The analog equation method.
A boundary-only integral equation method for nonlinear static and dynamic problems in general bodies

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Submitted 1 December, 2001

Abstract

In this paper the Analog Equation Method (AEM) a boundary-only method is presented for solving nonlinear static and dynamic problems in continuum mechanics. General bodies are considered, that is bodies whose properties may be position or direction dependent and their response is nonlinear. The nonlinearity may result from both nonlinear constitutive relations (material nonlinearity) and large deflections (geometrical nonlinearity). The quintessence of the method is the replacement of the coupled nonlinear partial differential equations with variable coefficients governing the response of the body by an equivalent set of linear uncoupled equations under fictitious sources. The fictitious sources are established using a BEM-based technique and the solution of the original problem is obtained from the integral representation of the solution of the substitute problem. A variety of static and dynamic problems are solved using the AEM are presented to illustrate the method and demonstrate its efficiency and accuracy.

1   Introduction

The boundary methods are known for their major advantage to restrict the discretization only to the boundary of the body. Among

them the most reputed one is the boundary integral equation or otherwise known as Boundary Element Method (BEM), a name resulting from the employed technique to solve the boundary integral equations. Although the BEM has been proven to be powerful alternative to the so called domain methods, such as FDM and FEM, when linear problems are encountered, this method has been criticized as not been capable of solving nonlinear problems, especially in nonhomogeneous bodies where the coefficients of the differential equations are variable. This is one of the reasons that many investigators are reluctant to be involved with BEM and use it as a computational tool.

Effort to develop BEM methods for nonlinear problems has been given by many BEM investigators. Almost all of these methods have not avoided domain discretization. The only method that can be considered as boundary-only is the dual reciprocity method (DRM) [1]. The term *'boundary-only'* is used in the sense that discretization and integration are limited only on the boundary, although collocation points inside the domain may be used to improve the solution. Nevertheless, DRM works when for a non standard linear partial differential equation or a nonlinear one it is possible to extract a standard linear partial differential operator $L(\cdot)$ and lump the remainder to the right-hand-side as a body-force term:

$$L(u) = b(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \tag{1}$$

where $b(\cdot)$ is, in general, a nonlinear function of its arguments.

Further, DRM can be employed if the fundamental solution of the adjoint differential equation can be established, namely, a partial singular solution of the equation

$$L^*(u^*) = \delta(P - Q) \tag{2}$$

where $L^*(\cdot)$ is the adjoint operator to $L^*(\cdot)$ and $\delta(P - Q)$ is the Dirac delta function.

On the basis of the aforementioned, it is apparent that DRM cannot be employed when

(**a**) The differential operator cannot be put in the form of the eqn. (1), e.g.

$$u_{xx}u_{yy} - u_{xy}^2 = f(x, y) \tag{3}$$

(b) The fundamental solution of eqn. (2) is not available, e.g. when the operator $L^*(\cdot)$ has variable coefficients.

Apparently, the efficiency of DRM decreases in the case of problems described by coupled nonlinear equations. Besides, different DRM formulations and consequently different computer programs are required for different body force terms as well as for different operators $L^*(\cdot)$, even the order of equations is the same.

In this paper a boundary-only method is presented for solving nonlinear static and dynamic problems. The method is alleviated from the restrictions characterizing DRM. Simple fundamental solutions are used which depend only on the order of differential equations, e.g. for second order differential equations the fundamental solution of the Laplace equation is employed for both static and dynamic problems. The method is based on the concept of the analog equation [2], according to which the nonlinear problem is replaced by an equivalent simple linear one under a fictitious source with the same boundary and initial conditions. The substitute problem is chosen so that the integral representation of the solution is known. The fictitious source is established by approximating it with a radial basis function series expansion as in the DRM and the solution of the original problem is computed from the integral representation of the substitute problem, which is used as mathematical formula. Without restricting the generality the method is illustrated by applying it to second order partial and ordinary differential equations.

The method has been already successfully employed to solve a variety of engineering problems described by partial differential equations, among them potential flow problems in bodies whose material constants depend on the field function (e.g. temperature dependent conductivity) [3], determination of surface with prescribed mean or total curvature [3], the soap bubble problem [4], nonlinear static and dynamic analysis of homogeneous isotropic and heterogeneous orthotropic membranes [5,6,7,8], finite elasticity problems, inverse problems [9], *equationless* problems in nonlinear bodies using only boundary data [10], nonlinear analysis of shells [11]. The method has been also applied to problems described by coupled nonlinear ordinary differential equations, e.g. finite deformation analysis of elastic cables [12,13], large deflection analysis of beams [14] and integration of nonlinear equations

of motion [15]. Some example problems are solved to demonstrate the applicability, efficiency and accuracy of the AEM.

2   Illustration of the AEM for 2nd order PDE's of hyperbolic type

2.1   Problem statement

Consider a non-homogeneous body occupying the two-dimensional domain Ω in the xy-plane (Fig. 1), whose dynamic response is governed by the following initial boundary value problem

$$\rho u_{tt} + c u_t + N(u) = g(x,y,t) \text{ in } \Omega, \ 0 \leq t \tag{4}$$

$$\beta_1 u + \beta_2 u_n = \beta_3 \text{ on } \Gamma \tag{5}$$

$$u(x,y,0) = u_1(x,y), \ u_t(x,y,0) = u_2(x,y) \text{ in } \Omega \tag{6a,b}$$

where $u = u(x,y,t)$ is the unknown field function and

$$N(u) = N(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, x, y) \tag{7}$$

is a nonlinear second order differential operator defined in Ω; $\Gamma = \bigcup_{i=0}^{i=K} \Gamma_i$ is the boundary where $\Gamma_i (i = 1, 2, ..., K)$ are K non-intersecting closed contours surrounded by the contour Γ_0. Moreover, $\beta_i = \beta_i(s), i = 1, 2, 3$ are functions specified on the boundary Γ with s being the arc length, while $u_1(x,y)$ and $u_2(x,y)$ are given functions denoting the initial deflection and velocity distributions, respectively. Finally, $\rho = \rho(x,y)$ and $c = c(x,y)$ are the mass and damping densities, respectively, and $g(x,y,t)$ the forcing function. The boundary condition (5) has been assumed linear for the convenience of the presentation of the method, although a nonlinear boundary condition could be considered.

2.2   The analog equation method

Let $u = u(x,y,t)$ be the sought solution to the problem (4)-(6). This function is two times continuously differentiable in Ω. Thus, if the
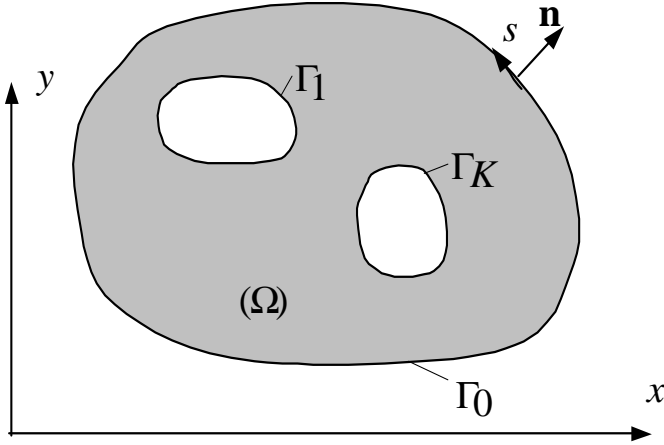
Figure 1: Multiply Connected Domain Ω and Boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_K$

Laplace operator $\nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is applied to it, we have

$$\nabla^2 u = b(x, y, t) \tag{8}$$

Eqn. (8) is a quasi-static equation and indicates that the solution of eqn. (4) at instant t could be established by solving this equation under the boundary condition (5), if the fictitious time dependent source $b(x, y, t)$ were known. Eqn. (8) is the *analog equation*, which together with the boundary condition (5) and the initial condition (6a,b) constitute the substitute problem.

The fictitious source can be established following a procedure similar to that presented by Katsikadelis and Nerantzaki [3] for the static problem. We assume

$$b = \sum_{j=1}^{M} \alpha_j f_j \tag{9}$$

where $f_j = f_j(x, y)$ is a set of approximation functions and $\alpha_j = \alpha_j(t)$ time dependent coefficients to be determined.

The solution of eqn. (8) at instant t can be written as a sum of the homogeneous solution $\bar{u} = \bar{u}(x, y, t)$ and a particular solution

$u_p = u_p(x, y, t)$ of the nonhomogeneous equation. Thus, we can write

$$u = \bar{u} + u_p \tag{10}$$

The particular solution is obtained from

$$\nabla^2 u_p = \sum_{j=1}^{M} \alpha_j f_j \tag{11}$$

which yields

$$u_p = \sum_{j=1}^{M} \alpha_j \hat{u}_j \tag{12}$$

where $\hat{u}_j (j = 1, 2, ..., M)$ is a particular solution of the equation

$$\nabla^2 \hat{u}_j = f_j \qquad j = 1, 2, ..., M \tag{13}$$

The particular solution of eqn. (13) can always be determined, if f_j is specified. The homogeneous solution $\bar{u}$ is obtained from the boundary value problem

$$\nabla^2 \bar{u} = 0 \text{ in } \Omega \tag{14}$$

$$\beta_1 \bar{u} + \beta_2 \bar{q} = \beta_3 - (\beta_1 \sum_{j=1}^{M} \alpha_j \hat{u}_j + \beta_2 \sum_{j=1}^{M} \alpha_j \hat{q}_j) \text{ on } \Gamma \tag{15}$$

where $\hat{q}_j = \partial \hat{u}_j / \partial n$.

The boundary value problem (14)-(15) is solved using the BEM. Thus, the integral representation of the solution $\bar{u}$ is given as

$$c\bar{u}(P, t) = - \int_T (u^* \bar{q} - \bar{u} q^*) ds \qquad P(x, y) \in \Omega \cup \Gamma \tag{16}$$

in which $u^* = \ln r / 2\pi$ is the fundamental solution to eqn. (14) and $q^* = u^*_{,n}$ its derivative normal to the boundary with $r = |Q - P| = \left[(\xi - x)^2 + (y - \eta)^2 \right]^{1/2}$ being the distance between any two points

$P(x,y)$ in $\Omega \cup \Gamma$, $Q(\xi,\eta)$ on Γ; c is a constant which takes the values $c = 1$ if $P \in \Omega$ and $c = \alpha/2\pi$ if $P \in \Gamma$; α is the interior angle between the tangents of boundary at point P. Note that it is $c = 1/2$ for points where the boundary is smooth.

On the basis of eqns. (10), (12) and (16) the solution of eqn. (8) is written as

$$cu = -\int_\Gamma (u^*\bar{q} - \bar{u}q^*)ds + \sum_{j=1}^{M} \alpha_j \hat{u}_j \qquad (17)$$

Differentiating the above equation for $P \in \Omega$ ($c = 1$) yields

$$u_x = -\int_\Gamma (u_x^*\bar{q} - \bar{u}q_x^*)ds + \sum_{j=1}^{M} (\hat{u}_j)_x \alpha_j,$$

$$u_y = -\int_\Gamma (u_y^*\bar{q} - \bar{u}q_y^*)ds + \sum_{j=1}^{M} (\hat{u}_j)_y \alpha_j, \qquad (18\text{a,b})$$

$$u_{yy} = -\int_\Gamma (u_{xx}^*\bar{q} - \bar{u}q_{xx}^*)ds + \sum_{j=1}^{M} (\hat{u}_j)_{xx} \alpha_j,$$

$$u_{xx} = -\int_\Gamma (u_{yy}^*\bar{q} - \bar{u}q_{yy}^*)ds + \sum_{j=1}^{M} (\hat{u}_j)_{yy} \alpha_j, \qquad (19\text{a,b})$$

$$u_{xy} = -\int_\Gamma (u_{xy}^*\bar{q} - \bar{u}q_{xy}^*)ds + \sum_{j=1}^{M} (\hat{u}_j)_{xy} \alpha_j \qquad (20)$$

The final step of AEM is to apply eqn. (4) to M discrete points inside Ω. We, thus, obtain a set of M equations

$$\rho^i u_{tt}^i + c^i u_t^i + N(u^i) = g^i \qquad (i = 1, 2, ..., M) \qquad (21)$$

Using eqns. (17) to (20) to evaluate u and its derivatives at points $i = 1, 2, ..., M$ and substituting them into eqn. (21) the following set of nonlinear ordinary differential equations, which play the role of the semidiscretized equations of motion

$$F_i(\alpha_j, \dot{\alpha}_j, \ddot{\alpha}_j) = g^i \qquad (i = 1, 2, ..., M) \qquad (22)$$

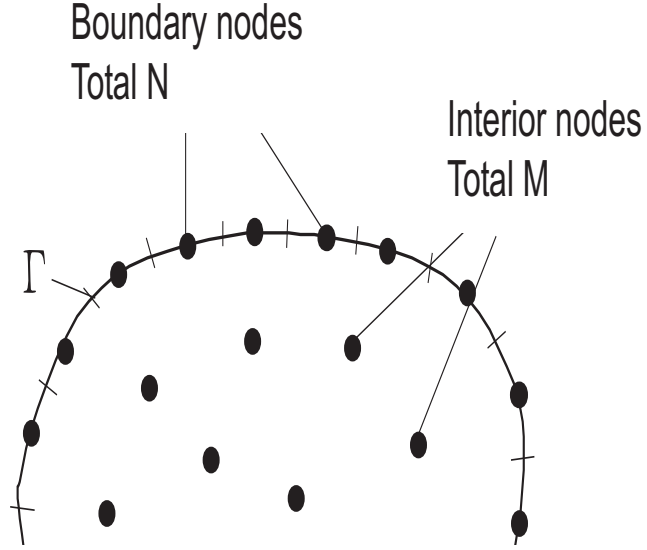which can be solved to yield the coefficients α_j. The AEM can be implemented only numerically.

Figure 2: Boundary discretization and domain nodal points

2.3   Numerical Implementation

The BEM with constant elements is used to approximate the boundary integrals in eqns. (17) to (20). If N is the number of the boundary nodal points (see Fig. 2), then eqn. (17) is written as

$$c^i \bar{u}^i = \sum_{k=1}^{N} \bar{H}_{ik} \bar{u}^k - \sum_{k=1}^{N} G_{ik} \bar{q}^k \tag{23}$$

where

$$\bar{H}_{ik} = \int_k q^*(r_{ik}) ds \tag{24}$$

$$G_{ik} = \int_k u^*(r_{ik}) ds \tag{25}$$

Applying eqn. (23) to all boundary nodal points and using matrix notation yields

$$[H]\{\bar{u}\} - [G]\{\bar{q}\} = 0 \qquad (26)$$

where

$$[H] = [\bar{H}] - [C] \qquad (27)$$

with $[C]$ being a diagonal matrix including the values of the coefficient c^i. The boundary condition (15), when applied to the N boundary nodal points, yields

$$(\beta_1)^i \bar{u}^i + (\beta_2)^i \bar{q}^i = (\beta_3)_i - \left[(\beta_1)^i \sum_{j=1}^{M} \alpha_j \hat{u}_j^i + (\beta_2)^i \sum_{j=1}^{M} \alpha_j \hat{q}_j^i \right] \qquad (28)$$

or using matrix notation

$$[\beta_1]\{\bar{u}\} + [\beta_2]\{\bar{q}\} = \{\beta_3\} - \left([\beta_1][\hat{U}] + [\beta_2][\hat{Q}] \right) \{\alpha\} \qquad (29)$$

in which $[\hat{U}] = [\hat{u}_j^i]$, $[Q] = [\hat{q}_j^i]$ are $N \times M$ known matrices; $[\beta_1], [\beta_2]$ are $N \times N$ known diagonal matrices and $\{\alpha\}$ the vector of the coefficients to be determined.

Eqns. (26) and (29) may be combined to express $\{\bar{u}\}$ and $\{\bar{q}\}$ in terms of $\{\alpha\}$. Thus, we may write

$$\begin{bmatrix} [H] & -[G] \\ [\beta_1] & [\beta_2] \end{bmatrix} \left\{ \begin{array}{c} \{\bar{u}\} \\ \{\bar{q}\} \end{array} \right\} = \begin{bmatrix} [0] \\ [T] \end{bmatrix} \{\alpha\} + \left\{ \begin{array}{c} \{0\} \\ \{\beta_3\} \end{array} \right\} \qquad (30)$$

where

$$[T] = - \left([\beta_1][\hat{U}] + [\beta_2][\hat{Q}] \right) \qquad (31)$$

Solving eqn. (30) yields

$$\{\bar{u}\} = [S_u]\{\alpha\} + \{d_u\} \qquad (32)$$

$$\{\bar{q}\} = [S_q]\{\alpha\} + \{d_q\} \qquad (33)$$

in which $[S_u], [S_q]$ are known $N \times M$ rectangular matrices and $\{d_u\}, \{d_q\}$ known vectors. Eqns. (17) to (20) when discretized and applied to the M nodal points inside Ω give

$$\{u\} = [H]\{\bar{u}\} - [G]\{\bar{q}\} + [\hat{U}]\{\alpha\} \tag{34}$$

$$\{u_x\} = [H_x]\{\bar{u}\} - [G_x]\{\bar{q}\} + [\hat{U}_x]\{\alpha\}$$
$$\{u_y\} = [H_y]\{\bar{u}\} - [G_y]\{\bar{q}\} + [\hat{U}_y]\{\alpha\} \tag{35a,b}$$

$$\{u_{xx}\} = [H_{xx}]\{\bar{u}\} - [G_{xx}]\{\bar{q}\} + [\hat{U}_{xx}]\{\alpha\}$$
$$\{u_{yy}\} = [H_{yy}]\{\bar{u}\} - [G_{yy}]\{\bar{q}\} + [\hat{U}_{yy}]\{\alpha\} \tag{36a,b}$$

$$\{u_{xy}\} = [H_{xy}]\{\bar{u}\} - [G_{xy}]\{\bar{q}\} + [\hat{U}_{xy}]\{\alpha\} \tag{37}$$

in which $[G], [H], [G_x], [H_{yy}], ..., [H_{xy}]$ are known $M \times M$ matrices originating from the integration of the kernel functions u^* and q^* and their respective derivatives; $[\hat{U}], [\hat{U}_x], ..., [\hat{U}_{xy}]$ are known matrices having dimensions , the elements of which result from the functions $\hat{u}_j$ and their derivatives.

Substituting eqns. (32) and (33) into eqns. (34)-(37) yields

$$\{u\} = [W]\{\alpha\} + \{w\} \tag{38}$$

$$\{u_x\} = [W_x]\{\alpha\} + \{w_x\}, \ \ \{u_y\} = [W_y]\{\alpha\} + \{w_y\} \tag{39a,b}$$

$$\{u_{xx}\} = [W_{xx}]\{\alpha\} + \{w_{xx}\}, \ \ \{u_{yy}\} = [W_{yy}]\{\alpha\} + \{w_{yy}\} \tag{40a,b}$$

$$\{u_{xy}\} = [W_{xy}]\{\alpha\} + \{w_{xy}\} \tag{41}$$

where $[W], [W_x], ..., [W_{xy}]$ are known matrices and $\{w\}, \{w_x\}, ..., \{w_{xy}\}$ known vectors.

Differentiating eqn. (38) with respect to time and taking into account that the vector $\{w\}$ is constant we obtain

$$\{\dot{u}\} = [W]\{\dot{\alpha}\} \tag{42}$$

$$\{\ddot{u}\} = [W]\{\ddot{\alpha}\} \tag{43}$$

Finally, writing eqn. (21) in matrix form and substituting eqns. (38) to (43) in it, we obtain the typical semidiscretized nonlinear equation of motion

$$[M]\{\ddot{\alpha}\} + [C]\{\dot{\alpha}\} + N(\{\alpha\}) = \{g\} \tag{44}$$

where $[M]$ and $[C]$ are generalized mass and damping matrices, respectively. The initial conditions for eqn. (44) are obtained from eqns. (38) and (42) on the base of eqns. (6a,b). Thus, we have

$$\{\alpha(0)\} = [W]^{-1}(\{u_1\} - \{w\}) \tag{45}$$

$$\{\dot{\alpha}(0)\} = [W]^{-1}\{u_2\} \tag{46}$$

The dynamic problem

For forced $(g(x, y, t) \neq 0)$ or free vibrations $(g(x, y, t) = 0)$, eqn. (44) is solved using any time step integration method taking into account the initial conditions (45) and (46). Once α_j are computed, the solution of the problem and its derivatives are evaluated from eqns. (38) to (41). For points not coinciding with the nodal points these quantities are computed from the discretized counter part of eqns. (17) to (20).

The static problem

In this case it is $\{\dot{\alpha}\} = \{\ddot{\alpha}\} = \{0\}$ and eqn. (44) becomes

$$N(\{\alpha\}) = \{g\} \tag{47}$$

from which the coefficients $\{\alpha\}$ are established by solving a system of nonlinear algebraic equations.

3   Examples

On the basis of the numerical procedure presented in section 2, a FOR-
TRAN code has been written and numerical results for example prob-
lems have been obtained, which illustrate the applicability, effectiveness
and accuracy of the AEM. The employed approximation functions f_j
are the multiquadrics, which are defined as

$$f_j = \sqrt{r^2 + c^2} \tag{48}$$

where c is an arbitrary constant and

$$r = \sqrt{(x - x_j)^2 + (y - y_j)^2} \qquad j = 1, 2, ..., M \tag{49}$$

with x_j, y_j being the collocation nodal points inside Ω. Using these
radial basis functions, the particular solution of eqn. (13) is obtained
as

$$\hat{u}_j = -\frac{c^3}{3} \ln\left(c\sqrt{r^2 + c^2} + c^2\right) + \frac{1}{9}(r^2 + 4c^2)\sqrt{r^2 + c^2} \tag{50}$$

3.1   Heat flow in bodies with nonlinear material properties

In this case the thermal conductivity k depends on the temperature
$u(x, y)$ [3]. If we assume that $k = k_0[1 + \beta(u - u_0)/u_0]$, where k_0, β and
u_0 are constants, the governing equation is written as

$$k\nabla^2 u + \beta(u_x^2 + u_y^2) = 0 \tag{51}$$

Numerical results for a square plane body with $k_0 = 1$, $\beta = 3$,
$u_0 = 300$ and unit side length $0 \leq x, y \leq 1$ under the mixed boundary
conditions $u(0, y) = 300$, $u(1, y) = 400$, $u_n(x, 0) = 0$, $u_n(x, 1) = 0$
are given in Table 1 as compared with those obtained using the Dual
Reciprocity Method and the Kirchhof's transformation method

Position		u				
x	Y	DRM	Kirchhof	AEM		
				M=25	M=69	M=225
0.1	0.5	314.15	314.00	314.44	314.17	314.19
0.3	0.5	338.34	337.82	339.23	337.76	338.83
0.5	0.5	358.49	358.11	359.51	358.11	359.00
0.7	0.5	376.27	376.08	376.62	376.68	375.97
0.9	0.5	392.43	392.36	392.24	392.40	391.93

Table 1: Temperature in a square plate with temperature dependent conductivity.

3.2 Determination of a surface with constant Gaussian curvature

A surface that passes through a skew closed space curve and has given Gaussian curvature K is determined from the following boundary value problem [3]

$$u_{xx}u_{yy} - u_{xy}^2 - K(1 + u_x^2 + u_y^2) = 0 \text{ in } \Omega \tag{52}$$

$$u = \hat{u} \text{ on } \Gamma \tag{53}$$

where Ω is the domain surrounded by the projection Γ of the curve on the x, y plane. Numerical results for the square domain $0 \le x, y \le 5$ with boundary conditions $u(0, y) = (50 - y^2)^{1/2}$, $u(5, y) = (25 - y^2)^{1/2}$, $u(x, 0) = (50 - x^2)^{1/2}$, $u(x, 5) = (25 - x^2)^{1/2}$ and Gaussian curvature $K = 1/50$ are given in Table 2 as compared with the exact ones

x	y=1.25		y=3.75	
	AEM	Exact	AEM	Exact
0.25	6.92	6.95	5.94	5.98
0.75	6.82	6.91	5.81	5.94
1.25	6.70	6.84	5.68	5.86
1.75	6.56	6.73	5.55	5.73
2.25	6.40	6.58	5.39	5.55
2.75	6.21	6.39	5.18	5.32
3.25	6.01	6.15	4.91	5.03
3.75	5.71	5.86	4.59	4.67
4.25	5.39	5.51	4.20	4.22
4.75	5.03	5.08	3.68	3.65

Table 2: Numerical results for in Example 3.2

3.3   The problem of minimal surface

This is the problem of determining a surface passing through one or more non-intersecting skew closed space curves and having a minimal area. The physical analog is the surface that a soap bubble assumes when constraint by bounding contours (Plateau's problem). The condition

$$\min A = \int_\Omega (1 + u_x^2 + u_y^2) dx dy \tag{54}$$

requires that the minimal surface $u(x, y)$ is a solution of the following boundary value problem [4]

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0 \text{ in } \Omega \tag{55}$$

$$u = \tilde{u} \text{ on } \Gamma \tag{56}$$

The catenoid
The minimal surface supported on the two concentric circles lying at different levels is known as the *catenoid*. The obtained solution for

Figure 3: The catenoid

$R = 5$, $z = 0$ and $R = 2$, $z = 3$ is determined and it shown in graphical form in Fig. 3.

Cross shaped membrane

The surface of a soap bubble that passes through the space curve, which is the intersection of the cylindrical surface $r = 5(\sin^4 \theta + \cos^4 \theta)$, $0 \leq \theta \leq 2\pi$, $z \geq 0$, and the sphere $x^2 + y^2 + z^2 = R^2$, $R = 5$ is determined. The obtained surface is shown in Fig. 5.

3.4 Large deflections of heterogeneous orthotropic membranes

Consider a thin flexible initially flat elastic membrane consisting of heterogeneous orthotropic linearly elastic material occupying the two-dimensional, in general multiply connected, domain Ω in the xy-plane bounded by the $K+1$ nonintersecting contours $\Gamma_0, \Gamma_1..., \Gamma_K$. The membrane is prestressed either by imposed displacement $\bar{u}, \bar{v}$ or by external forces $\tilde{T}_x, \tilde{T}_y$ acting along the boundary $\Gamma = \bigcup_{i=0}^{i=K} \Gamma_i$. Assuming non-linear kinematic relations, which retain the square of the slopes of the deflection surface, while the strain components remain still small com-

Figure 4: Plan form of the cross-shaped membrane

Figure 5: Cross-shaped membrane

pared with the unity, we obtain the following three coupled nonlinear differential equations in terms of the displacements [7]

$$(C_1 u_{,x} + C v_{,y})_{,x} + (C_{12} u_{,y} + C_{12} v_{,x})_{,y} =$$
$$= -\left(\tfrac{C_1}{2} w_{,x}^2 + \tfrac{C}{2} w_{,y}^2\right)_{,x} - (C_{12} w_{,x} w_{,y})_{,y} \tag{57a}$$

$$(C_2 v_{,y} + C u_{,x})_{,y} + (C_{12} u_{,y} + C_{12} v_{,x})_{,x} =$$
$$= -\left(\tfrac{C_2}{2} w_{,y}^2 + \tfrac{C}{2} w_{,x}^2\right)_{,y} - (C_{12} w_{,x} w_{,y})_{,x} \tag{57b}$$

$$\left[C_1\left(u_{,x} + \tfrac{1}{2} w_{,x}^2\right) + C\left(v_{,y} + \tfrac{1}{2} w_{,y}^2\right)\right] w_{,xx} +$$
$$+ \left[2 C_{12}\left(u_{,y} + v_{,x} + w_{,x} w_{,y}\right)\right] w_{,xy} + \tag{57c}$$
$$+ \left[C_2\left(v_{,y} + \tfrac{1}{2} w_{,y}^2\right) + C\left(u_{,x} + \tfrac{1}{2} w_{,x}^2\right)\right] w_{,yy} = -g$$

subjected to the boundary conditions

$$T_x = \tilde{T}_x \text{ or } u = \tilde{u} \tag{58a}$$

$$T_y = \tilde{T}_y \text{ or } v = \tilde{v} \tag{58b}$$

$$T_x w_{,x} + T_y w_{,y} = \tilde{V} \text{ or } w = \tilde{w} \tag{58c}$$

where $u = u(x, y)$, $v = v(x, y)$ are the in-plane displacement components and $w = w(x, y)$ the transverse deflection produced when the membrane is subjected to the load $g = g(x, y)$ acting in the direction normal to its plane. The position dependent coefficients $C_1 = C_1(x, y)$, $C_2 = C_2(x, y)$, $C = C(x, y)$ and $C_{12} = C_{12}(x, y)$ characterize the stiffness of the orthotropic membrane and are given as

$$C_1 = \tfrac{E_1 h}{1 - v_1 v_2}, \ C_2 = \tfrac{E_2 h}{1 - v_1 v_2},$$
$$C = \tfrac{E_1 v_2 h}{1 - v_1 v_2} = \tfrac{E_2 v_1 h}{1 - v_1 v_2}, \ C_{12} = G h \tag{59a,b,c,d}$$

in which E_1, E_2 and v_1, v_2 are the elastic moduli and the Poisson coefficients in the x and y directions, respectively, constraint by the relation $E_1v_1 = E_2v_2$ and G is the shear modulus.

The analog equations in this case are three uncoupled Poisson's equations, namely

$$\nabla^2 u = b_1(x,y), \ \ \nabla^2 v = b_2(x,y), \ \ \nabla^2 w = b_3(x,y) \qquad \text{(60a,b,c)}$$

The fictitious sources are established using the same procedure with that for on analog equations and the displacements as well as their derivatives are computed from the integral representations of the solution of the respective Poisson's equations.

Membrane of arbitrary shape

In this example, the heterogeneous orthotropic membrane of arbitrary shape was analyzed ($N = 80$, $M = 61$). Its boundary is defined by the curve $r = 5 \left| \sin \theta \right|^3 + 6 \left| \cos \theta \right|^3$, $0 \leqslant \theta \leqslant 2\pi$. The membrane is prestressed by $u_n = 0.05m$ in the direction normal to the boundary, while $u_t = 0$ in the tangential direction. The employed data are $h = 0.002m$, $g = 3kN/m^2$, $E_1 = E\sqrt{\lambda}$, $E_2 = E\sqrt{\lambda}$, $\nu_1 = 0.3$, $\nu_2 = \lambda\nu_1$ and $G = E/2(1 + \nu_1\sqrt{\lambda})$ where $E = 110000 + kr^2$, $r = (x^2 + y^2)^{1/2}$ and k a constant. The contours of the principle stress resultants N_1 for various values of k and λ are shown in Fig. 6 and Fig. 7.

3.5   Nonlinear vibrations of membranes

The free and forced vibrations of a homogeneous isotropic membrane have been studied. The governing equations result from eqns. (57) for $C_1 = C_2 = Eh/(1 - \nu^2)$, $C_{12} = E\nu h/(1 - \nu^2$ and including the inertia force in the third equation. Thus we have the following initial boundary value problem [6]

$$\frac{1-\nu}{2}\nabla^2 u + \frac{1+\nu}{2}(u_{,x} + v_{,y})_{,x} =$$
$$= -w_{,x}(w_{,xx} + \frac{1-\nu}{2}w_{,yy}) - \frac{1+\nu}{2}w_{,y}\,w_{,xy} \qquad \text{(61a)}$$

$$\frac{1-\nu}{2}\nabla^2 v + \frac{1+\nu}{2}(u_{,x} + v_{,y})_{,y} =$$
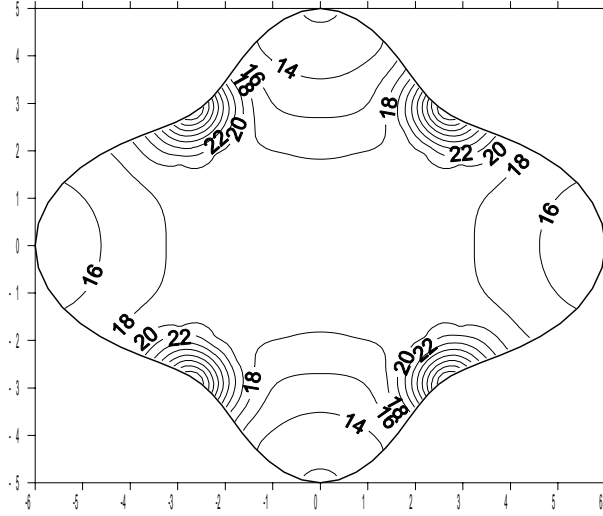$$= -w_{,y}(w_{,yy} + \frac{1-\nu}{2}w_{,xx}) - \frac{1+\nu}{2}w_{,x}\,w_{,xy} \qquad \text{(61b)}$$

Figure 6: Contour of $N_1(\lambda = 2, k = 0)$ in an orthotropic heterogeneous membrane of arbitrary shape

$$\rho\ddot{w} - C[(u_{,x} + \tfrac{1}{2}w_{,x}^2) + \nu(v_{,y} + \tfrac{1}{2}w_{,y}^2)]w_{,xx} -$$

$$-C(1 - \nu)(u_{,y} + v_{,x} + w_{,x}\, w_{,y})]w_{,xy} - \qquad \text{in } \Omega \qquad (61c)$$

$$-C[(v_{,y} + \tfrac{1}{2}w_{,y}^2) + \nu(u_{,x} + \tfrac{1}{2}w_{,x}^2)]w_{,yy} = g$$

$$T_x = \tilde{T}_x \text{ or } u = \tilde{u}, \qquad T_y = \tilde{T}_y \text{ or } v = \tilde{v},$$
$$\qquad (62)$$
$$T_x w_{,x} + T_y w_{,y} = \tilde{V} \text{ or } w = \tilde{w} \text{ on } \Gamma$$

$$w(x, y, 0) = \tilde{w}_o, \qquad \dot{w}(x, y, 0) = \dot{\tilde{w}}_o \text{ in } \Omega \qquad (63)$$

3.6   Square membrane

A uniformly prestressed ($N_x = N_y = 2.514 kN/m$, $N_{xy} = 0$) square membrane ($0 \le x \le a$) has been studied.   The employed data

Figure 7: Contour of $N_1(\lambda = 2,\ k = 5000)$ in an orthotropic heterogeneous membrane of arbitrary shape

are: $a = 5.0m$, $h = 0.002m$, $E = 1.1 \times 10^5\, kN/m^2$, $\nu = 0.3$ and $g_0 = 1.934\, kN/m^2$. The results obtained with $N = 100$ and $M = 49$ are compared with those from the one-term approximate series solution [16], which assumes $w(x, y, t) = w_0(t) \sin(\pi x/a) \sin(\pi y/a)$. In Fig. 8 and Fig. 9 results for the natural vibrations are shown for (i): $w(x, y, 0) = w_0 \sin(\pi x/a) \sin(\pi y/a)$ and $\dot{w}(x, y, 0) = 0$, ($w_0 = 0.446$); (ii): $w(x, y, 0) = $ deflection surface produced by the static load g_0 and $\dot{w}(x, y, 0) = 0$. Moreover, in Fig. 10, the dependence of the period T (T_0 is the period of the linear vibration) on the maximum amplitude is shown for both cases. It should be noted that the approximate solution gives very good results in case (i). Finally, the forced vibrations have been studied under the so-called "static" load $g = g_0 t/2t_1$ for $0 \leq t \leq t_1$ and $g = g_0/2$ for $t_1 \leq t$, ($t_1 = 10\,\mathrm{sec}$) with zero initial conditions. The response ratio $R(t) = w(0, 0, t)/w_{st}$ of the central deflection is shown in Fig. 11 as compared with that obtained by the one term approximate solution; w_{st} is the central static deflection obtained by AEM solution. Apparently, the "static" load produces smaller deflections in
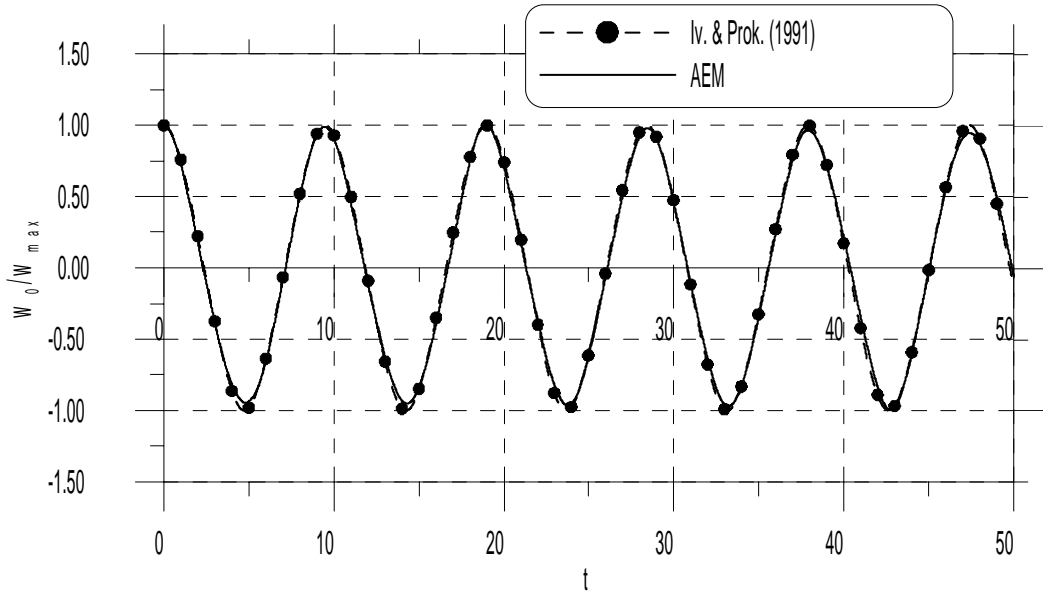
Figure 8: Central deflection. Case (i)

the AEM solution than in the one-term series solution. Therefore the increase of the period in the AEM solution was anticipated on the base of Fig.10.

4   Conclusions

1. As the method is boundary-only, it has all the advantages of the BEM, i.e. the discretization and integration are performed only on the boundary.

2. Simple static known fundamental solutions are employed for both static and dynamic problems. They depend only on the order of the differential equation and not specific differential operator which governs the problem under consideration.

3. The computer program is the same for both static and dynamic problems and depends only on the order of the differential equation and not specific differential operator which governs the problem under consideration.

Figure 9: Central deflection. Case (ii)

4. The deflections and the stress resultants are computed at any point using the respective integral representation as mathematical formulas.

5. Accurate numerical results for the displacements and the stress resultants are obtained using radial basis functions of multiquadric type.

6. The concept of the analog equation in conjunction with radial basis functions approximation of the fictitious sources renders BEM a versatile computational method for solving difficult nonlinear static and dynamic engineering problems for non-homogeneous bodies.

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Figure 10: Period versus max. central deflection

Figure 11: Response ratio under "static" load

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