STROH-LIKE FORMALISM FOR KIRCHHOFF ANISOTROPIC THERMOELASTIC PLATES

Xu Wang   Peter Schiavone

Stroh-like formalism for Kirchhoff anisotropic thermoelastic plates

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Abstract

A Stroh-like formalism is developed for the heat conduction and the coupled stretching and bending deformations of a laminated anisotropic thermoelastic thin plate based on Kirchhoff theory. For the heat conduction problem, a Stroh-like quartic formalism is developed. Two-dimensional generalized temperature and heat flux function vectors are introduced. The structure of the introduced $4{\times}4$ fundamental plate matrix for heat conduction is the same as that of the $8{\times}8$ fundamental elasticity matrix in the Stroh sextic formalism for generalized plane strain elasticity. Consequently, the orthogonality and closure relations for heat conduction in thin plates is established. For the thermoelastic problem, an inhomogeneous particular solution is derived rigorously. We obtain an octet formalism in which the general solution is composed of the well-known homogeneous solution developed by Cheng and Reddy (isothermal case) and the inhomogeneous particular solution arising from the thermal effect.

Keywords: Anisotropic thermoelastic plate; Kirchhoff theory; Quartic formalism; Octet formalism; Heat conduction; Thermoelasticity

1 Introduction

Laminated composite structures have found extensive engineering applications (for example, in aerospace, underwater structures and microelectronic

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devices) and have attracted many researchers' interest (see, for example, [1]). In terms of classical two-dimensional mathematical models for the deformation of thin plates, the Kirchhoff model for bending is well-established as are subsequent refinements of this theory [1-3]. In a laminated plate, however, the stretching and bending deformations are intrinsically coupled. Recently Cheng and Reddy [4-7] developed a new Stroh-like octet formalism to analyze the coupled stretching and bending deformations of Kirchhoff laminated anisotropic thin plates. Amazingly, this new formalism preserves almost all of the beautiful properties and identities in the Stroh sextic formalism for generalized plane strain elasticity [8]. The Stroh octet formalism established by Cheng and Reddy is confined to the isothermal case.

In this study, we try to incorporate thermal effects in the Kirchhoff model of laminated anisotropic plates. More specifically, we will extend the Stroh octet formalism to thermo-anisotropic thin elastic plates. To begin with, we develop a Stroh-like quartic formalism for the analysis of steady-state heat conduction of laminated anisotropic plates. Two-dimensional generalized temperature and heat flux function vectors are introduced. The heat flux resultants and flux moments can be determined by differentiation of the introduced two heat flux functions. The structure of the introduced fundamental plate matrix for heat conduction is the same as that of the fundamental elasticity matrix [8]. Thus the orthogonality and closure relations and several further identities can be directly derived. Secondly, we extend the Stroh octet formalism for Kirchhoff anisotropic plates by adding the inhomogeneous particular solution (due to thermal effects) to the general solution. The present octet formalism for a thermo-anisotropic elastic thin plate presents a counterpart of the Stroh sextic formalism for thermo-anisotropic elasticity in which the thickness of the elastic solid approaches infinity [8]. Finally, similarities and differences between the two formalisms are discussed.

2   The laminated anisotropic thin plate

Consider an undeformed plate of uniform thickness h in a Cartesian coordinate system (x_1, x_2, x_3) in which the mid-plane of the plate is taken to be $x_3=0$. The thickness h is much smaller than other typical in-plane dimensions of the plate. The plate is composed of an anisotropic, linear thermoelastic material that can be inhomogeneous and laminated in the thickness direction. As per the convention adopted by Cheng and Reddy [4-7], a repeated index implies, unless otherwise specified, summation over the range of the

index with Greek indices ranging from 1 to 2, lowercase Latin indices from 1 to 3, and uppercase Latin indices ranging from 1 to 4.

In the following two sections, we will first develop in Section 3 a Stroh-like quartic formalism for steady-state heat conduction in a laminated anisotropic thin plate, then in Section 4, we derive a Stroh octet formalism for a laminated thermo-anisotropic elastic thin plate.

3 Quartic formalism for heat conduction in anisotropic plates

The heat conduction in an anisotropic material is given characterized by

$$h_i = -k_{ij}T_{,j}, \tag{1}$$

where h_i and T are respectively the heat fluxes and temperature and k_{ij} are the heat conduction coefficients. In this study the Onsager's [9] reciprocal relations are invoked so that $k_{ij} = k_{ji}$.

If we assume that $h_3 = 0$ in the thin plate (this assumption is valid when the top and bottom surfaces of the plate are insulating), we will obtain from Eq.(1) that

$$T_{,3} = -\frac{k_{3\beta}}{k_{33}}T_{,\beta}. \tag{2}$$

Substituting the above into Eq.(1), we arrive at

$$h_\alpha = -\tilde{k}_{\alpha\beta}T_{,\beta}, \tag{3}$$

where $\tilde{k}_{\alpha\beta}$ are the reduced heat conduction coefficients and are given by

$$\tilde{k}_{\alpha\beta} = k_{\alpha\beta} - \frac{k_{\alpha3}k_{3\alpha}}{k_{33}}. \tag{4}$$

Since the balance of energy is given by

$$h_{i,i} = 0, \tag{5}$$

and h_3 is ignored in the plate, we will have

$$H_{\alpha,\alpha} = 0, \; P_{\alpha,\alpha} = 0, \tag{6}$$

where H_α and P_α are the flux resultants and flux moments defined by

$$H_\alpha = Qh_\alpha, \; P_\alpha = Qx_3h_\alpha, \tag{7}$$

with

$$Q = \int_{-h/2}^{h/2} (\cdots) dx_3. \tag{8}$$

The temperature field in the thin plate is assumed to take the following form

$$T = T_1 + x_3 T_2, \tag{9}$$

where T_1 and T_2 are independent of x_3. Equation (9) can be considered as the heat conduction version of the Kirchhoff assumption.

We further assume that T_1 and T_2 take the following forms

$$T_1 = e_1 g'(z), \ T_2 = e_2 g'(z), \tag{10}$$

where $g(z)$ is an arbitrary analytic function of $z = x_1 + \lambda x_2$, λ and e_α are unknown constants to be determined.

The in-plane heat fluxes can then be calculated from Eq. (3) as

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = - \begin{bmatrix} \tilde{k}_{11} + \lambda \tilde{k}_{12} & x_3(\tilde{k}_{11} + \lambda \tilde{k}_{12}) \\ \tilde{k}_{12} + \lambda \tilde{k}_{22} & x_3(\tilde{k}_{12} + \lambda \tilde{k}_{22}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} g''(z). \tag{11}$$

The flux resultants and flux moments can be determined as

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = - \begin{bmatrix} Q(\tilde{k}_{11} + \lambda \tilde{k}_{12}) & Qx_3(\tilde{k}_{11} + \lambda \tilde{k}_{12}) \\ Q(\tilde{k}_{12} + \lambda \tilde{k}_{22}) & Qx_3(\tilde{k}_{12} + \lambda \tilde{k}_{22}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} g''(z), \tag{12}$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = - \begin{bmatrix} Qx_3(\tilde{k}_{11} + \lambda \tilde{k}_{12}) & Qx_3^2(\tilde{k}_{11} + \lambda \tilde{k}_{12}) \\ Qx_3(\tilde{k}_{12} + \lambda \tilde{k}_{22}) & Qx_3^2(\tilde{k}_{12} + \lambda \tilde{k}_{22}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} g''(z). \tag{13}$$

The balance of energy in terms of flux resultants and flux moments can now be expressed into

$$\begin{bmatrix} Q(\tilde{k}_{11} + 2\lambda \tilde{k}_{12} + \lambda^2 \tilde{k}_{22}) & Qx_3(\tilde{k}_{11} + 2\lambda \tilde{k}_{12} + \lambda^2 \tilde{k}_{22}) \\ Qx_3(\tilde{k}_{11} + 2\lambda \tilde{k}_{12} + \lambda^2 \tilde{k}_{22}) & Qx_3^2(\tilde{k}_{11} + 2\lambda \tilde{k}_{12} + \lambda^2 \tilde{k}_{22}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{14}$$

which comprises two equations for the determination of the unknowns λ and e_α. For a non-trivial solution of $\begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$ we must have

$$\det \begin{bmatrix} Q(\tilde{k}_{11} + 2\lambda \tilde{k}_{12} + \lambda^2 \tilde{k}_{22}) & Qx_3(\tilde{k}_{11} + 2\lambda \tilde{k}_{12} + \lambda^2 \tilde{k}_{22}) \\ Qx_3(\tilde{k}_{11} + 2\lambda \tilde{k}_{12} + \lambda^2 \tilde{k}_{22}) & Qx_3^2(\tilde{k}_{11} + 2\lambda \tilde{k}_{12} + \lambda^2 \tilde{k}_{22}) \end{bmatrix} = 0, \tag{15}$$

which is a polynomial of degree 4 in λ and gives four roots for the eigenvalue λ. The associated eigenvector $\begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$ is determined from Eq.(14). It can be proved that Eq.(15) provides two pairs of complex conjugates of λ (see the appendix for a rigorous proof).

Equation (14) will be automatically satisfied by introducing ξ_1 and ξ_2 such that

$$\begin{bmatrix} -\lambda \xi_1 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} Q(\tilde{k}_{11} + \lambda \tilde{k}_{12}) & Qx_3(\tilde{k}_{11} + \lambda \tilde{k}_{12}) \\ Q(\tilde{k}_{12} + \lambda \tilde{k}_{22}) & Qx_3(\tilde{k}_{12} + \lambda \tilde{k}_{22}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (16)$$

$$\begin{bmatrix} -\lambda \xi_2 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} Qx_3(\tilde{k}_{11} + \lambda \tilde{k}_{12}) & Qx_3^2(\tilde{k}_{11} + \lambda \tilde{k}_{12}) \\ Qx_3(\tilde{k}_{12} + \lambda \tilde{k}_{22}) & Qx_3^2(\tilde{k}_{12} + \lambda \tilde{k}_{22}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \quad (17)$$

Using Eqs.(16) and (17), we can rewrite Eqs.(12) and (13) as

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} \lambda \xi_1 \\ -\xi_1 \end{bmatrix} g''(z), \quad \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \lambda \xi_2 \\ -\xi_2 \end{bmatrix} g''(z). \quad (18)$$

If we introduce heat flux functions

$$\Theta_1 = \xi_1 g'(z), \quad \Theta_2 = \xi_2 g'(z), \quad (19)$$

Eq.(18) is equivalent to

$$H_\alpha = \in_{\alpha\beta} \Theta_{1,\beta}, \quad P_\alpha = \in_{\alpha\beta} \Theta_{2,\beta}, \quad (20)$$

where $\in_{\alpha\beta}$ are the components of the two-dimensional permutation tensor.

Equations (10) and (19) can then be written as

$$\mathbf{T} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}^T = \mathbf{e} g'(z), \quad \mathbf{\Theta} = \begin{bmatrix} \Theta_1 & \Theta_2 \end{bmatrix}^T = \xi g'(z), \quad (21)$$

where

$$\mathbf{e} = \begin{bmatrix} e_1 & e_2 \end{bmatrix}^T, \quad \xi = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T. \quad (22)$$

T and $\mathbf{\Theta}$ are, respectively, the generalized temperature and heat flux function vectors.

Equations (16) and (17) can be rewritten as

$$\begin{bmatrix} -\hat{\mathbf{Q}} & \mathbf{0} \\ -\hat{\mathbf{R}}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix} = \lambda \begin{bmatrix} \hat{\mathbf{R}} & \mathbf{I} \\ \hat{\mathbf{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix}, \quad (23)$$

where $\hat{\mathbf{Q}}$, $\hat{\mathbf{R}}$ and $\hat{\mathbf{T}}$ are three 2×2 symmetric matrices defined by

$$\hat{\mathbf{Q}} = \begin{bmatrix} Q\tilde{k}_{11} & Qx_3\tilde{k}_{11} \\ Qx_3\tilde{k}_{11} & Qx_3^2\tilde{k}_{11} \end{bmatrix}, \ \hat{\mathbf{R}} = \begin{bmatrix} Q\tilde{k}_{12} & Qx_3\tilde{k}_{12} \\ Qx_3\tilde{k}_{12} & Qx_3^2\tilde{k}_{12} \end{bmatrix}, \qquad (24)$$

$$\hat{\mathbf{T}} = \begin{bmatrix} Qx_3\tilde{k}_{22} & Qx_3\tilde{k}_{22} \\ Qx_3\tilde{k}_{22} & Qx_3^2\tilde{k}_{22} \end{bmatrix}.$$

In addition $\hat{\mathbf{Q}}$ and $\hat{\mathbf{T}}$ are positive definite (see the appendix for proof). Equation (23) can be further transformed into the following standard eigenrelation:

$$\hat{\mathbf{N}} \begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix}, \qquad (25)$$

where

$$\hat{\mathbf{N}} = \mathbf{Y}^{-1}\mathbf{X}, \qquad (26)$$

with

$$\mathbf{X} = \begin{bmatrix} -\hat{\mathbf{Q}} & \mathbf{0} \\ -\hat{\mathbf{R}}^T & \mathbf{I} \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} \hat{\mathbf{R}} & \mathbf{I} \\ \hat{\mathbf{T}} & \mathbf{0} \end{bmatrix}. \qquad (27)$$

Here we call $\hat{\mathbf{N}}$ the fundamental plate matrix for heat conduction. Since $\mathbf{X}\hat{\mathbf{I}}\mathbf{Y}^T = \begin{bmatrix} -\hat{\mathbf{Q}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{T}} \end{bmatrix}$ with $\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$ is symmetric, we can conclude that

$$\hat{\mathbf{I}}\hat{\mathbf{N}} = (\hat{\mathbf{I}}\hat{\mathbf{N}})^T = \hat{\mathbf{N}}^T\hat{\mathbf{I}}, \qquad (28)$$

which implies that $\hat{\mathbf{I}}\hat{\mathbf{N}}$ is symmetric. Consequently $\hat{\mathbf{N}}$ must take the following form

$$\hat{\mathbf{N}} = \begin{bmatrix} \hat{\mathbf{N}}_1 & \hat{\mathbf{N}}_2 \\ \hat{\mathbf{N}}_3 & \hat{\mathbf{N}}_1^T \end{bmatrix}, \qquad (29)$$

where $\hat{\mathbf{N}}_1$, $\hat{\mathbf{N}}_2 = \hat{\mathbf{N}}_2^T$ and $\hat{\mathbf{N}}_3 = \hat{\mathbf{N}}_3^T$ are three 2×2 matrices. It is found that the standard eigenrelation (25) together with (29) is symbolically the same as that of the Stroh sextic formalism for generalized plane strain [8] and that of the Stroh octet formalism for a Kirchhoff anisotropic thin plate [4]. Equations (23)-(29) are presented to show a connection between the present quartic formalism for heat conduction in a laminated anisotropic thin plate and the octet formalism for the coupled stretching and bending deformations of a laminated anisotropic thin plate under the isothermal assumption (Cheng

and Reddy, 2002). In fact a simple algebraic manipulation of Eq. (23) leads
to

$$\hat{\mathbf{N}}_1 = -\hat{\mathbf{T}}^{-1}\hat{\mathbf{R}}^T, \ \hat{\mathbf{N}}_2 = \hat{\mathbf{T}}^{-1}, \ \hat{\mathbf{N}}_3 = \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1}\hat{\mathbf{R}}^T - \hat{\mathbf{Q}}. \tag{30}$$

Thus the structure of the fundamental plate matrix for heat conduction
is exactly the same as that of the fundamental elasticity matrix [8].

Equation (15) provides two pairs of complex conjugates of λ. The associated eigenvectors are also two pairs of complex conjugates. Thus the general
solution of the generalized temperature and heat flux function vectors are
given by

$$\mathbf{T} = \mathbf{E}\mathbf{g}'(z) + \bar{\mathbf{E}}\overline{\mathbf{g}'(z)}, \ \mathbf{\Theta} = \mathbf{\Xi}\mathbf{g}'(z) + \bar{\mathbf{\Xi}}\overline{\mathbf{g}'(z)}, \tag{31}$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}, \ \mathbf{\Xi} = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix},$$
$$\mathbf{g}'(z) = \begin{bmatrix} g_1'(Z_1) & g_2'(Z_2) \end{bmatrix}^T, \tag{32}$$
$$Z_\alpha = x_1 + \lambda_\alpha x_2, \ Im\{\lambda_\alpha\} > 0, \ (\alpha = 1, 2),$$

with

$$\begin{bmatrix} \hat{\mathbf{N}}_1 & \hat{\mathbf{N}}_2 \\ \hat{\mathbf{N}}_3 & \hat{\mathbf{N}}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{e}_\alpha \\ \xi_\alpha \end{bmatrix} = \lambda_\alpha \begin{bmatrix} \mathbf{e}_\alpha \\ \xi_\alpha \end{bmatrix}, \ (\alpha = 1, 2). \tag{33}$$

The obtained general solution (31) is valid when $\lambda_1 \neq \lambda_2$. Following [4],
[8], the following orthogonality relations establish, in view of the structure of
$\hat{\mathbf{N}}$ in Eq. (29) or more specifically in Eq. (30):

$$\begin{bmatrix} \mathbf{\Xi}^T & \mathbf{E}^T \\ \bar{\mathbf{\Xi}}^T & \bar{\mathbf{E}}^T \end{bmatrix} \begin{bmatrix} \mathbf{E} & \bar{\mathbf{E}} \\ \mathbf{\Xi} & \bar{\mathbf{\Xi}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{34}$$

Consequently, three 2×2 real matrices $\hat{\mathbf{S}}$, $\hat{\mathbf{H}}$ and $\hat{\mathbf{L}}$ can be introduced:

$$\hat{\mathbf{S}} = i(2\mathbf{E}\mathbf{\Xi}^T - \mathbf{I}), \ \hat{\mathbf{H}} = 2i\mathbf{E}\mathbf{E}^T, \ \hat{\mathbf{L}} = -2i\mathbf{\Xi}\mathbf{\Xi}^T, \tag{35}$$

which is a result of the closure relations. More identities can be easily derived
[4], [8].

4 Octet formalism for thermo-anisotropic elastic plates

The constitutive law for an anisotropic material incorporating the thermal
effect can be written into:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \gamma_{ij}T, \tag{36}$$

where σ_{ij} and $\varepsilon_{ij} = \frac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i})$ are the stresses and strains with $\tilde{u}_i$ the displacements, C_{ijkl} and γ_{ij} are, respectively, the elastic stiffnesses and thermal moduli.

The following fundamental assumptions that the transverse shear deformation and transverse normal stress component are neglected are fundamental in Kirchhoff thin plate theory

$$\varepsilon_{\alpha 3} = 0, \ \sigma_{33} = 0. \tag{37}$$

By utilizing the assumptions in Eq.(37), we can obtain ε_{33} from Eq.(36) as follows

$$\varepsilon_{33} = -\frac{C_{33\omega\rho}}{C_{3333}}\varepsilon_{\omega\rho} + \frac{\gamma_{33}}{C_{3333}}T. \tag{38}$$

Substituting the above expression of ε_{33} and the assumption that the temperature field is of the form $T = T_1 + x_3 T_2$ into Eq.(36) will yield

$$\sigma_{\alpha\beta} = \tilde{C}_{\alpha\beta\omega\rho}\varepsilon_{\omega\rho} - \tilde{\gamma}_{\alpha\beta}T_1 - x_3\tilde{\gamma}_{\alpha\beta}T_2, \tag{39}$$

where $\tilde{C}_{\alpha\beta\omega\rho}$ and $\tilde{\gamma}_{\alpha\beta}$ are, respectively, the reduced elastic stiffnesses and reduced thermal moduli, given by

$$\tilde{C}_{\alpha\beta\omega\rho} = C_{\alpha\beta\omega\rho} - C_{\alpha\beta 33}C_{33\omega\rho}/C_{3333}, \ \tilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} - C_{\alpha\beta 33}\gamma_{33}/C_{3333}. \tag{40}$$

In the absence of external loads on the upper and lower faces of the plate, the equilibrium equations are given by

$$N_{\alpha\beta,\beta} = 0, \ R_{\beta,\beta} = 0, \tag{41}$$

where the membrane stress resultants $N_{\alpha\beta}$, bending moments $M_{\alpha\beta}$ and transverse shearing forces R_β are defined by

$$N_{\alpha\beta} = Q\sigma_{\alpha\beta}, \ M_{\alpha\beta} = Qx_3\sigma_{\alpha\beta}, \ R_\beta = M_{\alpha\beta,\alpha}. \tag{42}$$

The modified Kirchhoff transverse shearing forces applied to free edges are defined by

$$V_1 = R_1 + M_{12,2}, \ V_2 = R_2 + M_{21,1}. \tag{43}$$

In Kirchhoff plate theory, the displacements are assumed to take the following forms

$$\tilde{u}_1 = u_1 + x_3\vartheta_1, \ \tilde{u}_2 = u_2 + x_3\vartheta_2, \ \tilde{u}_3 = w, \tag{44}$$

where the in-plane displacements u_1 and u_2, deflection w and slopes $\vartheta_1 = -w_{,1}$ and $\vartheta_2 = -w_{,2}$ on the mid-plane are independent of x_3.

The homogeneous solution in the absence of thermal effects, has been derived by [4]. In the following, we focus on the derivation of the inhomogeneous particular solution arising from the thermal effect.

The displacements on the mid-plane of the plate take the following forms

$$u_1 = c_1 g(z), \ u_2 = c_2 g(z), \ w = -c_3 \int g(z) \mathrm{d}z, \tag{45}$$

where c_i are unknown constants to be determined, and $z = x_1 + \lambda x_2$ with λ the eigenvalue for heat conduction.

The stresses can be calculated from Eqs.(39) and (45) as

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix}
=
\begin{bmatrix} \tilde{\mathbf{Q}} + \lambda \tilde{\mathbf{R}} & x_3(\tilde{\mathbf{Q}} + \lambda \tilde{\mathbf{R}}) \\ \tilde{\mathbf{R}}^T + \lambda \tilde{\mathbf{T}} & x_3(\tilde{\mathbf{R}}^T + \lambda \tilde{\mathbf{T}}) \end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \lambda c_3 \end{bmatrix} g'(z) \tag{46}
$$

$$
- \begin{bmatrix} \tilde{\gamma}_{11} & x_3\tilde{\gamma}_{11} \\ \tilde{\gamma}_{21} & x_3\tilde{\gamma}_{21} \\ \tilde{\gamma}_{12} & x_3\tilde{\gamma}_{12} \\ \tilde{\gamma}_{22} & x_3\tilde{\gamma}_{22} \end{bmatrix}
\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} g'(z),
$$

where the components of the three 2×2 matrices $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{T}}$ are given by

$$\tilde{Q}_{\alpha\omega} = \tilde{C}_{\alpha1\omega1}, \ \tilde{R}_{\alpha\omega} = \tilde{C}_{\alpha1\omega2}, \ \tilde{T}_{\alpha\omega} = \tilde{C}_{\alpha2\omega2}. \tag{47}$$

In addition $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{T}}$ are symmetric and positive definite.

The membrane stress resultants, bending moments and transverse shearing forces can then be obtained as

$$
\begin{bmatrix} N_{11} \\ N_{21} \\ N_{12} \\ N_{22} \end{bmatrix}
=
\begin{bmatrix} Q(\tilde{\mathbf{Q}} + \lambda \tilde{\mathbf{R}}) & Qx_3(\tilde{\mathbf{Q}} + \lambda \tilde{\mathbf{R}}) \\ Q(\tilde{\mathbf{R}}^T + \lambda \tilde{\mathbf{T}}) & Qx_3(\tilde{\mathbf{R}}^T + \lambda \tilde{\mathbf{T}}) \end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \lambda c_3 \end{bmatrix} g'(z) \tag{48}
$$

$$
- \begin{bmatrix} Q\tilde{\gamma}_{11} & Qx_3\tilde{\gamma}_{11} \\ Q\tilde{\gamma}_{21} & Qx_3\tilde{\gamma}_{21} \\ Q\tilde{\gamma}_{12} & Qx_3\tilde{\gamma}_{12} \\ Q\tilde{\gamma}_{22} & Qx_3\tilde{\gamma}_{22} \end{bmatrix}
\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} g'(z),
$$

$$
\begin{bmatrix} M_{11} \\ M_{21} \\ M_{12} \\ M_{22} \end{bmatrix} = \begin{bmatrix} Qx_3(\tilde{\mathbf{Q}} + \lambda\tilde{\mathbf{R}}) & Qx_3^2(\tilde{\mathbf{Q}} + \lambda\tilde{\mathbf{R}}) \\ Qx_3(\tilde{\mathbf{R}}^T + \lambda\tilde{\mathbf{T}}) & Qx_3^2(\tilde{\mathbf{R}}^T + \lambda\tilde{\mathbf{T}}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \lambda c_3 \end{bmatrix} g'(z) \quad (49)
$$

$$
- \begin{bmatrix} Qx_3\tilde{\gamma}_{11} & Qx_3^2\tilde{\gamma}_{11} \\ Qx_3\tilde{\gamma}_{21} & Qx_3^2\tilde{\gamma}_{21} \\ Qx_3\tilde{\gamma}_{12} & Qx_3^2\tilde{\gamma}_{12} \\ Qx_3\tilde{\gamma}_{22} & Qx_3^2\tilde{\gamma}_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} g'(z),
$$

$$
\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} \tilde{\lambda}^T Qx_3(\tilde{\mathbf{Q}} + \lambda\tilde{\mathbf{R}}) & \tilde{\lambda}^T Qx_3^2(\tilde{\mathbf{Q}} + \lambda\tilde{\mathbf{R}}) \\ \tilde{\lambda}^T Qx_3(\tilde{\mathbf{R}}^T + \lambda\tilde{\mathbf{T}}) & \tilde{\lambda}^T Qx_3^2(\tilde{\mathbf{R}}^T + \lambda\tilde{\mathbf{T}}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \lambda c_3 \end{bmatrix} g''(z)
$$

$$
- \begin{bmatrix} Qx_3\gamma^T\tilde{\lambda} & Qx_3^2\gamma^T\tilde{\lambda} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} g''(z), \qquad\qquad (50)
$$

where

$$
\tilde{\lambda} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \; \gamma = \gamma^T = \begin{bmatrix} \tilde{\gamma}_{11} & \tilde{\gamma}_{12} \\ \tilde{\gamma}_{21} & \tilde{\gamma}_{22} \end{bmatrix}. \qquad\qquad (51)
$$

Consequently, using Eqs.(48)-(50), the equilibrium equations (41) can be expressed as

$$
\begin{bmatrix} Q\mathbf{G} & Qx_3\mathbf{G}\tilde{\lambda} \\ \tilde{\lambda}^T Qx_3\mathbf{G} & \tilde{\lambda}^T Qx_3^2\mathbf{G}\tilde{\lambda} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} Q\gamma\tilde{\lambda} & Qx_3\gamma\tilde{\lambda} \\ \tilde{\lambda}^T Qx_3\gamma\tilde{\lambda} & \tilde{\lambda}^T Qx_3^2\gamma\tilde{\lambda} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},
$$
$$
(52)
$$

where $\mathbf{G}$ is a 2×2 symmetric matrix defined by

$$
\mathbf{G} = \tilde{\mathbf{Q}} + \lambda(\tilde{\mathbf{R}} + \tilde{\mathbf{R}}^T) + \lambda^2\tilde{\mathbf{T}}. \qquad\qquad (53)
$$

The first two equilibrium equations of (52) will be automatically satisfied

if we introduce $\widehat{\mathbf{d}} = \begin{bmatrix} d_1 & d_2 \end{bmatrix}^T$ such that

$$
\begin{bmatrix} -\lambda\widehat{\mathbf{d}} \\ \widehat{\mathbf{d}} \end{bmatrix} = \begin{bmatrix} Q(\tilde{\mathbf{Q}} + \lambda\tilde{\mathbf{R}}) & Qx_3(\tilde{\mathbf{Q}} + \lambda\tilde{\mathbf{R}}) \\ Q(\tilde{\mathbf{R}}^T + \lambda\tilde{\mathbf{T}}) & Qx_3(\tilde{\mathbf{R}}^T + \lambda\tilde{\mathbf{T}}) \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{c}} \\ \breve{\mathbf{c}} \end{bmatrix} - \begin{bmatrix} Q\tilde{\gamma}_{11} & Qx_3\tilde{\gamma}_{11} \\ Q\tilde{\gamma}_{21} & Qx_3\tilde{\gamma}_{21} \\ Q\tilde{\gamma}_{12} & Qx_3\tilde{\gamma}_{12} \\ Q\tilde{\gamma}_{22} & Qx_3\tilde{\gamma}_{22} \end{bmatrix} \mathbf{e},
$$

(54)

where $\mathbf{e}$ has been defined by Eq.(22), and

$$
\widehat{\mathbf{c}} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T, \ \breve{\mathbf{c}} = \begin{bmatrix} c_3 & c_4 \end{bmatrix}^T,
$$

(55)

with $c_4 = \lambda c_3$.

The third equilibrium equation of (52) will be automatically satisfied if we introduce $\breve{\mathbf{d}} = \begin{bmatrix} d_3 & d_4 \end{bmatrix}^T$ such that

$$
\begin{bmatrix} -\lambda\breve{\mathbf{d}} + \frac{1}{2}\begin{bmatrix} 0 \\ \tilde{\lambda}^T\breve{\mathbf{d}} \end{bmatrix} \\ \breve{\mathbf{d}} - \frac{1}{2}\begin{bmatrix} \tilde{\lambda}^T\breve{\mathbf{d}} \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} Qx_3(\tilde{\mathbf{Q}} + \lambda\tilde{\mathbf{R}}) & Qx_3^2(\tilde{\mathbf{Q}} + \lambda\tilde{\mathbf{R}}) \\ Qx_3(\tilde{\mathbf{R}}^T + \lambda\tilde{\mathbf{T}}) & Qx_3^2(\tilde{\mathbf{R}}^T + \lambda\tilde{\mathbf{T}}) \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{c}} \\ \breve{\mathbf{c}} \end{bmatrix}
$$

$$
- \begin{bmatrix} Qx_3\tilde{\gamma}_{11} & Qx_3^2\tilde{\gamma}_{11} \\ Qx_3\tilde{\gamma}_{21} & Qx_3^2\tilde{\gamma}_{21} \\ Qx_3\tilde{\gamma}_{12} & Qx_3^2\tilde{\gamma}_{12} \\ Qx_3\tilde{\gamma}_{22} & Qx_3^2\tilde{\gamma}_{22} \end{bmatrix} \mathbf{e}.
$$

(56)

By making use of Eqs.(54) and (56), Eqs.(48)-(50) can be rewritten as

$$
\begin{bmatrix} N_{11} \\ N_{21} \\ N_{12} \\ N_{22} \end{bmatrix} = \begin{bmatrix} -\lambda\widehat{\mathbf{d}} \\ \widehat{\mathbf{d}} \end{bmatrix} g'(z),
$$

(57)

$$
\begin{bmatrix} M_{11} \\ M_{21} \\ M_{12} \\ M_{22} \end{bmatrix} = \begin{bmatrix} -\lambda\breve{\mathbf{d}} + \frac{1}{2}\begin{bmatrix} 0 \\ \tilde{\lambda}^T\breve{\mathbf{d}} \end{bmatrix} \\ \breve{\mathbf{d}} - \frac{1}{2}\begin{bmatrix} \tilde{\lambda}^T\breve{\mathbf{d}} \\ 0 \end{bmatrix} \end{bmatrix} g'(z),
$$

(58)

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} -\lambda \tilde{\lambda}^T \breve{\mathbf{d}} \\ \tilde{\lambda}^T \breve{\mathbf{d}} \end{bmatrix} g''(z), \tag{59}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\lambda^2 d_4 \\ d_3 \end{bmatrix} g''(z). \tag{60}$$

If we introduce the stress functions

$$\phi_1 = d_1 g(z), \ \ \phi_2 = d_2 g(z), \ \ \psi_1 = d_3 g(z), \ \ \psi_2 = d_4 g(z), \tag{61}$$

then Eqs.(57)-(60) are equivalent to

$$N_{\alpha\beta} = - \in_{\beta\omega} \phi_{\alpha,\omega}, \ \ M_{\alpha\beta} = - \in_{\beta\omega} \psi_{\alpha,\omega} - \tfrac{1}{2} \in_{\alpha\beta} \psi_{\omega,\omega}, \\ R_\alpha = -\tfrac{1}{2} \in_{\alpha\beta} \psi_{\omega,\omega\beta}, \ \ V_\alpha = - \in_{\alpha\omega} \psi_{\omega,\omega\omega}. \tag{62}$$

Equations (45) and (61) can be written as

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \vartheta_1 & \vartheta_2 \end{bmatrix}^T = \mathbf{c}g(z), \ \ \boldsymbol{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \psi_1 & \psi_2 \end{bmatrix}^T = \mathbf{d}g(z), \tag{63}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}^T, \ \ \mathbf{d} = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \end{bmatrix}^T, \tag{64}$$

$$c_4 = \lambda c_3. \tag{65}$$

Equations (54) and (56) can be written as

$$\begin{bmatrix} -\mathbf{Q} & \tfrac{1}{2}\mathbf{I}_{43} \\ -\mathbf{R}^T & \mathbf{I} - \tfrac{1}{2}\mathbf{I}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{R} & \mathbf{I} - \tfrac{1}{2}\mathbf{I}_{44} \\ \mathbf{T} & \tfrac{1}{2}\mathbf{I}_{34} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} - \boldsymbol{\Omega}\mathbf{e}, \tag{66}$$

where

$$(\mathbf{I}_{33})_{KL} = \delta_{K3}\delta_{L3}, \ \ (\mathbf{I}_{34})_{KL} = \delta_{K3}\delta_{L4}, \ \ (\mathbf{I}_{43})_{KL} = \delta_{K4}\delta_{L3}, \ \ (\mathbf{I}_{44})_{KL} = \delta_{K4}\delta_{L4}, \tag{67}$$

with δ_{KL} the Kronecker delta, and

$$\mathbf{Q} = \begin{bmatrix} Q\tilde{\mathbf{Q}} & Qx_3\tilde{\mathbf{Q}} \\ Qx_3\tilde{\mathbf{Q}} & Qx_3^2\tilde{\mathbf{Q}} \end{bmatrix}, \ \ \mathbf{R} = \begin{bmatrix} Q\tilde{\mathbf{R}} & Qx_3\tilde{\mathbf{R}} \\ Qx_3\tilde{\mathbf{R}} & Qx_3^2\tilde{\mathbf{R}} \end{bmatrix}, \ \ \mathbf{T} = \begin{bmatrix} Q\tilde{\mathbf{T}} & Qx_3\tilde{\mathbf{T}} \\ Qx_3\tilde{\mathbf{T}} & Qx_3^2\tilde{\mathbf{T}} \end{bmatrix}, \tag{68}$$

$$\boldsymbol{\Omega}^T = \begin{bmatrix} Q\tilde{\gamma}_{11} & Q\tilde{\gamma}_{21} & Qx_3\tilde{\gamma}_{11} & Qx_3\tilde{\gamma}_{21} & Q\tilde{\gamma}_{12} & Q\tilde{\gamma}_{22} & Qx_3\tilde{\gamma}_{12} & Qx_3\tilde{\gamma}_{22} \\ Qx_3\tilde{\gamma}_{11} & Qx_3\tilde{\gamma}_{21} & Qx_3^2\tilde{\gamma}_{11} & Qx_3^2\tilde{\gamma}_{21} & Qx_3\tilde{\gamma}_{12} & Qx_3\tilde{\gamma}_{22} & Qx_3^2\tilde{\gamma}_{12} & Qx_3^2\tilde{\gamma}_{22} \end{bmatrix}. \tag{69}$$

In addition $\mathbf{Q}$ and $\mathbf{T}$ defined in Eq.(68) are symmetric and positive definite matrices [4]. There are only seven independent relations in Eq.(66) in view of the fact that the fourth and seventh equations of Eq.(66) are identical. Adding the relation $c_4 = \lambda c_3$ with two arbitrary but unequal multipliers α and β, respectively to the fourth and seventh equations of (66) as in [4], we finally obtain the following standard form

$$\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} - (\mathbf{Y}_1 + \mathbf{Y}_2)^{-1} \mathbf{\Omega e}, \tag{70}$$

where

$$\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix} = (\mathbf{Y}_1 + \mathbf{Y}_2)^{-1}(\mathbf{X}_1 + \mathbf{X}_2), \tag{71}$$

$$\mathbf{X}_1 = \begin{bmatrix} -\mathbf{Q} & \mathbf{0} \\ -\mathbf{R}^T & \mathbf{I} \end{bmatrix}, \quad \mathbf{Y}_1 = \begin{bmatrix} \mathbf{R} & \mathbf{I} \\ \mathbf{T} & \mathbf{0} \end{bmatrix}, \tag{72}$$

$$\mathbf{X}_2 = \begin{bmatrix} \alpha\mathbf{I}_{44} & \frac{1}{2}\mathbf{I}_{43} \\ \beta\mathbf{I}_{34} & -\frac{1}{2}\mathbf{I}_{33} \end{bmatrix}, \quad \mathbf{Y}_2 = \begin{bmatrix} \alpha\mathbf{I}_{43} & -\frac{1}{2}\mathbf{I}_{44} \\ \beta\mathbf{I}_{33} & \frac{1}{2}\mathbf{I}_{34} \end{bmatrix}.$$

In addition, $\mathbf{N}_2$ and $-\mathbf{N}_3$ are symmetric and positive semi-definite [7]. A comparison of Eq.(72) with Eq.(27) reveals the interesting fact that $\mathbf{X}_1$ and $\mathbf{X}$ have the same structure, as do $\mathbf{Y}_1$ and $\mathbf{Y}$ despite the fact that their dimensions are different.

$\mathbf{u}$ and $\mathbf{\Phi}$ are the generalized displacement and stress function vectors. Now the general solution, which is composed of the homogeneous solution derived by Cheng and Reddy [4] and the inhomogeneous particular solution derived above, can be finally given by

$$\begin{aligned} \mathbf{u} = \begin{bmatrix} u_1 & u_2 & \vartheta_1 & \vartheta_2 \end{bmatrix}^T = \mathbf{Af}(z) + \bar{\mathbf{A}}\overline{\mathbf{f}(z)} + \mathbf{Cg}(z) + \bar{\mathbf{C}}\overline{\mathbf{g}(z)}, \\ \mathbf{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \psi_1 & \psi_2 \end{bmatrix}^T = \mathbf{Bf}(z) + \bar{\mathbf{B}}\overline{\mathbf{f}(z)} + \mathbf{Dg}(z) + \bar{\mathbf{D}}\overline{\mathbf{g}(z)}, \end{aligned} \tag{73}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}, \\ \mathbf{C} &= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 \end{bmatrix}, \\ \mathbf{f}(z) &= \begin{bmatrix} f_1(z_1) & f_2(z_2) & f_3(z_3) & f_4(z_4) \end{bmatrix}^T, \\ z_K &= x_1 + p_K x_2, \quad Im\{p_K\} > 0, \quad (K = 1, 2, 3, 4), \end{aligned} \tag{74}$$

with

$$\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_K \\ \mathbf{b}_K \end{bmatrix} = p_K \begin{bmatrix} \mathbf{a}_K \\ \mathbf{b}_K \end{bmatrix}, \ (K = 1, 2, 3, 4), \qquad (75)$$

$$\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_\alpha \\ \mathbf{d}_\alpha \end{bmatrix} = \lambda_\alpha \begin{bmatrix} \mathbf{c}_\alpha \\ \mathbf{d}_\alpha \end{bmatrix} - (\mathbf{Y}_1 + \mathbf{Y}_2)^{-1} \boldsymbol{\Omega} \mathbf{e}_\alpha, \ (\alpha = 1, 2). \quad (76)$$

The above general solution is valid when all the eigenvalues are unequal, i.e., $\lambda_1 \neq \lambda_2 \neq p_1 \neq p_2 \neq p_3 \neq p_4$. Even though the standard eigenrelation (75) is symbolically the same as that of the Stroh sextic formalism as observed by Cheng and Reddy [4], the second term on the right-hand side of Eq.(76) is symbolically different from its counterpart in thermo-anisotropic elasticity [8].

5 Eigenvalues for heat conduction

The eigenvalues corresponding to the heat conduction can be determined by solving Eq.(15), which is a quartic equation in λ. For some special cases discussed below, the eigenvalues can be given explicitly.

5.1 Plate symmetric about its mid-plane

If the plate is symmetric about its mid-plane, we have $Qx_3\tilde{k}_{\alpha\beta} = 0$. Consequently the eigenvalues can be explicitly determined from Eq.(15) as

$$\begin{aligned} \lambda_1 &= \frac{-Q\tilde{k}_{12} + \mathrm{i}\sqrt{(Q\tilde{k}_{11})(Q\tilde{k}_{22}) - (Q\tilde{k}_{12})^2}}{Q\tilde{k}_{22}}, \\ \lambda_2 &= \frac{-Qx_3^2\tilde{k}_{12} + \mathrm{i}\sqrt{(Qx_3^2\tilde{k}_{11})(Qx_3^2\tilde{k}_{22}) - (Qx_3^2\tilde{k}_{12})^2}}{Qx_3^2\tilde{k}_{22}}. \end{aligned} \qquad (77)$$

5.2 Anti-symmetric angle-ply laminated plate

For an anti-symmetric angle-ply laminated plate [4],

$$Q\tilde{k}_{12} = 0, \ Qx_3\tilde{k}_{11} = Qx_3\tilde{k}_{22} = 0, \ Qx_3^2\tilde{k}_{12} = 0. \qquad (78)$$

Consequently the eigenvalues can be determined from Eq.(15) as [10-11],

$$\lambda_1, \lambda_2 = \begin{cases} \mathrm{i}\sqrt{\rho_1}([\frac{1}{2}(s_1 + 1)]^{\frac{1}{2}} \pm [\frac{1}{2}(s_1 - 1)]^{\frac{1}{2}}), \ \text{if } s_1 \geq 1 \\ \sqrt{\rho_1}(\pm[\frac{1}{2}(1 - s_1)]^{\frac{1}{2}} + \mathrm{i}[\frac{1}{2}(1 + s_1)]^{\frac{1}{2}}), \ \text{if } -1 < s_1 < 1 \end{cases} \qquad (79)$$

where

$$\rho_1 = \sqrt{\frac{(Q\tilde{k}_{11})(Qx_3^2\tilde{k}_{11})}{(Q\tilde{k}_{22})(Qx_3^2\tilde{k}_{22})}}, \quad s_1 = \frac{(Q\tilde{k}_{11})(Qx_3^2\tilde{k}_{22}) + (Q\tilde{k}_{22})(Qx_3^2\tilde{k}_{11}) - 4(Qx_3\tilde{k}_{12})^2}{2\sqrt{(Q\tilde{k}_{11})(Qx_3^2\tilde{k}_{11})(Q\tilde{k}_{22})(Qx_3^2\tilde{k}_{22})}}.$$

$$(80)$$

5.3 Orthotropic material with three axes of symmetry along the coordinate axes

In this case, $\tilde{k}_{12} = 0$. Consequently the eigenvalues can be similarly determined from Eq.(15) as

$$\lambda_1, \lambda_2 = \begin{cases} i\sqrt{\rho_2}([\frac{1}{2}(s_2+1)]^{\frac{1}{2}} \pm [\frac{1}{2}(s_2-1)]^{\frac{1}{2}}), & \text{if } s_2 \geq 1 \\ \sqrt{\rho_2}(\pm[\frac{1}{2}(1-s_2)]^{\frac{1}{2}} + i[\frac{1}{2}(1+s_2)]^{\frac{1}{2}}), & \text{if } -1 < s_2 < 1 \end{cases} \quad (81)$$

where

$$\rho_2 = \sqrt{\frac{(Q\tilde{k}_{11})(Qx_3^2\tilde{k}_{11}) - (Qx_3\tilde{k}_{11})^2}{(Q\tilde{k}_{22})(Qx_3^2\tilde{k}_{22}) - (Qx_3\tilde{k}_{22})^2}},$$

$$(82)$$

$$s_2 = \frac{(Q\tilde{k}_{11})(Qx_3^2\tilde{k}_{22}) + (Q\tilde{k}_{22})(Qx_3^2\tilde{k}_{11}) - 2(Qx_3\tilde{k}_{11})(Qx_3\tilde{k}_{22})}{2\sqrt{[(Q\tilde{k}_{22})(Qx_3^2\tilde{k}_{22}) - (Qx_3\tilde{k}_{22})^2][(Q\tilde{k}_{11})(Qx_3^2\tilde{k}_{11}) - (Qx_3\tilde{k}_{11})^2]}}.$$

5.4 Transversely isotropic material with the axis of symmetry at the x_3-axis

In this case we have $\tilde{k}_{11} = \tilde{k}_{22}$ and $\tilde{k}_{12} = 0$. Consequently Eq.(15) becomes

$$(1 + \lambda^2)^2 \left[(Q\tilde{k}_{11})(Qx_3^2\tilde{k}_{11}) - (Qx_3\tilde{k}_{11})^2\right] = 0. \quad (83)$$

Due to the Schwarz integral inequality [4], we have

$$\lambda_1 = \lambda_2 = i. \quad (84)$$

In addition there are two independent eigenvectors associated with the double eigenvalues of i. We conclude that $\hat{N}$ is semisimple, and the general solution (31) is still valid.

6   Conclusions

A Stroh-like formalism has been developed for the decoupled steady-state thermal conduction and the thermal stress analyses of a laminated anisotropic thin plate. For the heat conduction problem, a Stroh-like quartic formalism is derived through the introduction of a pair of two-dimensional generalized temperature and heat flux function vectors. The structure of the introduced 4×4 fundamental plate matrix for heat conduction is the same as that of the 8×8 fundamental elasticity matrix [8]. Thus the orthogonality and closure relations and further identities can be similarly derived. For the thermal stress analysis, a Stroh-like octet formalism is derived. The resulting general solution (73) consists of the homogeneous solution obtained by Cheng and Reddy [4]and the inhomogeneous particular solution derived in this work to account for the thermal effect. It is expected that the derived Stroh-like formalism for Kirchhoff laminated anisotropic thermoelastic thin plates can be conveniently used in the analysis of inclusion and crack problems in these laminated structures under thermomechanical loadings. The solution procedure can be outlined as: (i) first the two-dimensional complex function vector $\mathbf{g}(z)$ characterizing the temperature field in the laminated anisotropic thin plate can be derived using Eq. (31) and appropriate boundary conditions for heat conduction; (ii) the four-dimensional complex function vector $\mathbf{f}(z)$ characterizing the thermoelastic field in the laminated anisotropic thin plate can then be derived using Eq.(73) and appropriate boundary conditions for thermoelasticity.

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Appendix

Since k_{ij} is positive definite (Ting, 1996), the following inequality follows

$$\tilde{k}_{\alpha\beta}y_\alpha y_\beta > 0, \tag{A1}$$

where $\begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ is any non-zero real vector.

If λ is real, taking $y_\alpha = (\delta_{\alpha 1} + \lambda\delta_{\alpha 2})e_1$ into Eq.(A1) yields

$$(\tilde{k}_{11} + 2\lambda\tilde{k}_{12} + \lambda^2\tilde{k}_{22})e_1^2 > 0, \tag{A2}$$

and e_1 is an arbitrary real and non-zero number. Replacing e_1 by $e_1 + x_3e_2$ in Eq.(A2) and integrating it through the plate thickness gives

$$Q(\tilde{k}_{11} + 2\lambda\tilde{k}_{12} + \lambda^2\tilde{k}_{22})(e_1 + x_3e_2)^2 > 0, \tag{A3}$$

or equivalently

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} Q(\tilde{k}_{11} + 2\lambda\tilde{k}_{12} + \lambda^2\tilde{k}_{22}) & Qx_3(\tilde{k}_{11} + 2\lambda\tilde{k}_{12} + \lambda^2\tilde{k}_{22}) \\ Qx_3(\tilde{k}_{11} + 2\lambda\tilde{k}_{12} + \lambda^2\tilde{k}_{22}) & Qx_3^2(\tilde{k}_{11} + 2\lambda\tilde{k}_{12} + \lambda^2\tilde{k}_{22}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} > 0, \tag{A4}$$

which violates Eq.(15). Therefore λ cannot be real.

Similarly by substituting $y_\alpha = \delta_{\alpha 1}(e_1 + x_3 e_2)$ and $y_\alpha = \delta_{\alpha 2}(e_1 + x_3 e_2)$ into Eq.(A1) respectively and integrating through the plate thickness, we will finally arrive at

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} Q\tilde{k}_{11} & Qx_3\tilde{k}_{11} \\ Qx_3\tilde{k}_{11} & Qx_3^2\tilde{k}_{11} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} > 0,$$

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} Q\tilde{k}_{22} & Qx_3\tilde{k}_{22} \\ Qx_3\tilde{k}_{22} & Qx_3^2\tilde{k}_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} > 0, \qquad \text{(A5)}$$

which implies that $\hat{\mathbf{Q}}$ and $\hat{\mathbf{T}}$ defined by Eq.(24) are positive definite.

Thus the coefficients of λ^0 and λ^4 of the polynomial of degree four in λ arising from Eq.(15) are positive.

Since the coefficients of λ^0 and λ^4 arising from Eq.(15) are non-zero, we will have four non-zero roots of λ. Since all the coefficients of the quartic equation in λ are real, we can conclude that Eq.(15) provides two pairs of complex conjugates of λ.

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Formalizam tipa Stroh-a za Kirchhoff-ove anizotropne termoelastične ploče

Formalizam tipa Stroha, baziran na Kirchhoff-ovoj teoriji, je razvijen za provodjenje toplote i spregnute deformacije zatezanja i savijanja laminatne anizotropne termoelastične tanke ploče. Za problem provodjenja toplote, formalizam tipa Stroha četvrtog stepena je razvijen. Vektori dvodimenzionalne generalisane temperature i funkcije toplotnog fluksa su uvedeni. Struktura uvedene 4×4 osnovne matrice ploče za provodenja toplote je ista kao 8×8 fundamentalna matrica elastičnosti u Stroh-ovom formalizmu šestog stepena elastičnost generalisane ravanske deformacije. Shodno tome, relacije upravnosti i zatvaranja za provodenje toplote u tankim pločama su uspostavljene. Za problem termoelastičnosti, nehomogeno posebno rešenje je izvedeno rigorozno. Dobijamo formalizam osmog stepena u kome je opšte rešenje sastavljeno od poznatog homogenog rešenja razvijenog Cheng-om i Reddy-jem (izotermni slučaj) kao i nehomogenog posebnog rešenja koje dolazi iz termičkog efekta.