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GENERAL SOLUTION AND FUNDAMENTAL SOLUTION FOR TWO-DIMENSIONAL PROBLEM IN ORTHOTROPIC THERMOELASTIC MEDIA WITH VOIDS

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General solution and fundamental solution for two-dimensional problem in orthotropic thermoelastic media with voids

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Abstract

The aim of the present paper is to study the fundamental solution in orthotropic thermoelastic media with voids. With this objective, firstly the two-dimensional general solution in orthotropic thermoelastic media with voids is derived. On the basis of general solution,the fundamental solution for a steady point heat source on the surface of a semi-infinite orthotropic thermoelastic material with voids is constructed by six newly introduced harmonic functions. The temperature and voids distribution and components of displacement and stress are expressed in terms of elementary functions. From the present investigation, a special case of interest is also deduced and compared with the previous results obtained. Since all the components are expressed in terms of elementary functions, it is convenient to use them.

Keywords: Fundamental solution, orthotropic, thermoelastic, voids, semi–infinite.

1 Introduction

Fundamental solutions or Green's functions play an important role in the solution of numerous problems in the mechanics and physics of solids. They are a basic building block of many further works. For example, fundamental solutions can be used to construct many analytical solutions of practical

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problems when boundary conditions are imposed. They are essential in the boundary element method as well as the study of cracks, defects and inclusions. Many researchers have been investigated the Green's function for elastic solid in isotropic and anisotropic elastic media, notable among them are Freedholm[1], Lifshitz and Rozentsveig [2], Elliott[3], Kröner[4], Synge [5], Lejcek [6],Pan and Chou [7] and Pan and Yuan [8].

When thermal effects are considered, Sharma [9] investigated the fundamental solution in transversely isotropic thermoelastic material in an integral form.Chen et al. [10] derived the three dimensional general solution in transversely isotropic thermoelastic materials. Hou et al. [11, 12] investigated the Green's function for two and three-dimensional problem for a steady Point heat source in the interior of a semi-infinite thermoelastic materials. Also, Hou et al. [13] investigated the two dimensional general solutions and fundamental solutions in orthotropic thermoelastic materials.

The theory of linear elastic material with voids is one of the most important generalizations of the classical theories of elasticity. The theory has practical utility in investigating various type of geological, biological and synthetic porous material for which the elastic theory is inadequate. This theory is concerned with elastic materials consisting of a distribution of small pores (voids), in which the voids volume is included among the kinematics variables and in the limiting case of vanishing this volume, the theory reduces to classical theory of elasticity.

Iesan [14] has established a linear theory of thermoelastic material with voids. He presented the basic field equations and discussed the condition of propagation of acceleration waves in a homogeneous isotropic thermoelastic material with voids. He shows that transverse waves propagates without effecting the temperature and porosity of the material.Dhaliwal and Wang [15] have also formulated a thermoelastic theory for elastic material with voids to include heat flux among the constitutive variables and assumes an evolution equation for the heat flux. Chirita and Scalia [16] and Pompei and Scalia [17] have investigated the spatial and temporal behavior of the transient solutions for the initial-boundary value problems associated with the linear theory of thermoelastic material with voids by using surface power function method. Scalia, Pompei and Chirita [18] studied the steady time harmonic oscillations within the context of thermoelasticity for materials with voids. Singh and Tomar [19] investigated the propagation of plane waves in an infinite thermoelastic medium with voids.

Kumar and Chawla [20, 21] the derived the two dimensional fundamental solution and Green's function for two dimensional problem in orthotropic thermoelastic diffusion media. Also Kumar and Chawla [22, 23] derived the fundamental solution and Green's function in orthotropic piezothermoelastic diffusion media. However, the important fundamental solution for two-dimensional problem for a steady point heat source in orthotropic thermoelastic material with voids has not been discussed so far. This type of technique is very useful in finding the fundamental solution for different theories that is micropolar thermoelastic, microstretch thermoelastic, micropolar thermoelastic material with voids etc. The solution of these types of problem has not been discussed so far in the literature.

The Fundamental solution for two-dimensional in orthotropic thermoelastic medium with voids is investigated in this paper. Based on the two-dimensional general solution of orthotropic thermoelastic media with voids,the fundamental solutions for a steady point heat source acting on the surface of a thermoelastic material with voids is obtained by six newly introduced harmonic functions. From the present investigation, some special cases of interest are also deduced.

2 Basic equations

Following Iesan [14], the basic equations for anisotropic thermoelastic material with voids, in the absence of body forces, extrinsic equilibrated body force and heat sources are

Constitutive relations

$$t_{ij} = c_{ijkm}e_{km} + D_{ijk}\phi_{,k} + B_{ij}\phi - \beta_{ij}T, \tag{1}$$

Equations of motion

$$\rho\ddot{u}_i = c_{ijkm}u_{i,j} + B_{ij}\phi_{,j} - \beta_{ij}T_{,j}, \tag{2}$$

Equilibrated equation

$$\rho\chi\ddot{\phi} = -\omega_0\dot{\phi} - \varsigma\phi + bT + A_{ij}\phi_{,ji} - B_{ij}e_{ij}, \tag{3}$$

Equation of heat conduction

$$\rho C^*\dot{T} + T_0(\beta_{ij}\dot{e}_{ij} + b\dot{\phi}) = K_{ij}T_{,ij}. \tag{4}$$

Here $c_{ijkm}, A_{ij}, B_{ij}, \beta_{ij}, \omega_0, \varsigma, b$ are the constitutive coefficients, T is the temperature distribution from the reference temperature T_0, ϱ is the density, χ

is the equilibrated inertia, ϕ is the volume fraction field, $e_{ij} = (u_{i,j} + u_{j,i})/2$ are the components of the strain tensor, u_i are components of displacement vector, C^* is the specific heat at constant strain and the above coefficients have the following symmetries:

$$c_{ijkm} = c_{kmij} = c_{jikm},$$
$$B_{ij} = B_{ji}, \quad \beta_{ij} = \beta_{ji}, \quad A_{ij} = A_{ji}, \quad K_{ij} = K_{ji}. \tag{5}$$

If the material symmetry is of a type that possesses a center of symmetry, then D_{ijk} is identically zero.

3 Formulation of the problem

We consider homogenous orthotropic thermoelastic material with voids. Let us take $Oxyz$ as the frame of reference in Cartesian coordinates, the origin O being any point on the plane boundary.

For two-dimensional static problem, we assume the displacement vector, temperature change and void parameter are, respectively, of the form

$$\vec{u} = (u, 0, w), \quad \mathrm{T}(x, z, t), \quad \phi(x, z, t). \tag{6}$$

We define the dimensionless quantities as

$$(x', z', u'w') = \frac{\omega_1^*}{v_1}(x, z, u, w), \qquad T' = \frac{a_1}{c_{11}}T,$$
$$\phi' = \frac{\omega_1^{*2}}{v_1^2}\chi\phi, \;\; \sigma'_{ij} = \frac{\sigma_{ij}}{a_1 \mathrm{T}_0}, \;\; H' = \frac{a_1 v_1}{c_{11} K_1 \omega_1^*}H,$$

where

$$v_1^2 = \varsigma_1, \qquad \omega_1^* = \frac{a\, c_{11}}{K_1}. \tag{7}$$

Equations (1) - (4) for orthotropic thermoelastic material with voids, with the aid of (7), after suppressing the primes, yield

$$\left(\frac{\partial^2}{\partial x^2} + \delta_1 \frac{\partial^2}{\partial z^2}\right) u + \left(\delta_2 \frac{\partial^2}{\partial x \partial z}\right) w + \eta_1 \left(\frac{\partial}{\partial x}\right)\phi - \left(\frac{\partial}{\partial x}\right) T = 0, \tag{8}$$

$$\left(\delta_2 \frac{\partial^2}{\partial x \partial z}\right) u + \left(\delta_1 \frac{\partial^2}{\partial x^2} + \delta_3 \frac{\partial^2}{\partial z^2}\right) w + \eta_2 \left(\frac{\partial}{\partial z}\right) C - \varepsilon_1 \left(\frac{\partial}{\partial z}\right) T = 0, \tag{9}$$

$$\left(\frac{\partial^2}{\partial x^2} + \varepsilon_2 \frac{\partial^2}{\partial z^2}\right) T = 0, \tag{10}$$

$$\left(\delta_4 \frac{\partial}{\partial x}\right) u + \left(\delta_5 \frac{\partial}{\partial z}\right) w + \left(\frac{\partial^2}{\partial x^2} + \delta_6 \frac{\partial^2}{\partial z^2} - \delta_7\right)\phi + \delta_8 T = 0. \tag{11}$$

Here

$$(\delta_1, \delta_2, \delta_3) = \frac{1}{c_{11}}(c_{55}, c_{13} + c_{55}, c_{33}),$$
$$\varepsilon_1 = \frac{a_3}{a_1},\ \varepsilon_2 = \frac{K_3}{K_1},\ (\eta_1, \eta_2) = \frac{v_1^2}{c_{11}\omega_1^{*2}}(B_1,\ B_3),$$
$$(\delta_4, \delta_5) = -\frac{\chi}{\alpha_1}(\gamma_1, \gamma_3),\ \delta_6 = \frac{\alpha_3}{\alpha_1},\ \ \delta_7 = \frac{\xi_1 v_1^2}{\omega_1^{*2}},\ \ \delta_8 = \frac{mc_{11}}{a_1\alpha_1}.$$

The equations (8)-(11) can be written as

$$D\{u, w, \phi, T\}^{tr} = 0. \tag{12}$$

where D is the differential operator matrix given by

$$\begin{bmatrix} \frac{\partial^2}{\partial x^2} + \delta_1\frac{\partial^2}{\partial z^2} & \delta_2\frac{\partial^2}{\partial x \partial z} & \eta_1\frac{\partial}{\partial x} & -\frac{\partial}{\partial x} \\ \delta_2\frac{\partial^2}{\partial x \partial z} & \delta_1\frac{\partial^2}{\partial x^2} + \delta_4\frac{\partial^2}{\partial z^2} & \eta_2\frac{\partial}{\partial z} & -\varepsilon_1\frac{\partial}{\partial z} \\ -\delta_4\frac{\partial}{\partial x} & -\delta_5\frac{\partial}{\partial z} & \frac{\partial^2}{\partial x^2} + \delta_5\frac{\partial^2}{\partial x^2} - \delta_7 & \delta_8 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial x^2} + \varepsilon_3\frac{\partial^2}{\partial z^2} \end{bmatrix} \tag{13}$$

Equation (12) is a homogeneous set of differential equations in u, w, T, ϕ. The general solution by the operator theory as follows

$$\begin{aligned} u &= A_{i1}F + \bar{A}_{i1}G, \qquad w = A_{i2}F + \bar{A}_{i2}G, \\ T &= A_{i3}F + \bar{A}_{i3}G, \qquad \phi = \bar{A}_{i4}F + A_{i4}G, \quad (i = 1, 2, 3, 4) \end{aligned} \tag{14}$$

where A_{ij} are algebraic cofactors of the matrix D,of which the determinant is

$$\begin{aligned} |D| &= \left(a^*\frac{\partial^6}{\partial z^6} + b^*\frac{\partial^6}{\partial x^2\partial z^4} + c^*\frac{\partial^6}{\partial x^4\partial z^2} + d^*\frac{\partial^6}{\partial x^6}\right) \times \left(\frac{\partial^2}{\partial x^2} + \varepsilon_3\frac{\partial^2}{\partial z^2}\right) \\ &+ \left(\bar{a}\frac{\partial^4}{\partial z^4} + \bar{b}\frac{\partial^4}{\partial x^2\partial z^2} + \bar{c}\frac{\partial^4}{\partial z^4}\right) \times \left(\frac{\partial^2}{\partial x^2} + \varepsilon_3\frac{\partial^2}{\partial z^2}\right), \end{aligned} \tag{15}$$

$$\begin{aligned} &a^* = \delta_1\delta_4\delta_5,\ \ b^* = \delta_1(\delta_4 - \delta_8) + \delta_5(\delta_4 - \delta_2^2), \\ &c^* = \delta_1\delta_5(1 + \delta_1) + (\delta_1^2 - \delta_5^2),\ \ d^* = \delta_1, \\ &\bar{a} = \delta_1(\eta_2\delta_6 - \delta_3\delta_7),\ \ \bar{c} = \delta_6\eta_1 - \delta_1\delta_8 \\ &\bar{b} = \delta_4(1 - \delta_8) + \eta_2\delta_7 + \delta_2(\delta_2\delta_8 - \delta_5\eta_2) + \eta_1(\delta_5\delta_3 - \delta_2\delta_7) \end{aligned}$$

The functions F and G in equation (14) satisfy the following homogeneous equation

$$|D|\,F = 0 \qquad \text{and} \qquad |D|\,G = 0. \tag{16}$$

It can be seen that if $i = 1, 2, 3$ are taken in equation (14), three general solutions are obtained in which $T = 0$. These solutions are identical to those without thermal effect and are not discussed here. Therefore if $i = 4$ should be taken in equation (14), the following solutions are obtained

$$u = \left(p_1\frac{\partial^4}{\partial x^4} + q_1\frac{\partial^4}{\partial z^2\partial x^2} + r_1\frac{\partial^4}{\partial z^4}\right)\frac{\partial F}{\partial x} + \left(\bar{p}_1\frac{\partial^2}{\partial x^2} + \bar{q}_1\frac{\partial^2}{\partial z^2}\right)\frac{\partial G}{\partial x}, \tag{17a}$$

$$w = \left(p_2\frac{\partial^4}{\partial x^4} + q_2\frac{\partial^4}{\partial z^2\partial x^2} + r_2\frac{\partial^4}{\partial z^4}\right)\frac{\partial F}{\partial z} + \left(\bar{p}_2\frac{\partial^2}{\partial x^2} + \bar{q}_2\frac{\partial^2}{\partial z^2}\right)\frac{\partial G}{\partial z}, \tag{17b}$$

$$\phi = \left(p_3\frac{\partial^4}{\partial x^4} + q_3\frac{\partial^4}{\partial z^2\partial x^2} + r_3\frac{\partial^4}{\partial z^4}\right)F, \tag{17c}$$

$$\begin{aligned} T = {} & \left(a^*\frac{\partial^6}{\partial z^6} + b^*\frac{\partial^6}{\partial x^2\partial z^4} + c^*\frac{\partial^6}{\partial x^4\partial z^2} + d^*\frac{\partial^6}{\partial x^6}\right)F \\ & + \left(\bar{a}\frac{\partial^4}{\partial z^4} + \bar{b}\frac{\partial^4}{\partial x^2\partial z^2} + \bar{c}\frac{\partial^4}{\partial z^4}\right)G, \end{aligned} \tag{17d}$$

where

$$\begin{aligned}
p_1 &= -\delta_1, \quad q_1 = \varepsilon_1\delta_2 - \delta_1\delta_5 - \delta_4, \quad r_1 = \delta_5(\delta_2\varepsilon_1 - \delta_4),\\
\bar{q}_1 &= \delta_2(\delta_8\eta_2 - \delta_7\varepsilon_1) - \delta_9(\delta_2\eta_2 + \delta_4\eta_1) + \delta_6(\varepsilon_1\eta_1 - \eta_2) + \delta_4\delta_8,\\
p_2 &= \varepsilon_1 - \delta_2, \quad q_2 = \delta_5(\varepsilon_1 - \delta_2) + \varepsilon_1\delta_2, \quad r_2 = \delta_1\delta_5\varepsilon_1,\\
\bar{p}_1 &= \delta_1(\delta_8 - \eta_1\delta_9), \quad \bar{p}_2 = \eta_2\delta_8 + \delta_7(\delta_2 - \varepsilon_1) + \eta_1(\delta_6\varepsilon_1 - \delta_2\delta_9)\\
p_3 &= \delta_1(\delta_9 - \delta_6), \quad q_3 = \delta_9(\delta_1^2 - \delta_2^2) + \varepsilon_1(\delta_2\delta_6 - \delta_7) + \delta_4(\delta_8 - \delta_5) + \delta_2\delta_7,\\
r_3 &= \delta_1(\delta_3\delta_8 - \varepsilon_1\delta_7).
\end{aligned}$$

Equation (16) can be rewritten as

$$\prod_{j=1}^{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2}\right)F = 0, \tag{18a}$$

$$\prod_{j=1}^{3}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2}\right)G = 0. \tag{18b}$$

where $z_j = s_j z,\ s_4 = \sqrt{K_1/K_3}$ and s_j $(j = 1,2,3)$ are three roots (with positive real parts) of the following algebraic equation

$$a^* s^6 - b^* s^4 + c^* s^2 - d^* = 0. \tag{19}$$

and $\bar{z}_j = s_j z,\ \bar{s}_3 = \sqrt{K_1/K_3}$ and $s_j(j = 1,2)$ are three roots (with positive real parts) of the following algebraic equation

$$\bar{a}s^4 - bs^2 + \bar{c} = 0. \tag{20}$$

As known from the generalized *Almansi theorem* (proved by Ding et al. [24]), the function F and G can be expressed in terms of four harmonic functions

i $\quad F = F_1 + F_2 + F_3 + F_4$ for distinct $s_j \quad (j = 1,2,3,4)$

$$G = G_1 + G_2 + G_3 \text{ for distinct } \bar{s}_j(j = 1,2,3,). \tag{21a}$$

ii $\quad F = F_1 + F_2 + F_3 + zF_4$ for $s_1 \neq s_2 \neq s_3 = s_4.$

$$G = G_1 + G_2 + zG_3 \ \ for \ \ \bar{s}_1 \neq \bar{s}_2 = s_3. \tag{21b}$$

iii $\quad F = F_1 + F_2 + zF_3 + z^2F_4$ for $s_1 \neq s_2 = s_3 = s_4. \quad$ (21c)

iv $\quad F = F_1 + zF_2 + z^2F_3 + z^3F_4 \ \ for \ \ s_1 = s_2 = s_3 = s_4.$

$$G = G_1 + zG_2 + z^2G_3 \quad \bar{s}_1 = \bar{s}_2 = \bar{s}_3, \tag{21d}$$

where $F_j\,(j = 1,2,3,4)$ and $G_j\,(j = 1,2,3)$ satisfy the following harmonic equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2}\right) F_j = 0, \qquad (j = 1,2,3,4) \tag{22a}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2}\right) G_j = 0, \qquad (j = 1,2,3). \tag{22b}$$

The general solution for the case of distinct roots, can be derived as follows

$$\begin{aligned} u &= \sum_{j=1}^{4} p_{1j}\frac{\partial^5 F_j}{\partial x \partial z_j^4} + \sum_{j=1}^{3} \bar{p}_{1j}\frac{\partial^3 G_j}{\partial x \partial z_j^2}, \\ w &= \sum_{j=1}^{4} s_j p_{2j}\frac{\partial^5 F_j}{\partial z_j^5} + \sum_{j=1}^{3} s_j \bar{p}_{2j}\frac{\partial^3 F_j}{\partial z_j^3} \\ \phi &= \sum_{j=1}^{4} p_{3j}\frac{\partial^4 F_j}{\partial z_j^4}, \qquad T = p_{44}\frac{\partial^6 F_4}{\partial z_4^6} + \bar{p}_{44}\frac{\partial^4 G_4}{\partial z_4^4}, \end{aligned} \tag{23}$$

where

$$\begin{aligned}
p_{kj} &= p_k - q_k s_j^2 + r_k s_j^4, \quad (k=1,2) \\
p_{44} &= d^* s_4^6 - b^* s_4^4 + c^* s_4^2 - a^*, \\
\bar{p}_{kj} &= -p_k + q_k s_j^2, \quad (k=1,2) \\
\bar{p}_{44} &= \bar{a} - \bar{b}\bar{s}_3^2 + \bar{c}\bar{s}_3^4.
\end{aligned}$$

In the similar way general solution for the other three cases can be derived. Equation (23) can be further simplified by taking

$$p_{1j}\frac{\partial^4 F_j}{\partial z_j^4} = \psi_j, \qquad \text{and} \qquad \bar{p}_{1j}\frac{\partial^2 G_j}{\partial \bar{z}_j^2} = \bar{\psi}_j. \tag{24}$$

$$\begin{aligned}
u &= \sum_{j=1}^{4} p_{1j}\frac{\partial \psi_j}{\partial x} + \sum_{j=1}^{3} \bar{p}_{1j}\frac{\partial \bar{\psi}_j}{\partial x}, \\
w &= \sum_{j=1}^{4} s_j P_{1j}\frac{\partial \psi_j}{\partial z_j} + \sum_{j=1}^{3} \bar{s}_j \bar{P}_{2j}\frac{\partial \bar{\psi}_j}{\partial \bar{z}_j} \\
\phi &= \sum_{j=1}^{4} P_{2j}\psi_j, \quad T = P_{34}\frac{\partial^2 \psi_4}{\partial z_4^2} + \bar{p}_{23}\frac{\partial^2 \bar{\psi}_4}{\partial \bar{z}_4^2},
\end{aligned} \tag{25}$$

where

$$P_{1j} = p_{2j}/p_{1j}, \quad P_{2j} = p_{3j}/p_{1j}, \quad P_{34} = p_{44}/p_{14}.$$

$$\bar{P}_{1j} = \bar{p}_{2j}/\bar{p}_{1j}, \quad \bar{P}_{23} = \bar{p}_{33}/\bar{p}_{13}.$$

The functions $\psi_j\,(j = 1,2,3,4)$ and $\bar{\psi}_j\,(j = 1,2,3)$ satisfy the harmonic equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi_j = 0, \; j = 1,2,3,4 \tag{26a}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\bar{\psi}_j = 0, \; j = 1,2,3. \tag{26b}$$

By making use of (5)-(7) in equation (1) and after suppressing the primes, with the aid of (25), we obtain

$$\sigma_{xx} = \sum_{j=1}^{4} \left(-f_1 + f_1 s_j^2 P_{1j} - h_1 P_{2j} - f_1 P_{3j}\right) \frac{\partial^2 \psi_j}{\partial z_j^2} \tag{27a}$$

$$+ \sum_{j=1}^{3} \left(-f_1 + f_1 \bar{s}_j^2 \bar{P}_{1j} - f_1 P_{3j}\right) \frac{\partial^2 \psi_j}{\partial z_j^2},$$

$$\sigma_{zz} = \sum_{j=1}^{4} \left(-f_2 + h_2 s_j^2 P_{1j} - h_3 P_{2j} - h_4 P_{3j}\right) \frac{\partial^2 \psi_j}{\partial z_j^2} \tag{27b}$$

$$+ \sum_{j=1}^{3} \left(-f_2 + h_2 \bar{s}_j^2 \bar{P}_{1j} - h_4 \bar{P}_{3j}\right) \frac{\partial^2 \bar{\psi}_j}{\partial z_j^2},$$

$$\sigma_{zx} = \sum_{j=1}^{4} h_5 (1 + P_{1j}) s_j \frac{\partial^2 \psi_j}{\partial x \partial z_j} + \sum_{j=1}^{4} h_5 (1 + \bar{P}_{1j}) \bar{s}_j \frac{\partial^2 \bar{\psi}_j}{\partial x \partial z_j}. \tag{27c}$$

Substituting the values of σ_{xx}, σ_{zz} and σ_{zx} from (27) into equations (1)-(3), with the aid of (5) and (6), leads to

$$f_1 - f_1 s_j^2 P_{1j} + f_1 P_{3j} + h_1 P_{2j} = h_5 (1 + P_{1j}) s_j^2, \tag{28a}$$

$$- f_2 + h_2 s_j^2 P_{1j} - h_3 P_{2j} - h_4 P_{3j} = h_5 (1 + P_{1j}), \tag{28b}$$

$$\left(1 - \varepsilon_3 s_j^2\right) P_{3j} = 0 \quad (j = 1, 2, 3, 4) \tag{28c}$$

$$f_1 - f_1 \bar{s}_j^2 \bar{P}_{1j} + f_1 \bar{P}_{3j} + h_1 \bar{P}_{2j} = h_5 (1 + \bar{P}_{1j}) \bar{s}_j^2, \tag{29a}$$

$$- f_2 + h_2 \bar{s}_j^2 \bar{P}_{1j} - h_3 \bar{P}_{2j} - h_4 \bar{P}_{3j} = h_5 (1 + \bar{P}_{1j}) \tag{29b}$$

$$\left(1 - \varepsilon_3 \bar{s}_j^2\right) \bar{P}_{2j} = 0 \quad (j = 1, 2, 3). \tag{29c}$$

The general solutions (27) with the help of (28) and (29) can be simplified as

$$\sigma_{xx} = -\sum_{j=1}^{4} s_j^2 w_{1j} \frac{\partial^2 \psi_j}{\partial z_j^2} - \sum_{j=1}^{3} \bar{s}_j^2 \bar{w}_{1j} \frac{\partial^2 \bar{\psi}_j}{\partial \bar{z}_j^2},$$

$$\sigma_{zx} = \sum_{j=1}^{4} s_j w_{1j} \frac{\partial^2 \psi_j}{\partial x \partial z_j}, \tag{30}$$

$$\sigma_{zz} = \sum_{j=1}^{4} w_{1j} \frac{\partial^2 \psi_j}{\partial z_j^2} + \sum_{j=1}^{3} \bar{w}_{1j} \frac{\partial^2 \bar{\psi}_j}{\partial \bar{z}_j^2},$$

where

$$\begin{aligned} w_{1j} &= \frac{f_1 - f_2 s_j^2 P_{1j} + f_1 P_{3j} + h_1 P_{2j}}{s_j^2} = h_5(1 + P_{1j}) \\ &= -f_2 + h_2 s_j^2 P_{1j} - h_3 P_{2j} - h_4 P_{3j}. \end{aligned} \tag{31}$$

$$\begin{aligned} \bar{w}_{1j} &= \frac{f_1 - f_1 \bar{s}_j^2 \bar{P}_{1j} + f_1 \bar{P}_{3j} + h_1 \bar{P}_{2j}}{\bar{s}_j^2} = h_5(1 + \bar{P}_{1j}) \\ &= -f_2 + h_2 \bar{s}_j^2 \bar{P}_{1j} - h_3 \bar{P}_{2j} - h_4 \bar{P}_{3j}. \end{aligned} \tag{32}$$

4 Fundamental solution for a point heat source in a semi - infinite orthotropic thermoelastic material with voids

We consider a semi-infinite orthotropic thermoelastic material with voids $z \geq 0$. A point heat source H is applied at the origin and the surface $z = 0$ is free, equilibrated and thermally insulated. The complete geometry of the problem is shown in Figure 1. The general solutions given by equation (25) and (30) are then applied and derived in this section.

Introduce the harmonic functions as

$$\psi_j = A_j \left[\frac{1}{2}(z_j^2 - x^2) \left(\log r_j - \frac{3}{2} \right) - x z_j \tan^{-1} \frac{x}{z_j} \right] j = 1, 2, 3, 4 \tag{33}$$

where A_j (j=1, 2, 3, 4) are arbitrary constants to be determined and

$$r_j = \sqrt{x^2 + z_j^2}, \tag{34}$$

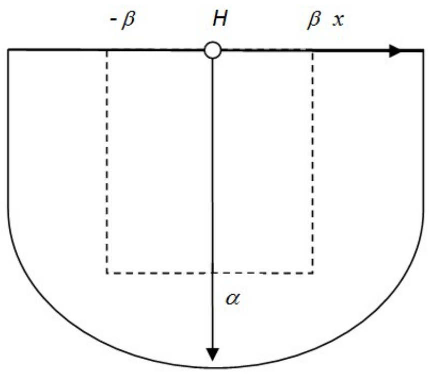

Figure 1: Geometry of the problem

as well as

$$\bar{\psi}_j = \bar{A}_j \left[\frac{1}{2}(\bar{z}_j^2 - x^2) \left(\log \bar{r}_j - \frac{3}{2} \right) - x\bar{z}_j \tan^{-1} \frac{x}{\bar{z}_j} \right], \quad j = 1, 2, 3, \tag{35}$$

where $\bar{A}_j$ $(j = 1, 2, 3)$ are arbitrary constants to be determined and

$$\bar{r}_j = \sqrt{x^2 + \bar{z}_j^2}. \tag{36}$$

Here $\bar{A}_3$ can be written as linear combination of A_4 i.e. $\bar{A}_3 = \eta A_4$ where η is some arbitrary constant.

The boundary conditions on the surface $z = 0$ are

$$\sigma_{zz} = \sigma_{zx} = 0, \quad \frac{\partial T}{\partial z} = 0, \quad \frac{\partial \phi}{\partial z} = 0. \tag{37}$$

When the mechanical, thermal condition and volume fraction field for a rectangle of $0 \leq z \leq \alpha$ and $-\beta \leq x \leq \beta$ $(\beta > 0)$ are considered (Figure 1), the following three equations can be obtained

$$\int_{-\beta}^{\beta} \sigma_{zz}(x,\alpha)dx + \int_{0}^{\alpha} [\sigma_{zx}(\beta,z) - \sigma_{zx}(-\beta,z)]dz = 0, \tag{38a}$$

$$-\varepsilon_2 \int_{-\beta}^{\beta} \left[\frac{\partial T}{\partial z}(x,\alpha)\right] dx - \int_{0}^{\alpha} \left[\frac{\partial T}{\partial x}(\beta,z) - \frac{\partial T}{\partial x}(-\beta,z)\right] dz = H. \tag{38b}$$

$$\int_{-\beta}^{\beta} \frac{\partial \phi}{\partial z}(x,\alpha)dx + \int_{0}^{\alpha} \left[\frac{\partial \phi}{\partial x}(\beta,z) - \frac{\partial \phi}{\partial x}(-\beta,z)\right] dz = 0, \tag{38c}$$

Substituting the values of ψ_j and $\bar{\psi}_j$ from equations (33) and (35) into equations (25) and (30), we arrive at the expressions for components of displacement, mass concentration, temperature change and stress components as follows:

$$u = -\sum_{j=1}^{4} A_j \left[x(\log r_j - 1) + z_j \tan^{-1}\frac{x}{z_j}\right] - \sum_{j=1}^{3} \bar{A}_j \left[x(\log \bar{r}_j - 1) + \bar{z}_j \tan^{-1}\frac{x}{\bar{z}_j}\right], \tag{39a}$$

$$w = \sum_{j=1}^{4} s_j P_{1j} A_j \left[z_j(\log r_j - 1) - x \tan^{-1}\frac{x}{z_j}\right] + \sum_{j=1}^{3} \bar{s}_j \bar{P}_{1j} \bar{A}_j \left[\bar{z}_j(\log \bar{r}_j - 1) - x \tan^{-1}\frac{x}{\bar{z}_j}\right], \tag{39b}$$

$$T = A_4 P_{34} \log r_4 + \bar{A}_3 \bar{P}_{23} \log \bar{r}_3, \tag{39c}$$

$$\phi = \sum_{j=1}^{4} A_j P_{2j} \log r_j, \tag{39d}$$

$$\sigma_{xx} = -\sum_{j=1}^{4} s_j^2 w_{1j} A_j \log r_j - \sum_{j=1}^{3} \bar{s}_j^2 \bar{w}_{1j} \bar{A}_j \log \bar{r}_j, \tag{39e}$$

$$\sigma_{zz} = \sum_{j=1}^{4} w_{1j} A_j \log r_j + \sum_{j=1}^{3} \bar{w}_{1j} \bar{A}_j \log \bar{r}_j, \tag{39f}$$

$$\sigma_{zx} = -\sum_{j=1}^{4} s_j w_{1j} A_j \tan^{-1} \frac{x}{z_j} - \sum_{j=1}^{4} \bar{s}_j \bar{w}_{1j} \bar{A}_j \tan^{-1} \frac{x}{\bar{z}_j}. \tag{39g}$$

Inserting the values of $\sigma_{zz}, \sigma_{zx}, T$ and ϕ from equations (39 c,d,f,g) into the boundary conditions (37), we get

$$\sum_{j=1}^{4} w_{1j} A_j = 0, \tag{40a}$$

$$\sum_{j=1}^{3} \bar{w}_{1j} \bar{A}_j = 0, \tag{40b}$$

$$\sum_{j=1}^{4} s_j w_{1j} A_j = 0, \tag{40c}$$

$$\sum_{j=1}^{3} \bar{s}_j \bar{w}_{1j} \bar{A}_j = 0, \tag{40d}$$

It should be noted that $\frac{\partial T}{\partial z}$ and $\frac{\partial \phi}{\partial z}$ are automatically satisfied at the surface $z = 0$.

Substituting the values of σ_{zz} and σ_{zx} from equations (39 f,g) into equation (38a), we obtain

$$\sum_{j=1}^{4} w_{1j} A_j I_3 + \sum_{j=1}^{4} \bar{w}_{1j} \bar{A}_j I_4 = 0, \tag{41}$$

where

$$\begin{aligned} I_3 = {} & \left[x \left(\log \sqrt{x^2 + s_j^2 \alpha^2} - 1 \right) + s_j \alpha \tan^{-1} \frac{x}{s_j \alpha} \right]_{x=-\beta}^{x=\beta} \\ & - 2 \left[z_j \tan^{-1} \frac{\beta}{s_j z} + b \log \sqrt{\beta^2 + s_j^2 z^2} \right]_{z=0}^{z=\alpha} = 2\beta(\log \beta - 1), \end{aligned} \tag{42a}$$

and

$$I_4 = \left[x \left(\log \sqrt{x^2 + \bar{s}_j^2 \alpha^2} - 1 \right) + \bar{s}_j \alpha \tan^{-1} \frac{x}{\bar{s}_j \alpha} \right]_{x=-\beta}^{x=\beta} - 2 \left[\bar{z}_j \tan^{-1} \frac{\beta}{\bar{s}_j z} + \beta \log \sqrt{\beta^2 + \bar{s}_j^2 z^2} \right]_{z=0}^{z=\alpha} = 2\beta(\log \beta - 1). \tag{42b}$$

By virtue of the equations (42), the equations (41) degenerate into equations (40 a,b) i.e. equations (38a) and (41) are satisfied automatically.

Some useful integrals are given below

$$\int \frac{\partial T}{\partial z} dx = s_4 P_{34} A_4 \int \frac{z_4}{r_4^2} dx = s_4 P_{34} A_4 \tan^{-1} \frac{x}{z_4}, \tag{43a}$$

$$\int \frac{\partial T}{\partial x} dz = P_{34} A_4 \int \frac{x}{r_4^2} dz = -\frac{P_{34}}{s_4} A_4 \tan^{-1} \frac{x}{z_4}, \tag{43b}$$

$$\frac{\partial \phi}{\partial z} = \sum_{j=1}^{4} A_j s_j^2 P_{2j} \int \frac{z}{x^2 + s_j^2 z^2} dx = \sum_{j=1}^{4} A_j s_j P_{2j} \tan^{-1} \frac{x}{s_j z}, \tag{43c}$$

$$\int \frac{\partial \phi}{\partial x} dz = \sum_{j=1}^{4} A_j P_{2j} \int \frac{x}{x^2 + s_j^2 z^2} dz = -\sum_{j=1}^{4} \frac{A_j}{s_j} P_{2j} \tan^{-1} \frac{x}{z_j}. \tag{43d}$$

Substituting (39c) into equation (38b), with the aid of $s_4 = \sqrt{K_1/K_3} = \bar{s}_3$ and the integrals (43 a,b), we obtain

$$P_{34} A_4 I_5 + \bar{P}_{23} \bar{A}_3 I_6 = \frac{H}{\sqrt{K_3 K_1}}, \tag{44}$$

with

$$I_5 = -\left[\tan^{-1} (\frac{x}{s_4 \alpha}) \right]_{x=-\beta}^{x=\beta} + \left[\tan^{-1} (\frac{\beta}{s_4 z}) \right]_{z=0}^{z=\alpha} = -\pi. \tag{45a}$$

$$I_6 = -\left[\tan^{-1} (\frac{x}{\bar{s}_4 \alpha}) \right]_{x=-\beta}^{x=\beta} + \left[\tan^{-1} (\frac{\beta}{\bar{s}_4 z}) \right]_{z=0}^{z=\alpha} = -\pi. \tag{45b}$$

The constant A_4 can be determined from equations (44) and (45 a,b), as follows

$$A_4 = -\frac{H}{\pi (P_{34} + \alpha \bar{P}_{23}) \sqrt{K_3 K_1}}. \tag{46}$$

Substituting the value of ϕ from equation (39d) in equation (38c) and with the aid of the integrals (43 c,d), we obtain

$$\sum_{j=1}^{4} r_j A_j P_{2j} = 0, \tag{47}$$

where

$$r_j = \left[s_j^2 \tan^{-1}\left(\frac{x}{s_j\alpha}\right)\right]_{x=-\beta}^{x=\beta} - \left[2\tan^{-1}\left(\frac{\beta}{s_j z}\right)\right]_{z=0}^{z=\alpha}. \tag{48}$$

On simplifying this, we get

$$r_j = 2(s_j^2-1)\tan^{-1}(\frac{\beta}{s_j\alpha}) + \pi.$$

Thus, the six constants A_j $(j = 1, 2, 3, 4)$, $\bar{A}_j$ $(j = 1, 2, 3)$ can be determined by six equations including equations (40a)–(40d), (46) and (47).

5 Special Cases

Case I : *The absence of voids effect*: In the absence of voids effect (39a)–(39g) reduce to:

$$u = -\sum_{j=1}^{3} A_j \left[x(\log r_j - 1) + z_j \tan^{-1}\frac{x}{z_j}\right], \tag{49a}$$

$$w = \sum_{j=1}^{3} s_j P_{1j} A_j \left[z_j(\log r_j - 1) - x\tan^{-1}\frac{x}{z_j}\right], \tag{49b}$$

$$T = A_3 P_{23} \log r_3, \tag{49c}$$

$$\sigma_{xx} = -\sum_{j=1}^{3} s_j^2 w_{1j} A_j \log r_j, \tag{49d}$$

$$\sigma_{zz} = \sum_{j=1}^{3} w_{1j} A_j \log r_j, \tag{49e}$$

$$\sigma_{zx} = -\sum_{j=1}^{3} s_j w_{1j} A_j \tan^{-1}\frac{x}{z_j}. \tag{49f}$$

The above results are similar as those obtained by Hou et al. [13].

Case II: *Negligible void and thermal effects*: In the absence of void and thermal effects, we also obtain the corresponding results for orthotropic elastic medium.

6 Concluding remarks

The Fundamental solution for two-dimensional problem in orthotropic thermoelastic media with voids has been derived.After applying the dimensionless quantities and using the operator theory, the two-dimensional general solution in orthotropic thermoelastic media with voids is derived firstly. On the basis of general solution,the Fundamental solution for a steady point heat source on the surface of a semi-infinite orthotropic thermoelastic material with voids is constructed by six newly introduced harmonic functions. The components of displacement, stress, temperature change and voids are expressed in terms of elementary functions. From the present investigation, some special cases of interest are also deduced and compared with the previous results obtained. Since all the components are expressed in terms of elementary functions, it is convenient to use them.This type of solution technique is very useful in many theories of thermoelastic that is micropolar thermoelastic, microstretch thermoelastic, micropolar thermoelastic material with voids etc. The solution of these types of problem has not been discussed so far in the literature.

References

[1] I. Freedholm, Sur Les Equations de L'EquilbreD'um Crops SolideElastique, Acta Mathe.,23(1900), 1-42.

[2] I.M. Lifshitz, L.N. Rozentsveig, Contruction of Green tensor for the fundamental equation of elasticity theory in the case of an unbounded elastically anisotropic medium, Zhurnal Eksperimental,noi I Teioretical-Fiziki, 17 (1947), 783-791.

[3] H.A. Elliott, Three dimensional stress distribution in hexagonal aeolotropic crystals, Proc. Cambridge Philos.Soc.,44(1948),522-533.

[4] E. Kröner, Das fundamental integral der anisotropen elastischen differential gleichungen, Z.Phys.136(1953), 402-410.

[5] J.L. Synge, The Hyper circle in Mathematical Physics,Cambridge University Press, London, UK, 1957.

[6] J. Lejcek , The Green's function of the theory of elasticity in an isotropic hexagonal medium, Czech. J. Phys., B19 (1969), 799-803.

[7] Y.C. Pan, T.W. Chou, Point forces solution for an infinite transversely isotropic solid, ASME J. Appl. Mech., 43(1976), 608-612.

[8] E. Pan, F.G. Yuan, Three dimensional Green's function in anisotropic biomaterials, Int. J. Solid Struct., 37(2000), 5329-5351.

[9] B. Sharma, Thermal stresses in transversely isotropic semi-infinite elastic solids, ASME J. Appl. Mech., 23(1958), 86-88.

[10] [10] W.Q. Chen, H.J. Ding, D.S. Ling, Thermoelastic field of transversely isotropic elastic medium containing a penny-shaped crack: exact fundamental solution, int. J. Solid Struct., 41(2004), pp.69-83.

[11] P.F. Hou, A.Y.T. Leung, C.P. Chen, Green functions for semi-infinite transversely isotropic thermoelastic materials, ZAMM Z. Angew. Math. Mech., 1(2008), 33-41.

[12] P.F. Hou, L. Wang, T. Yi, 2D Green's functions for semi-infinite orthotropic thermoelastic plane, Appl. Math. Model., 33(2009),1674-1682.

[13] P.F. Hou, H. Sha, C.P. Chen, 2D general solution and fundamental solution for orthotropic thermoelastic materials,Engineer. Anal.Bound. Elem., 35 (2011), 56-60.

[14] Iesan, D., A theory of thermoelastic materials with voids, Acta Mechanica, 60(1986)67-89.

[15] Dhaliwal, R.S. and Wang, J., A heat flux dependent theory of themoelastic with voids,Acta Mechanica, 110(1995)33-39.

[16] Chirita, S. and Scalia, A., On the spatial and temporial behavior in linear thermoelastic material with voids, Journal of Thermal Stresses 24 (2001),433-455.

[17] Pompei,A. and Scalia, A., On the asymptotic spatial behavior in the linear themoelasticity of materials with voids,Journal of Thermal Stresses 25 (2002),183-193.

[18] Scalia A, Pompei A and Chirita, S., On the behavior of steady time harmonic oscillations thermoelastic materials with voids,Journal of Thermal Stresses 27 (2004),pp.209-226.

[19] Singh, J. and Tomar, S.K., Plane waves in thermoelastic material with voids, Mechanics of material, 39(2007) 932-940.

[20] Kumar, R. and Chawla, V., A study of fundamental solution in orthotropic thermodiffusive elastic media, International Communication in Heat and Mass Transfer 27 (2011),456-462.

[21] Kumar, R. and Chawla, V., Green's functions in orthotropic thermoelstic diffusion media, Engineering Analysis with Boundary Elements, 36(2012),1272-1277.

[22] General Steady-State Solution and Green's Function in Orthotropic Piezothermoelastic Diffusion Medium, Archives of Mechanics, 64 (2012)555-579.

[23] Fundamental Solution for Two-Dimensional Problem in Orthotropic Piezo thermoelastic Diffusion Media, Material Physics and mechanics, 16(2013) 159-174.

[24] Ding, H.J, Chen, B., and Liang, J., General solutions for coupled equations in piezoelectric media; Int. J. solids Struct., 33(1996) 2283-2298.

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Opšte i osnovno rešenje dvodimenzionog problema ortotropne termoelastične sredine sa šupljinama

Proučava se osnovno rešenje ortotropne termoelastične sredine sa šupljinama. Uz ovaj cilj, pre svega je izvedeno dvodimenzionalno opšte rešenje ortotropne termoelastične sredine sa šupljinama. Na osnovu opšteg rešenja, osnovno rešenje za stalan tačkasti izvor toplote na površini polu-beskonačne ortotropne termoelastične sredine sa šupljinama je izgradjeno pomoću šest novouvedenih harmonijskih funkcija. Raspored temperature i šupljina kao i komponente pomeranja i napona su izražene elementarnim funkcijama. Iz prikazanog istraživanja jedan poseban interesantan slučaj je takodje zaključen i uporedjen sa prethodno dobijenim rezultatima. Budući da su sve komponente izražene elementarnim funkcijama, podesne su za korišćenje.

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