Theoret.Appl.Mech. TEOPM7, Vol.41, No.4, pp. 267–282, Belgrade 2014 *

SINGULARITIES INTERACTING WITH INTERFACES INCORPORATING SURFACE ELASTICITY UNDER PLANE STRAIN DEFORMATIONS

Xu Wang **Peter Schiavone**

*doi:10.2298/TAM1404267W Math. Subj. Class.: 74A50; 74E05; 74M25.

According to: *Tib Journal Abbreviations (C) Mathematical Reviews*, the abbreviation TEOPM7 stands for TEORIJSKA I PRIMENJENA MEHANIKA.

Theoret. Appl. Mech., Vol.41, No.4, pp. 267–282, Belgrade 2014

Singularities interacting with interfaces incorporating surface elasticity under plane strain deformations

Xu Wang* **Peter Schiavone†**

Abstract

We consider problems involving singularities such as point force, point moment, edge dislocation and a circular Eshelby's inclusion in isotropic bimaterials in the presence of an interface incorporating surface/interface elasticity under plane strain deformations and derive elementary solutions in terms of exponential integrals. The surface mechanics is incorporated using a version of the continuum-based surface/interface model of Gurtin and Murdoch. The results indicate that the stresses in the two half-planes are dependent on two interface parameters.

Keywords:Plane-strain deformation; Surface elasticity; Bimaterial interface; Exponential integrals; Singularity

1 Introduction

Solutions to problems involving pointwise singularities (for example, point force, point moment, edge dislocation and a circular Eshelby inclusion) interacting with a material interface are fundamental to the development of theories in micromechanics [14, 20, 21]. The majority of previous studies have assumed that the interface is perfect so that tractions and displacements are continuous across the material interface. For nanoscaled structures with high surface to volume ratios, however, the assumption of a perfect interface

*School of Mechanical and Power Engineering, East China University of Science and Technology, 130 Meilong Road, Shanghai 200237 China, e-mail: xuwang@ecust.edu.cn

†Department of Mechanical Engineering, University of Alberta, 4-9 Mechanical Engineering Building Edmonton, Alberta Canada T6G 2G8, e-mail: p.schiavone@ualberta.ca

is insufficient and surface or interface elasticity must be taken into account [1, 10, 18]. The theory of surface elasticity was established on the basis of rational continuum mechanics by Gurtin, Murdoch and co-workers [5]-[7]. Recently, the Gurtin-Murdoch model was further clarified and developed by Ru [17]. This model has been successfully used to study various defect problems in nanostructured systems (see, for example, [3, 8],[9]-[13],[18, 19],[22]-[26]). Most recent advances in this particular surface theory can be found in [2, 4, 15].

In this work, we endeavor to study the plane elasticity problem associated with a singularity of arbitrary type interacting with a bimaterial interface incorporating surface/interface elasticity. The surface mechanics is incorporated using a version of the surface/interface model of Gurtin and Murdoch. An elementary and elegant solution to the interaction problem in terms of exponential integrals is derived. Three special cases are discussed in detail.

2 Coupled bulk-surface/interface elasticity

2.1 The bulk elasticity

We assume that subscripts *i,j,k = 1,2,3* and we sum over repeated indices. In the absence of body forces, the equilibrium equations and the constitutive relations describing the deformations of a linearly elastic, homogeneous and isotropic bulk solid are given by

$$\sigma_{ij,j} = 0,\ \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij},\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1}$$

where λ and μ are the Lame constants of the material, σ_{ij} and ε_{ij} are the components of the stress and strain tensors, respectively, u_i is the ith component of the displacement vector $\mathbf{u}$ in $\Re^3$ and δ_{ij} is the Kronecker delta.

For plane-strain deformations of an isotropic elastic material, the nontrivial stresses, displacements and stress functions $\varphi_1,\ \varphi_2$ can be expressed in terms of two analytic functions $\phi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + ix_2$ as [16]

$$\begin{aligned}
&\sigma_{11} + \sigma_{22} = 2\left[\phi'(z) + \overline{\phi'(z)}\right],\\
&\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right],\\
&2\mu(u_1 + iu_2) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)},\\
&\varphi_1 + i\varphi_2 = i\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right],
\end{aligned} \tag{2}$$

where $\kappa = \frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\nu$ with ν $(0 \leq \nu \leq 1/2)$ being the Poisson's ratio. In addition, the stresses are related to the stress functions through [21]

$$\begin{aligned} \sigma_{11} &= -\varphi_{1,2},\ \sigma_{12} = \varphi_{1,1}, \\ \sigma_{21} &= -\varphi_{2,2},\ \sigma_{22} = \varphi_{2,1}. \end{aligned} \tag{3}$$

For the boundary value problem discussed in this work, it is more convenient to use the following two analytic functions [20]

$$\Phi(z) = \phi'(z),\ \Omega(z) = [z\phi'(z) + \psi(z)]'. \tag{4}$$

2.2 The surface/interface elasticity

The equilibrium conditions on the interface incorporating interface/surface elasticity can be expressed [5]-[7],[17]:

$$\begin{aligned} &[\sigma_{\alpha j} n_j \underline{e}_\alpha] + \sigma^s_{\alpha\beta,\beta}\underline{e}_\alpha = 0, \quad \text{(tangential direction)} \\ &[\sigma_{ij} n_i n_j] = \sigma^s_{\alpha\beta}\kappa_{\alpha\beta}, \qquad\quad \text{(normal direction)} \end{aligned} \tag{5}$$

where $\alpha,\ \beta$=1,3; n_i is the unit normal vector to the interface, $[*]$ denotes the jump of the quantities across the interface, $\sigma^s_{\alpha\beta}$ is the surface stress tensor and $\kappa_{\alpha\beta}$ is the curvature tensor of the surface. In addition, the constitutive equations on the isotropic interface are given by

$$\sigma^s_{\alpha\beta} = \sigma_0\delta_{\alpha\beta} + 2(\mu^s - \sigma_0)\varepsilon^s_{\alpha\beta} + (\lambda^s + \sigma_0)\varepsilon^s_{\gamma\gamma}\delta_{\alpha\beta} + \sigma_0(\nabla_s \mathbf{u})_{\alpha\beta}, \tag{6}$$

where $\varepsilon^s_{\alpha\beta}$ is the surface strain tensor, σ_0 is the surface tension, λ^s and μ^s are the two surface Lame parameters and ∇_s is the surface gradient.

3 Singularities interacting with a bimaterial interface incorporating surface elasticity

We consider the plane-strain deformations of two bonded dissimilar isotropic half-planes. The upper half-plane $x_2 > 0$ is occupied by isotropic material 1 and the lower half-plane $x_2 < 0$ is occupied by a second isotropic material 2. A pointwise singularity is located at $z = s$, $Im\{s\} = d > 0$ in the upper half-plane. The singularity can be a point load, a point moment, an edge dislocation or a circular Eshelby inclusion [20]. Throughout the paper, the subscripts 1 and 2 (or the superscripts (1) and (2)) will be used to identify the respective quantities in the upper and lower half-planes. Our task below is to derive $\Phi_1(z), \Omega_1(z)$ defined in the upper half-plane and $\Phi_2(z), \Omega_2(z)$ defined in the lower half-plane.

By utilizing Eqs. (5) and (6) and assuming a coherent interface ($\varepsilon^s_{\alpha\beta} = \varepsilon^{(1)}_{\alpha\beta} = \varepsilon^{(2)}_{\alpha\beta}$), the boundary conditions on the bimaterial interface $x_2 = 0$ and $-\infty < x_1 < +\infty$ are given by

$$\left.\begin{aligned} &u_1^{(1)} + iu_2^{(1)} = u_1^{(2)} + iu_2^{(2)}, \\ &\sigma_{12}^{(1)} + i\sigma_{22}^{(1)} - (\sigma_{12}^{(2)}) + i\sigma_{22}^{(2)}) \\ &\qquad = -J_0 u_{1,11}^{(2)} - i\sigma_0 u_{2,11}^{(2)}, \end{aligned}\right\} \quad x_2 = 0, \ \ -\infty < x_1 < +\infty, \tag{7}$$

where $J_0 = \lambda^s + 2\mu^s - \sigma_0 \geq 0$ ([11]-[12]). The first condition in Eq. (7) indicates that the displacements are continuous across the interface, and the second condition in Eq. (7) can be equivalently expressed into

$$\begin{aligned} &\sigma_{12}^{(1)} + i\sigma_{22}^{(1)} - (\sigma_{12}^{(2)} + i\sigma_{22}^{(2)}) \\ &\qquad = -\frac{J_0 + \sigma_0}{2}(u_{1,11}^{(2)} + iu_{2,11}^{(2)}) - \frac{J_0 - \sigma_0}{2}(u_{1,11}^{(2)} - iu_{2,11}^{(2)}). \end{aligned} \tag{8}$$

The continuity condition of displacements across the bimaterial interface in Eq. $(7)_1$ can be expressed in terms of the four analytic functions in the bimaterial as

$$\frac{1}{2\mu_1}\left[\kappa_1 \Phi_1^+(z) - \bar{\Omega}_1^-(z)\right] = \frac{1}{2\mu_2}\left[\kappa_2 \Phi_2^-(z) - \bar{\Omega}_2^+(z)\right], \ Im\{z\} = 0. \tag{9}$$

It readily follows from the above expression that

$$\begin{aligned} \Phi_1(z) &= -\tfrac{\Gamma}{\kappa_1}\bar{\Omega}_2(z) + \Phi_0(z) + \tfrac{1}{\kappa_1}\bar{\Omega}_0(z), \\ \bar{\Omega}_1(z) &= -\Gamma\kappa_2\Phi_2(z) + \kappa_1\Phi_0(z) + \bar{\Omega}_0(z), \end{aligned} \tag{10}$$

where $\Gamma = \mu_1/\mu_2$ and $\Phi_0(z)$ and $\Omega_0(z)$ are the known complex potentials for a singularity located at $z = s, \ Im\{s\} = d > 0$ in an infinite homogeneous plane of material 1 and are specifically given by

$$\Phi_0(z) = \sum_{m=1}^{M} \frac{A_m}{(z-s)^m}, \quad \Omega_0(z) = \sum_{m=1}^{M} \frac{B_m}{(z-s)^m}, \tag{11}$$

where the coefficients A_m and B_m may depend on s [20]. The specific expressions of $\Phi_0(z)$ and $\Omega_0(z)$ for a point force, a point moment, an edge dislocation and a circular Eshelby inclusion are presented in [20].

The interface condition in Eq. (8) can also be expressed in terms of the four analytic functions in the bimaterial as

$$\begin{aligned} & i\left[\Phi_1^+(z)+\overline{\Omega}_2^-(z)\right]-i\left[\Phi_2^-(z)+\overline{\Omega}_2^+(z)\right] \\ & =-\frac{J_0+\sigma_0}{4\mu_2}[\kappa_2\Phi_2'-(z)-\overline{\Omega}_2'^+(z)] \\ & -\frac{J_0-\sigma_0}{4\mu_2}[\kappa_2\overline{\Phi}_2'^+(z)-\Omega_2'^-(z)], \\ & Im\{z\} \; = \; 0. \end{aligned} \tag{12}$$

Substituting Eq. (10) into Eq. (12) and eliminating $\Phi_1^+(z)$ and $\bar{\Omega}_1^-(z)$, we can obtain the following expression

$$\begin{aligned} & -i(\Gamma\kappa_2+1)\Phi_2^-(z)+\frac{\kappa_2(J_0+\sigma_0)}{4\mu_2}\Phi_2'^+\ (z) \\ & -\frac{J_0-\sigma_0}{4\mu_2}\Omega_2'^-(z)+i(1+\kappa_1)\Phi_0(z) \\ & =\frac{i(\Gamma+\kappa_1)}{\kappa_1}\overline{\Omega}_2^+(z)+\frac{J_0+\sigma_0}{4\mu_2}\overline{\Omega}_2'^+(z) \\ & -\frac{\kappa_2(J_0-\sigma_0)}{4\mu_2}\overline{\Phi}_2'^+(z)-\frac{i(1+\kappa_1)}{\kappa_1}\overline{\Omega}_0(z), \\ & Im\{z\}=0. \end{aligned} \tag{13}$$

By applying Liouville's theorem, we arrive at the following set of coupled first-order differential equations for $\Phi_2(z)$ and $\Omega_2(z)$

$$\begin{aligned} & \begin{bmatrix} \Gamma\kappa_2+1 & 0 \\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2} \end{bmatrix} \begin{bmatrix} \Phi_2(z) \\ \Omega_2(z) \end{bmatrix} \\ & +\frac{i}{4\mu_2}\begin{bmatrix} \kappa_2(J_0+\sigma_0) & -(J_0-\sigma_0) \\ -(J_0-\sigma_0) & \frac{J_0+\sigma_0}{\kappa_2} \end{bmatrix} \begin{bmatrix} \Phi_2'(z) \\ \Omega_2'(z) \end{bmatrix} \\ & =\begin{bmatrix} (1+\kappa_1)\Phi_0(z) \\ \frac{(1+\kappa_1)}{\kappa_1\kappa_2}\Omega_0(z) \end{bmatrix}, \\ & Im\{z\}<0. \end{aligned} \tag{14}$$

In order to solve the coupled differential equations in Eq. (14), we first consider the following eigenvalue problem:

$$\begin{bmatrix} \kappa_2(J_0+\sigma_0) & -(J_0-\sigma_0) \\ -(J_0-\sigma_0) & \frac{J_0+\sigma_0}{\kappa_2} \end{bmatrix} \mathbf{v} = 4d\mu_2\lambda \begin{bmatrix} \Gamma\kappa_2+1 & 0 \\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2} \end{bmatrix} \mathbf{v}, \tag{15}$$

where λ is the eigenvalue and $\mathbf{v}$ the associated eigenvector. It is pointed out that the 2×2 matrix $\begin{bmatrix} \kappa_2(J_0+\sigma_0) & -(J_0-\sigma_0) \\ -(J_0-\sigma_0) & \frac{J_0+\sigma_0}{\kappa_2} \end{bmatrix}$ is real, symmetric and positive semi-definite; the 2×2 matrix $\begin{bmatrix} \Gamma\kappa_2+1 & 0 \\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2} \end{bmatrix}$ is real, symmetric and positive definite. The two eigenvalues $\lambda_1,\ \lambda_2$ and the two eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2$ of Eq. (15) can be explicitly determined as

$$\begin{aligned} \lambda_1 &= \frac{[\kappa_2(\Gamma+\kappa_1)+\kappa_1(\Gamma\kappa_2+1)]\,(J_0+\sigma_0)+\sqrt{\Re}}{8d\mu_2(\Gamma+\kappa_1)(\Gamma\kappa_2+1)} \geq 0, \\ \lambda_2 &= \frac{[\kappa_2(\Gamma+\kappa_1)+\kappa_1(\Gamma\kappa_2+1)]\,(J_0+\sigma_0)-\sqrt{\Re}}{8d\mu_2(\Gamma+\kappa_1)(\Gamma\kappa_2+1)} \geq 0, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} J_0-\sigma_0 \\ \kappa_2(J_0+\sigma_0)-4d\mu_2\lambda_1(\Gamma\kappa_2+1) \end{bmatrix}, \\ \mathbf{v}_2 &= \begin{bmatrix} J_0-\sigma_0 \\ \kappa_2(J_0+\sigma_0)-4d\mu_2\lambda_2(\Gamma\kappa_2+1) \end{bmatrix}. \end{aligned} \tag{17}$$

where

$$\Re \equiv [\kappa_2(\Gamma+\kappa_1)-\kappa_1(\Gamma\kappa_2+1)]^2\,(J_0+\sigma_0)^2+4\kappa_1\kappa_2(\Gamma+\kappa_1)(\Gamma\kappa_2+1)(J_0-\sigma_0)^2.$$

It is seen from Eq. (16) that λ_1 and λ_2, ($\lambda_1 \geq \lambda_2 \geq 0$) are both dimensionless and are termed interface parameters which are controlled by the distance from the singularity to the bimaterial interface. The interface becomes perfect when $\lambda_1 = \lambda_2 \to 0$. In addition, the following orthogonal relationships can be simply derived from Eq. (15)

$$\begin{bmatrix}\mathbf{v}_1^T\\ \mathbf{v}_2^T\end{bmatrix}\begin{bmatrix}\Gamma\kappa_2+1 & 0\\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2}\end{bmatrix}\begin{bmatrix}\mathbf{v}_1\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}\delta_1 & 0\\ 0 & \delta_2\end{bmatrix}$$
$$\begin{bmatrix}\mathbf{v}_1^T\\ \mathbf{v}_2^T\end{bmatrix}\begin{bmatrix}\kappa_2(J_0+\sigma_0) & -(J_0-\sigma_0)\\ -(J_0-\sigma_0) & \frac{J_0+\sigma_0}{\kappa_2}\end{bmatrix}\begin{bmatrix}\mathbf{v}_1\mathbf{v}_2\end{bmatrix} \tag{18}$$
$$=4d\mu_2\begin{bmatrix}\delta_1\lambda_1 & 0\\ 0 & \delta_2\lambda_2\end{bmatrix}$$

where

$$\delta_1=\mathbf{v}_1^T\begin{bmatrix}\Gamma\kappa_2+1 & 0\\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2}\end{bmatrix}\mathbf{v}_1=(\Gamma\kappa_2+1)(J_0-\sigma_0)^2$$
$$+\frac{(\Gamma+\kappa_1)\left[\kappa_2(J_0+\sigma_0)-4d\mu_2\lambda_1(\Gamma\kappa_2+1)\right]^2}{\kappa_1\kappa_2}>0,$$
$$\delta_2=\mathbf{v}_2^T\begin{bmatrix}\Gamma\kappa_2+1 & 0\\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2}\end{bmatrix}\mathbf{v}_2=(\Gamma\kappa_2+1)(J_0-\sigma_0)^2 \tag{19}$$
$$+\frac{(\Gamma+\kappa_1)\left[\kappa_2(J_0+\sigma_0)-4d\mu_2\lambda_2(\Gamma\kappa_2+1)\right]^2}{\kappa_1\kappa_2}>0.$$

Now we introduce two new analytic functions $Y_1(z)$ and $Y_2(z)$ defined by

$$\begin{bmatrix}\Phi_2(z)\\ \Omega_2(z)\end{bmatrix}=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\begin{bmatrix}Y_1(z)\\ Y_2(z)\end{bmatrix}. \tag{20}$$

Substituting the above into Eq. (14), pre-multiplying both sides by $\begin{bmatrix}\mathbf{v}_1^T\\ \mathbf{v}_2^T\end{bmatrix}$ and utilizing the orthogonal relationships in Eq. (18), we can finally arrive at the following two decoupled first-order differential equations

$$Y_1(z)+id\lambda_1Y_1'(z)=\sum_{m=1}^{M}\frac{C_m}{(z-s)^m},$$
$$Y_2(z)+id\lambda_2Y_2'(z)=\sum_{m=1}^{M}\frac{D_m}{(z-s)^m}, \tag{21}$$

where the coefficients C_m and D_m are related to A_m and B_m through

$$\begin{aligned} C_m &= \frac{(1+\kappa_1)(J_0-\sigma_0)}{\delta_1} A_m \\ &\quad + \frac{(1+\kappa_1)\left[\kappa_2(J_0+\sigma_0)-4d\mu_2\lambda_1(\Gamma\kappa_2+1)\right]}{\delta_1\kappa_1\kappa_2} B_m, \\ D_m &= \frac{(1+\kappa_1)(J_0-\sigma_0)}{\delta_2} A_m \\ &\quad + \frac{(1+\kappa_1)\left[\kappa_2(J_0+\sigma_0)-4d\mu_2\lambda_2(\Gamma\kappa_2+1)\right]}{\delta_2\kappa_1\kappa_2} B_m. \end{aligned} \tag{22}$$

The solutions to Eq. (21) can be concisely expressed in terms of the exponential integrals as

$$\begin{aligned} Y_1(z) &= \sum_{m=1}^{M} \tfrac{i^m C_m}{(d\lambda_1)^m} \exp\left[\tfrac{i(z-s)}{d\lambda_1}\right] \left[\tfrac{i(z-s)}{d\lambda_1}\right]^{1-m} E_m\left[\tfrac{i(z-s)}{d\lambda_1}\right], \\ Y_2(z) &= \sum_{m=1}^{M} \tfrac{i^m D_m}{(d\lambda_2)^m} \exp\left[\tfrac{i(z-s)}{d\lambda_2}\right] \left[\tfrac{i(z-s)}{d\lambda_2}\right]^{1-m} E_m\left[\tfrac{i(z-s)}{d\lambda_2}\right], \end{aligned} \tag{23}$$

where the exponential integral $E_m(z)$ is defined by

$$E_m(z) = \int_1^\infty \frac{\exp(-zt)}{t^m}\mathrm{d}t = z^{m-1}\int_z^\infty \frac{\exp(-t)}{t^m}\mathrm{d}t, \ m \geq 1. \tag{24}$$

In addition, $E_m(z)$ satisfies the following recurrence relations

$$\begin{aligned} E'_m(z) &= -E_{m-1}(z), \\ mE_{m+1}(z) &= \exp(-z) - zE_m(z). \end{aligned} \tag{25}$$

It is seen from Eq. (23) that $E_m(z)$ should be an ingredient of the solution for a pole of order m. Consequently, the original four analytic functions $\Phi_2(z)$, $\Omega_2(z)$ defined in the lower half-plane and $\Phi_1(z)$, $\Omega_1(z)$ defined in the upper half-plane can be further obtained from Eqs. (10) and (20) as

$$\begin{aligned} \begin{bmatrix} \Phi_2(z) \\ \Omega_2(z) \end{bmatrix} &= \mathbf{v}_1 \sum_{m=1}^{M} \frac{i^m C_m}{(d\lambda_1)^m} \exp\left[\frac{i(z-s)}{d\lambda_1}\right] \left[\frac{i(z-s)}{d\lambda_1}\right]^{1-m} E_m\left[\frac{i(z-s)}{d\lambda_1}\right] \\ &\quad + \mathbf{v}_2 \sum_{m=1}^{M} \frac{i^m D_m}{(d\lambda_2)^m} \exp\left[\frac{i(z-s)}{d\lambda_2}\right] \left[\frac{i(z-s)}{d\lambda_2}\right]^{1-m} E_m\left[\frac{i(z-s)}{d\lambda_2}\right], \\ &\quad Im\{z\} < 0, \end{aligned} \tag{26}$$

$$
\begin{aligned}
\Phi_1(z) &= -\frac{\Gamma}{\kappa_1}\Bigg([\kappa_2(J_0+\sigma_0) - 4d\mu_2\lambda_1(\Gamma\kappa_2+1)] \\
&\times \sum_{m=1}^{M} \frac{-i^m \bar{C}_m}{(d\lambda_1)^m} \exp\left[\frac{-i(z-\bar{s})}{d\lambda_1}\right]\left[\frac{-i(z-\bar{s})}{d\lambda_1}\right]^{1-m} \\
&\times E_m\left[\frac{-i(z-\bar{s})}{d\lambda_1}\right] + [\kappa_2(J_0+\sigma_0) - 4d\mu_2\lambda_2(\Gamma\kappa_2+1)] \\
&\times \sum_{m=1}^{M} \frac{(-\mathrm{i})^m \bar{D}_m}{(d\lambda_2)^m} \exp\left[\frac{-i(z-\bar{s})}{d\lambda_2}\right]\left[\frac{-i(z-\bar{s})}{d\lambda_2}\right]^{1-m} \\
&\times E_m\left[\frac{-i(z-\bar{s})}{d\lambda_2}\right]\Bigg) + \frac{1}{\kappa_1}\bar{\Omega}_0(z) + \Phi_0(z),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\Omega_1(z) &= -\Gamma\kappa_2(J_0-\sigma_0)\Bigg(\sum_{m=1}^{M} \frac{(-\mathrm{i})^m \bar{C}_m}{(d\lambda_1)^m} \exp\left[\frac{-i(z-\bar{s})}{d\lambda_1}\right] \\
&\times \left[\frac{-i(z-\bar{s})}{d\lambda_1}\right]^{1-m} E_m\left[\frac{-i(z-\bar{s})}{d\lambda_1}\right] \\
&+ \sum_{m=1}^{M} \frac{(-\mathrm{i})^m \bar{D}_m}{(d\lambda_2)^m} \exp\left[\frac{-i(z-\bar{s})}{d\lambda_2}\right]\left[\frac{-i(z-\bar{s})}{d\lambda_2}\right]^{1-m} \\
&\times E_m\left[\frac{-i(z-\bar{s})}{d\lambda_2}\right]\Bigg) + \kappa_1\bar{\Phi}_0(z) + \Omega_0(z),
\end{aligned}
$$

$$Im(z) > 0.$$

It is verified that when $\lambda_1 = \lambda_2 \to 0$ for a perfect bimaterial interface ($J_0 = \sigma_0 = 0$) or for a singularity far from the interface ($d \to \infty$), the results in Eqs. (26) and (27) reduce to those by Suo [20]. The stress field can be determined by substituting the analytic functions $\Phi_2(z)$, $\Omega_2(z)$ and $\Phi_1(z)$, $\Omega_1(z)$ into the following

$$
\begin{aligned}
\sigma_{11} + \sigma_{22} &= 2\left[\Phi(z) + \overline{\Phi(z)}\right], \\
\sigma_{22} + i\sigma_{12} &= \overline{\Phi(z)} + \Omega(z) + (\bar{z} - z)\Phi'(z).
\end{aligned}
\tag{28}
$$

It is seen that the stress fields in the two half-planes are dependent on the two interface parameters λ_1 and λ_2.

4 Discussion of special cases

In this section, we discuss three special cases: (i) $\sigma_0 = 0$ and $J_0 > 0$; (ii) $J_0 = 0$ and $\sigma_0 > 0$; (iii) $J_0 = \sigma_0 > 0$.

Case 4.1. $\sigma_0 = 0$ and $J_0 > 0$

When the surface tension is zero ($\sigma_0 = 0$) and $J_0 > 0$, the two eigenvalues λ_1, λ_2 and the two eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ of Eq. (15) are now given by

$$\lambda_1 = \frac{J_0\left[\kappa_2(\Gamma+\kappa_1)+\kappa_1(\Gamma\kappa_2+1)\right]}{4d\mu_2(\Gamma+\kappa_1)(\Gamma\kappa_2+1)} > 0, \qquad \lambda_2 = 0, \tag{29}$$

and

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -\frac{\kappa_1(\Gamma\kappa_2+1)}{\Gamma+\kappa_1} \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ \kappa_2 \end{bmatrix}. \tag{30}$$

In addition, the following orthogonal relationships are valid

$$\begin{aligned} &\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \Gamma\kappa_2+1 & 0 \\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \\ &\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \kappa_2 J_0 & -J_0 \\ -J_0 & \frac{J_0}{\kappa_2} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1\mathbf{v}_2 \end{bmatrix} = 4d\mu_2 \begin{bmatrix} \delta_1\lambda_1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \delta_1 &= \mathbf{v}_1^T \begin{bmatrix} \Gamma\kappa_2+1 & 0 \\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2} \end{bmatrix} \mathbf{v}_1 \\ &= \frac{(\Gamma\kappa_2+1)\left[\kappa_1(\kappa_2+1)+\Gamma\kappa_2(\kappa_1+1)\right]}{\kappa_2(\Gamma+\kappa_1)} > 0, \\ \delta_2 &= \mathbf{v}_2^T \begin{bmatrix} \Gamma\kappa_2+1 & 0 \\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2} \end{bmatrix} \mathbf{v}_2 \\ &= \frac{\kappa_1(\kappa_2+1)+\Gamma\kappa_2(\kappa_1+1)}{\kappa_1} > 0. \end{aligned} \tag{32}$$

The following two decoupled first-order differential equations can then be finally derived

$$\begin{aligned} &Y_1(z) + id\lambda_1 Y_1'(z) = \sum_{m=1}^{M} \frac{C_m}{(z-s)^m}, \\ &Y_2(z) = \sum_{m=1}^{M} \frac{D_m}{(z-s)^m}, \end{aligned} \tag{33}$$

where $Y_1(z)$ and $Y_2(z)$ have been defined in Eq. (20), C_m and D_m are related to A_m and B_m through

$$\begin{aligned} C_m &= \tfrac{1+\kappa_1}{\delta_1}A_m - \tfrac{(1+\kappa_1)(\Gamma\kappa_2+1)}{\delta_1\kappa_2(\Gamma+\kappa_1)}B_m, \\ D_m &= \tfrac{1+\kappa_1}{\delta_2}A_m + \tfrac{1+\kappa_1}{\delta_2\kappa_1}B_m. \end{aligned} \tag{34}$$

In fact, Eq. (33) contains only a single differential equation for $Y_1(z)$ and the function $Y_2(z)$ is given simply by Eq. $(33)_2$ since $\lambda_2 = 0$. In this case, the stresses in the bimaterial are dependent on the surface parameter λ_1. Furthermore, if $\mu_1 = 0$ (or $\Gamma = 0$), λ_1 in Eq. (29) becomes

$$\lambda_1 = \frac{J_0(\kappa_2+1)}{4d\mu_2}, \tag{35}$$

which is found to be consistent with the result by Yoon et al. [27].

Case 4.2. $J_0 = 0$ and $\sigma_0 > 0$

When $J_0 = 0$ and $\sigma_0 > 0$, the two eigenvalues λ_1, λ_2 and the two eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ of Eq. (15) are

$$\lambda_1 = \frac{\sigma_0\left[\kappa_2(\Gamma+\kappa_1)+\kappa_1(\Gamma\kappa_2+1)\right]}{4d\mu_2(\Gamma+\kappa_1)(\Gamma\kappa_2+1)} > 0,\ \lambda_2 = 0, \tag{36}$$

and

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{\kappa_1(\Gamma\kappa_2+1)}{\Gamma+\kappa_1} \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\kappa_2 \end{bmatrix}. \tag{37}$$

In addition, the following orthogonal relationships hold

$$\begin{aligned} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \Gamma\kappa_2+1 & 0 \\ 0 & \frac{\Gamma+\kappa_1}{\kappa_1\kappa_2} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} &= \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \\ \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \kappa_2\sigma_0 & \sigma_0 \\ \sigma_0 & \frac{\sigma_0}{\kappa_2} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} &= 4d\mu_2 \begin{bmatrix} \delta_1\lambda_1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{38}$$

where the expressions of δ_1 and δ_2 are identical to those in Eq. (32).

The form of the resulting two decoupled first-order differential equations is similar to Eq. (33). Now C_m and D_m should be redefined as

$$\begin{aligned} C_m &= \tfrac{1+\kappa_1}{\delta_1}A_m + \tfrac{(1+\kappa_1)(\Gamma\kappa_2+1)}{\delta_1\kappa_2(\Gamma+\kappa_1)}B_m, \\ D_m &= \tfrac{1+\kappa_1}{\delta_2}A_m - \tfrac{1+\kappa_1}{\delta_2\kappa_1}B_m. \end{aligned} \tag{39}$$

In this case, the stresses in the bimaterial are also dependent on the surface parameter λ_1.

Case 4.3. $J_0 = \sigma_0 > 0$

In the case of $J_0 = \sigma_0 > 0$, the two differential equations in Eq. (14) are in fact decoupled and can be explicitly written into

$$\begin{aligned}
&\Phi_2(z) + \frac{i\kappa_2\sigma_0}{2\mu_2(\Gamma\kappa_2+1)}\Phi_2'(z) = \frac{1+\kappa_1}{\Gamma\kappa_2+1}\Phi_0(z),\\
&\Omega_2(z) + \frac{i\kappa_1\sigma_0}{2\mu_2(\Gamma+\kappa_1)}\Omega_2'(z) = \frac{1+\kappa_1}{\Gamma+\kappa_1}\Omega_0(z),\\
&Im\{z\} < 0.
\end{aligned} \tag{40}$$

Thus the two interface parameters λ_1 and λ_2 are

$$\begin{aligned}
\lambda_1 &= \max\left\{\frac{\kappa_2\sigma_0}{2d\mu_2(\Gamma\kappa_2+1)}, \frac{\kappa_1\sigma_0}{2d\mu_2(\Gamma+\kappa_1)}\right\} > 0,\\
\lambda_2 &= \min\left\{\frac{\kappa_2\sigma_0}{2d\mu_2(\Gamma\kappa_2+1)}, \frac{\kappa_1\sigma_0}{2d\mu_2(\Gamma+\kappa_1)}\right\} > 0.
\end{aligned} \tag{41}$$

5 Conclusions

In this work, we address the interaction problem associated with a singularity near a bimaterial interface under plane strain deformations. In contrast to previous studies, the surface elasticity of the interface is incorporated via a version of the Gurtin-Murdoch model. The original boundary value problem is reduced to a set of two coupled first-order differential equations in Eq. (14) for the two analytic functions $\Phi_2(z)$ and $\Omega_2(z)$ defined in the lower half-plane which is free of the action of the singularity. Two independent first-order differential equations have been obtained in Eq. (21) by using a decoupling strategy, and are solved analytically using exponential integrals $E_m(z),\ m \geq 1$.

The obtained solutions can be further used to study, for example, a crack with surface elasticity interacting with a bimaterial interface which also incorporates the effects of surface elasticity. In this case, both edge dislocation and point force solutions are needed to simulate the crack [26].

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No: 11272121) and through a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (Grant No. RGPIN 155112).

References

[1] T. Chen, G.J. Dvorak and C.C Yu, *Size-dependent elastic properties of unidirectional nano-composites with interface stresses.* Acta Mech. 188, (2007), 39-54.

[2] P. Chhapadia, P. Mohammadi and P. Sharma, *Curvature-dependent surface energy and implications for nanostructures.* J. Mech. Phy. Solids 59, (2011), 2103-2115.

[3] H.L. Duan, J. Wang, Z.P. Huang and B.L. Karihaloo, *Size-dependent effective elastic constants of solids containing nano-inhomogeneities with interface stress.* J. Mech. Phys. Solids 53, (2005),1574-1596.

[4] X. Gao, Z. Huang, J. Qu and D. Fang, *A curvature-dependent interfacial energy-based interface stress theory and its applications to nano-structured materials:(I) General theory.* J. Mech. Phy. Solids 66, (2014), 59-77.

[5] M.E. Gurtin and A. Murdoch, *A continuum theory of elastic material surfaces. Arch. Ration. Mech. An.* 57, (1975), 291-323.

[6] M.E. Gurtin and A. Murdoch, *Surface stress in solids.* Int. J. Solids Struct. 14, (1978), 431-440.

[7] M.E. Gurtin, J. Weissmuller and F. Larche, *A general theory of curved deformable interface in solids at equilibrium*, Philos. Mag. A 78, (1998), 1093-1109.

[8] Z.P. Huang and L. Sun, *Size-dependent effective properties of a heterogeneous material with interface energy effect: from finite deformation theory to infinitesimal strain analysis.* Acta Mechanica 182, (2006), 195-210.

[9] C.I. Kim, C.Q Ru and P. Schiavone, *A clarification of the role of crack-tip conditions in linear elasticity with surface effects.* Math. Mech. Solids 18, (2013), 59−66.

[10] C.I. Kim, P. Schiavone and C.Q Ru, *The effects of surface elasticity on an elastic solid with mode-III crack: complete solution.* ASME J. Appl. Mech. 77, (2010), 021011-1−021011-7.

[11] C.I. Kim, P. Schiavone and C.Q Ru, *Analysis of plane-strain crack problems (mode I and mode II) in the presence of surface elasticity.* J. Elasticity 104, (2011) 397−420.

[12] C.I. Kim, P. Schiavone and C.Q Ru, *Effect of surface elasticity on an interface crack in plane deformations.* Proc. Roy. Soc. Lond. A 467, (2011) 3530-3549.

[13] C.I. Kim, P. Schiavone and C.Q. Ru, *The effect of surface elasticity on mode-III interface crack.* Arch. Mech. 63, (2011), 267−286.

[14] V.A. Lubarda, Energy analysis of dislocation arrays near bimaterial interfaces, Int. J. Solids Struct., 34, (1997) 10531073.

[15] P. Mohammadi, L.P. Liu, P. Sharma and R.V. Kukta, *Surface energy, elasticity and the homogenization of rough surfaces.* J. Mech. Phy. Solids, 61, (2013), 325-340.

[16] N.I. Muskhelishvili, N.I., Some Basic Problems of the Mathematical Theory of Elasticity. P. Noordhoff Ltd., Groningen, 1953.

[17] C.Q. Ru, *Simple geometrical explanation of Gurtin-Murdoch model of surface elasticity with clarification of its related versions.* Sci. China 53, (2010), 536-544.

[18] P. Sharma and S. Ganti, *Size-dependent Eshelby's tensor for embedded nano-inclusions incorporating surface/interface energies.* ASME J. Appl. Mech. 71, (2004), 663-671.

[19] P. Sharma, S. Ganti and N. Bhate, N., *Effect of surfaces on the size-dependent elastic state of nanoinhomogeneities.* Appl. Phys. Lett. 82, (2003), 535-537.

[20] Z. Suo, *Singularities interacting with interfaces and cracks.* Int. J. Solids Struct. 25, (1989), 1133-1142.

[21] T.C.T. Ting, Anisotropic Elasticity-Theory and Applications. Oxford University Press, New York 1996.

[22] G.F. Wang and T.J. Wang, Deformation around a nanosized elliptical hole with surface effect. Appl. Phys. Lett. 89, (2006), 161901.

[23] X. Wang and P. Schiavone, *Interaction of a screw dislocation with a nano-sized arbitrary shaped inhomogeneity with interface stresses under anti-plane deformations* Proc. Roy. Soc. Lond. A (2014) (In press).

[24] X. Wang and P. Schiavone, *A screw dislocation interacting with an anisotropic elliptical nano-inhomogeneity with interface stresses in anti-plane elasticity* J. Elasticity, (2014), (In press).

[25] X. Wang and P. Schiavone, *Two circular inclusions with arbitrarily varied surface effects* Acta Mech., (2014), (In press).

[26] X. Wang and P. Schiavone, *Interaction between an edge dislocation and a crack with surface elasticity* ASME J. Appl. Mech., **82**, 021006-1 - 021006-8 Feb 2015.

[27] J. Yoon, C.Q. Ru and A. Mioduchowski, *Effect of a thin surface coating layer on thermal stresses within an elastic half-plane.* Acta Mech. 185, (2006), 227−243.

Submitted in December 2014.

Medjudejstvo singulariteta i medjupovrši koje uključuje elastičnost površi pri ravanskoj deformaciji

Razmatramo probleme koji uključuju singularitete kao što su: koncentrisane sile, koncentrisane momente, ivičnu dislokaciju i kružni Eshelby-jev uključak u izotropnim dvomaterijalima u prisustvu medjupovrši uključujućo elastičnost površi / medjupovrši pri ravanskoj deformaciji. Pritom izvodimo elementarna rešenja eksponencijalnim integralima. Mehanika površi je ugradjena koristeći model površi / medjupovrši u kontinuumu zasnovanog Gurtin-om i Murdoch-om. Rezultati pokazuju da naponi u dve poluravni zavise od dva parametra medjupovrši.

doi:10.2298/TAM1404267W Math. Subj. Class.: 74A50; 74E05; 74M25.