha)} t^{2\alpha-1} \right),
\end{aligned}
$$

$$
\begin{aligned}
u_3 = \frac{1}{72} e^{\frac{1}{2} x} &\left(\frac{12}{\Gamma(3\alpha)} t^{3\alpha-1} - \frac{3}{\Gamma(3\alpha-1)} t^{3\alpha-2} \right. \\
&- \frac{12}{\Gamma(3\alpha+1)} t^{3\alpha} \right),
\end{aligned}
$$

and so on, in the same manner the remaining components can be obtained.

The series solution is

$$
\begin{aligned}
u\left(x,t\right) = \frac{1}{72} e^{\frac{1}{2}x} \bigg(& 96 - \frac{48}{\Gamma\left(1+\alpha\right)} t^{\alpha} + \frac{24}{\Gamma\left(1+2\alpha\right)} t^{2\alpha} \\
& - \frac{12}{\Gamma\left(2\alpha\right)} t^{2\alpha-1} - \frac{12}{\Gamma\left(3\alpha\right)} t^{3\alpha-1} \\
& - \frac{3}{\Gamma\left(3\alpha-1\right)} t^{3\alpha-2} - \frac{12}{\Gamma\left(3\alpha+1\right)} t^{3\alpha} \dots \bigg).
\end{aligned}
\tag{36}
$$

5 Conclusion

In this study, Variational iteration method with new approach has been successfully employed to obtain exact and approximate analytical solutions of nonlinear fractional Fokker-Plank and Fornberg-Whitham equations. The method has been used in a direct way without linearization, perturbation or any restrictive assumption.

References

[1] K. B. Oldham and J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York, 1974.

[2] K. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[3] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[4] S. Momani, Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using decomposition method, Appl. Math. Comput. 165 (2005) 459-472.

[5] A. Al-rabtah, V. S. Eturk and S. Momani, Solution of fractional oscillatory by using differential transform method, Comput. Math. Appl. 59(2010) 1356-1362.

[6] S. Momani and Z. Odibat, Homotopy pertubation method for nonlinear partial differential equations of fractional order, Phys. Lett. A 365 (2007) 345-350.

[7] Z. Odibat and S. Momani, The Variational iteration method: an efficient scheme for handeling fractional partial differential equations in fluid mechanics, Comput. Math. Appl. 58 (2009) 2199-2208.

[8] M. Inc, The Approximate and exact solutions space and time fractional Bergers equations with initial conditions by variational iteration method, J. Math. Appl. 345 (1) (2008) 476-484.

[9] M. Safari, D. D. Ganji and M. Moslemi, Application of He's variational iterational method and Adomian's decomposition method to fractional KdV-Bergers- Kuramoto equation, Comput. Math. Appl. 58 (2009) 2091-2097.

[10] F. Abidi and K. Omrani, The Homotopy analysis method for solving the Fornberg-Whitham equation and comparison with Adomian's decomposition method, Comput. Math. Appl. 59 (2010) 2743-2750.

[11] P. K. Gupta and M. Singh, Homotopy perturbation method for fractional Fornberg-Whitham Equation, Comput. Math. Appl. 61 (2011) 250-254.

[12] J. Lu, An analytical approach to the Fornberg-Whitham type equation by using vational iterational method, Comput. Math. Appl. 61 (2011) 2010-2013.

[13] J. H. He, Approximate analytical solution for seepage flow with fractional derivative in porous media, Comput. Methods Appl. Mech. Eng.167 (1998) 57-68.

[14] S. Momani and Z. Odibat, Numerical solutions of space –time fractional advection-dispersion equation, Numer. Methods Partial Differential equations 24 (2008) 1416-1429.

[15] Z. Obidat and S. Momani, Numerical methods for nonlinear partial differential equations of fractional order, Appl. Math. Model.32 (2008) 28 - 39.

[16] S. Momani and Z. Obidat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Phys. Lett. A 365 (2007) 345 - 350.

[17] Z. Odibat and S. Momani, Numerical solution of Fokker-Plank equation with space and time- fractional derivative, Phys. Lett. A 369 (2007) 349-358.

[18] B. Fornberg and G. B. Whitham, A numerical and theoretical study of certain nonlinear wave phenomena, Phil. Trans. R. Soc. Lond. A v289 (1978) 373 - 404.

[19] Q. M. Hassan, M. Jamshed, M. Shakeel and S. T. Mohyud-Din, Homotopy analytical method using modified Riemann-Liouville derivative for solving space and time fractional KdV equations, Int. J. Mod. Math. Sci., 5 (1) (2013), 14-26.

O linearnim i nelinearnim frakcionim parcijalnim diferencijalnim jednačinama

Varijacioni iteracioni metod (VIM) je primenjen za dobijanje analitičkih rešenja nelinearnih frakcionih parcijalnih diferencijalnih jednačina. Iteracioni postupak se zasniva na relativno novom pristupu koji se zove Jumarie-ov frakcioni izvod. Nekoliko primera su rešeni u cilju razjašnjenja efikasnosti predloženog metoda i rezultati su uporedjeni sa tačnim rešenjima, otkrivajuci visoku tačnost i efikasnost metoda.