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Dissipation of general viscous fluid distribution in Einstein and Barber theories

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Abstract

An attempt has been taken to investigate the problem of general viscous fluid distribution in the space-time governed by the metric $ds^2 = dt^2 - dx^2 - dy^2 + f(t-x, y, z)(dt - dx)^2$ in both the theories of gravitation proposed by Einstein (1915) and Barber (1982). It is observed that in both the theories the field equations are reducible to Laplace equation and viscous fluid distribution does not survive. Moreover, the vacuum models can be constructed by an arbitrary harmonic function in y and z coordinates and the solutions governing the models represent plane gravitational wave propagating in positive x-direction.

1 Introduction

The viscosity mechanism in cosmology can account for high entropy of the present universe (Weinberg, 1971, 1972). Bulk viscosity associated with the grand unified theory (phase transition) may lead to inflationary cosmology, which is used to overcome lacunae of several important problems in the standard big bang cosmology. There are several processes, which are expected to give rise to viscous effects.

These are the decoupling of neutrinos during the radiation era and the decomposition of matter and radiation during the recombination era.

Murphy (1973) shown that the big bang singularity can be avoided by the introduction of bulk viscosity. Banerjee and Santos (1984) studied the viscous and non-viscous fluids in Bianchi type II, VIII and IX space - times under the restriction that the ratio of shear to expansion is constant. Mohanty and Pradhan (1990) investigated the problem of interactions of a gravitational field with bulk viscous fluid in Robertson - Walker space - time. Mohanty and Pattanaik (1991) also studied the anisotropic cosmological models with constant bulk viscous coefficient.

In this paper, we have taken an attempt to study the interacting viscous fluid containing both bulk and shear viscous coefficients in both the theories proposed by Einstein and Barber. Einstein's field equations are derived in section 2, and their vacuum solution is obtained in section 3. In section 4, Barber's field equations are set up and their vacuum solution is derived in section 4. In section 5, some remarks on the solutions are given.

2 Einstein field equations

The metric considered here can be put in the form

$$ds^2 = (1+f)dt^2 - (1-f)dx^2 - 2f dt dx - dy^2 - dz^2, \tag{1}$$

where $f = f(t-x, y.z)$.

The energy-momentum tensor of general viscous fluid is given by

$$T_{ij} = (\rho + \bar{p})v_i v_j - \bar{p} g_{ij} + \eta_s u_{ij}, \tag{2a}$$

$$\bar{p} = p - \left(\eta_b - \frac{2}{3}\eta_s\right) v^a_{;a}, \tag{2b}$$

$$v_i v^i = 1, \tag{2c}$$

$$u_{ij} = v_{;ij} + v_{;ji} - v_i v^a v_{j;a} - v_j v^a v_{i;a} \tag{2d}$$

where ρ is the matter density, p the pressure, v^i the four velocity, and η_b and η_s are the bulk and shear viscosity coefficients respectively. Here semicolon represents covariant derivative with respect to g_{ij}.

In comoving coordinate frame i.e. $v_i = \left(0, 0, 0, \frac{1}{\sqrt{1-f}}\right)$, the Einstein′s field equations

$$R_{ij} - \frac{1}{2}Rg_{ij} + \lambda g_{ij} = -kT_{ij} \tag{3}$$

with (1) and (2) are obtained as

$$\eta_s f_2 = 0, \tag{4}$$

$$\eta_s f_3 = 0, \tag{5}$$

$$\bar{p} = \frac{\lambda}{k}, \tag{6}$$

$$\frac{1}{2}(f_{22} + f_{33}) + \lambda(1-f) = k\left[\bar{p}(1-f) - \eta_s \frac{f_1}{\sqrt{(1-f)}}\right],$$

$$\frac{1}{2}(f_{22} + f_{33}) - \lambda f = k\left[\bar{p}f - \eta_s \frac{f f_1}{2(1-f)^{3/2}}\right], \tag{8}$$

and

$$\frac{1}{2}(f_{22} + f_{33}) - \lambda(1+f) = k\left[(p+\bar{p})\frac{1}{1-f} - \bar{p}(1-f)\right] \tag{9}$$

for a non-zero field.

Here after wards the subscripts 1,2,3 and 4 after a field variable represent partial derivative with respect to x, y, z and t respectively.

Now (4) and (5) yield the following two cases:

CaseI, $\eta_s = 0$:

In this case, the set of field equations become

$$f_{22} + f_{33} = 0 \tag{10}$$

$$\rho = -\bar{p} = -\frac{\lambda}{k}, \quad for \quad f \neq 1 \tag{11}$$

CaseII, $f_2 = 0 = f_3$:
In this case, the set of field equations reduce to

$$\eta_s = 0 \tag{12}$$

$$\rho = -\bar{p} = -\frac{\lambda}{k}, \quad for \quad f \neq 1 \tag{13}$$

The energy conservation equations $T^{ij}_{;j} = 0$ for the metric (1) take the form

$$\bar{p} = 0 \tag{14}$$

Subsequently (11) and (13) yield

$$\rho = 0 = \lambda$$

Further in both the cases,

$$\bar{p} = p + \eta_b \frac{f_1}{2\sqrt{1-f}} = 0 \quad \text{which implies} \quad p = 0 = \eta_b.$$

Thus in both the aforesaid cases, the cosmological constant λ and viscous fluid do not survive. The first case leads to pure vacuum case governed by a harmonic function in y and z coordinates (Mohanty and Mishra, 2000) whereas in the second case even though the metric potential is an arbitrary function of $t - x$, the vacuum field equations are identically satisfied.

3 Einstein vacuum solutions

It is evident that the field (10) being Laplace equation, admits the following solutions where the field f assumed to be a separable function of $t - x, y$ and z in the form

$$f(t - x, y, z) = h(t - x)Y(y)Z(z).$$

Immediately one gets from (10)

$$\frac{Y''}{Y} = \frac{Z''}{Z} = M(const)$$

which yields the following three cases corresponding to tree different vacuum models:

CaseI, $M > 0$:

In this case, (10) admits a solution

$$f = h(t-x)(a_1 e^{py} + a_2 e^{-py})(a_3 \cos pz + a_4 \sin pz) \tag{15}$$

where $M = p^2$, and a_i, $i = 1, 2, 3, 4$ are constants of integration.

CaseII, $M < 0$:

In this case, the solution of (10) can be derived as

$$f = h(t-x)(b_1 \cos py + b_2 \sin py)(b_3 e^{pz} + b_4 e^{-pz}) \tag{16}$$

where $M = -p^2$, and b_i, $i = 1, 2, 3, 4$ are constants of integration.

CaseIII, $M = 0$:

In this case, the solution of (10) can be derived as

$$f = h(t-x)(c_1 y + c_2)(c_3 z + c_4) \tag{17}$$

where c_i, $i = 1, 2, 3, 4$ are constants of integration.

So, for the Einstein vacuum model, the metric potential $f(t-x, y, z)$ takes one of the forms given in (15), (16) and (17) referred to as models I, II and III respectively.

If f is assumed in the form $f = h(t-x) \pm Y(y)Z(z)$, then we get analogous solution as obtained above.

Further we assume the field

$$f = f(t-x, \varphi(r)) \tag{18}$$

in a separable form $f = h(t-x)\varphi(r)$ where φ depends only on the distance $r(\neq 0)$ of an arbitrary event $(t-x, y, z)$ from a fixed event $\xi = (t-x, y_1, z_1)$ in (y, z)-plane.

So, $r = \sqrt{(y-y_1)^2 + (z-z_1)^2}$.

Thus from (10), we have

$$\varphi'' + \frac{1}{r}\varphi' = 0 \tag{19}$$

which on two fold integrations yields

$$\varphi(r) = c_5 \log r + c_6 \tag{20}$$

where c_5 and c_6 are integration constants.

So, the metric potential becomes

$$f(t-x, y, z) = h(t-x)(c_5 \log r + c_6). \tag{21}$$

4 Barber's field equations and their consequences

The field equations in Barber's (1982) second self-creation theory are

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi\phi^{-1} T_{ij} \tag{22}$$

$$\Box\phi = \frac{8\pi\lambda}{3} T \tag{23}$$

where λ is a coupling to be determined from experiments, ϕ is a Barber's scalar and T is the trace of energy-momentum tensor.

By taking the help of (2), the Einstein-Barber's field equations (22) and (23) for the metric (2) in the comoving coordinate system can be written in the following explicit form:

$$\phi^{-1}\left[\frac{\eta_s f f_2}{2(1-f)^{3/2}}\right] = 0 \tag{24}$$

$$\phi^{-1}\left[\frac{\eta_s f f_3}{2(1-f)^{3/2}}\right] = 0 \tag{25}$$

$$\phi^{-1}\bar{p} = 0 \tag{26}$$

$$\frac{1}{2}(f_{22} + f_{33}) = 8\pi\phi^{-1}\left[\bar{p}(1-f) - \eta_s \frac{f_1}{\sqrt{1-f}}\right] \tag{27}$$

$$\frac{1}{2}(f_{22}+f_{33}) = -8\pi\phi^{-1}\left[\bar{p}f - \eta_s \frac{ff_1}{2(1-f)^{3/2}}\right] \tag{28}$$

$$\frac{1}{2}(f_{22}+f_{33}) = 8\pi\phi^{-1}\left[(\rho+\bar{p})\frac{1}{1-f} - \bar{p}(1-f)\right] \tag{29}$$

and

$$\Box\phi = -(1+f)\phi_{11} - \phi_{22} - \phi_{33} + (1-f)\phi_{44} - 2f\phi_{14} = \frac{8\pi\lambda}{3}(\rho - 3\bar{p}) \tag{30}$$

As before for non-zero field and finite Barber's scalar (24) and (25) yield the following two cases:

CaseI, $\eta_s = 0$:

In this case the field equations (27)-(30) reduce to

$$f_{22} + f_{33} = 0 \tag{31}$$

$$\rho = 0 \tag{32}$$

and

$$(1+f)\phi_{11} + \phi_{22} + \phi_{33} - (1-f)\phi_{44} + 2f\phi_{14} = 0 \tag{33}$$

Now from (2b) and (26), we have

$$\bar{p} = \rho + \eta_b \frac{f_1}{2\sqrt{1-f}} = 0,$$

which on physical ground yields

$$p = 0 = \eta_b.$$

Thus the viscous fluid distribution involving bulk and shear coefficients does not survive in Barber's theory when the space - time is described by (1).

CaseII, $f_2 = 0 = f_3$:

In this case, the field equations are identically satisfied and the viscous fluid does not survive i.e.

$$p = \rho = \eta_s = \eta_b = 0,$$

even though the field is an arbitrary function of $t - x$.

5 Barber's vacuum solution

If the functional form of ϕ can be assumed as that of f i.e.

$$\phi = \phi(t - x, y, z),$$

then (33) yields

$$\phi_{22} + \phi_{33} = 0 \tag{34}$$

In view of the interaction of Barber's scalar with the gravitational field, it can be assumed that

$$f = F(\phi) \tag{35}$$

Then (31) leads to

$$f_{22} + f_{33} = F''(\phi_2^2 + \phi_3^2) + F'(\phi_{22} + \phi_{33}) = 0 \tag{36}$$

which yields the following four cases:

CaseI, $F''= 0, F'= 0$:

In this case, we get

$$f = F(\phi) = const.$$

Thus the Barber's scalar is arbitrary whereas space-time is flat. Thus it leads to inconsistency in Barber's theory.

CaseII, $\phi_{22}+\phi_{33}= 0,\ F''= 0$:

In this case, we have

$$f = a\phi + b,$$

where a and b are constants of integrations, and both f and ϕ are harmonic functions (in y and z coordinates) whose explicit functional forms can be determined as before. Hence any harmonic function in y and z coordinates can generate Einstein-Barber's vacuum model in the space-time (1).

CaseIII, $\phi_2^2+\phi_3^2= 0,\ F'= 0$:

In this case, we obtain

$$f = const \ \text{ and } \ \phi = const.$$

So, the space-time reduces to purely flat space-time in Einstein theory.

CaseIV, $\phi_2^2+\phi_3^2= 0,\ \phi_{22}+\phi_{33} = 0$:

In this case, we have

$$\phi = const.$$

Hence the Barber's theory leads to Einstein theory in vacuo.

6 Remarks

In Einstein theory, the models obtained do not have singularity either at $y = 0$ or $z = 0$ or at both $y = 0 = z$. At $y = \infty$ and $z = \infty$, the models I and II admit singularity respectively whereas model III admits singularity at both. No remarks can be given as regards to singularity with respect to first spatial and temporal coordinates, because the model depends on an arbitrary function of $t - x$. But all the solutions represent forward moving plane gravitational wave. The viscous fluid distribution with bulk and shear coefficients does not survive in the space-time (1) and reduces to pure vacuum case. At $r = 0$, the model governed by (21) admits a singularity.

Further it is shown that both the metric potential 'f'and the Barber's scalar 'ϕ', exist and satisfy Laplace equation in Barber's theory. Moreover it is interesting to note that an arbitrary harmonic function in y and z coordinates can generate both Einstein and Barber'sss vacuum models when the space-time is described by the metric (1).

The non-zero components of curvature tensor for the metric (1) can be obtained as:

$$R_{1212} = R_{2424} = -R_{2124} = -\frac{f_{22}}{2}$$

$$R_{1213} = R_{1234} = R_{1324} = R_{4243} = -\frac{f_{23}}{2}$$

$$R_{1313} = R_{3434} = -R_{3134} = -\frac{f_{33}}{2}$$

If we assume f to be independent of y and z , or linear in them, there will be no field i.e. the space-time is flat (the curvature tensor vanishes) and Barber's theory. Also the field in the form

$$f(t-x,y,z) = yzf_1(t-x) + \frac{1}{2}(y^2 - z^2)f_2(t-x),$$

which is homogeneous and quadratic in y and z coordinates, corresponding to a plane wave propagating in positive x-direction and the curvature tensor in such a field depends only on $t-x$:

$$R_{1212} = R_{2424} = -R_{2124} = -\frac{f_2(t-x)}{2}$$

$$R_{1213} = R_{1234} = R_{1324} = R_{4243} = -\frac{f_1(t-x)}{2}$$

$$R_{1313} = R_{3434} = -R_{3134} = \frac{f_2(t-x)}{2}$$

Thus the metric contains two arbitrary functions $f_1(t-x)$ and $f_2(t-x)$ corresponding to elliptically and circularly polarized wave.

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Disipacija opšte raspodele viskoznog fluida u teorijama Ajnštajna i Barbera

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Učinjen je pokušaj da se prouči problem opšte prostorno-vremenske raspodele viskoznog fluida sa metrikom $ds^2 = dt^2 - dx^2 - dy^2 + f(t-x, y, z)(dt - dx)^2$ u gravitacionim teorijama Ajnštajna (1915) i Barbera(1982). Primećujemo da se u obe teorije jednačine polja mogu svesti na Laplasovu jednačinu, a da se raspodela viskoznog fluida ne može održati. Štaviše, oba vakuumska modela se mogu konstruisati proizvoljnom harmonijskom funkcijom koordinata y i z, dok rešenja koja modele odredjuju predstavljaju ravanski gravitacioni talas koji se prostire u x-pravcu.