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Homogenization methods and the mechanics of generalized continua - part 2

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Abstract

The need for generalized continua arises in several areas of the mechanics of heterogeneous materials, especially in homogenization theory. A generalized homogeneous substitution medium is necessary at the global level when the structure made of a composite material is subjected to strong variations of the mean fields or when the intrinsic lengths of non–classical constituents are comparable to the wavelength of variation of the mean fields. In the present work, a systematic method based on polynomial expansions is used to replace a classical composite material by Cosserat and micromorphic equivalent ones. In a second part, a mixture of micromorphic constituents is homogenized using the multiscale asymptotic method. The resulting macroscopic medium is shown to be a Cauchy, Cosserat, microstrain or a full micromorphic continuum, depending on the hierarchy of the characteristic lengths of the problem.

1 Introduction

1.1 Scope of this work

The renewal of the mechanics of generalized continua in the last twenty years can be associated with the strong developments of the mechanics

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of heterogeneous materials [1]. The need for extended continua arises in two main areas of the mechanics of heterogeneous materials : strain localization phenomena and fracture, on the one hand, and homogenization theory, on the other hand. Generalized continua are characterized by the introduction of additional degrees of freedom (higher order continua), or the use of higher order gradients of the displacement field (higher grade theories). One may also include the fully nonlocal theories that rely on the introduction of integral formulations of constitutive equations. The basic balance and constitutive equations governing generalized continua are well–established [2], even though developments especially in the nonlinear case are still reported. The present work deals with higher order continua, mainly Cosserat, micromorphic, and incidently couple–stress and microstrain continua.

Physical motivations for introducing higher order stress tensors or directors were always put forward by the authors of these brillant theories, advocating the discrete or more generally the heterogeneous nature of solids as in [3]. Scale transition schemes have been designed to construct homogeneous generalized continua starting from a given underlying composite microstructure. Homogenization methods are well–suited for this purpose. The first attempt of derivation of a second grade elastic medium from classical composite materials by means of homogenization theory goes back to [4]. This represents the ideal way of identifying the numerous material parameters introduced by such theories.

Extended homogenization methods are necessary to construct an homogeneous effective generalized continuum to replace a heterogeneous Cauchy medium. For, the classical Cauchy continuum is not sufficient when the heterogeneous material is subjected to strong gradients of deformation, which means that the mean fields vary from unit cell to unit cell. Perturbation, asymptotic, self–consistent or variational methods are available [4, 5, 6, 7, 8, 9, 10, 11, 12]. An alternative approach based on polynomial expansions has been proposed [13, 14, 15] to identify second grade and Cosserat homogeneous substitution media (HSM), with the advantage that it can be applied to any type of local behaviour, linear or nonlinear.

If the constituents of the considered heterogeneous material are regarded themselves as generalized continua, specific homogenization techniques must be used to determine the nature of the effective medium.

Variational methods are applied in [16] to heterogeneous couple–stress media. The problem of the homogenization of Cosserat media is solved in [17] using the multiscale asymptotic method [18].

A general treatment of the two previous identified problems of homogenization theory applied to generalized continua has been tackled in [14]. The present work is a second step in this endeavour and therefore a continuation of [14]. In [14], the construction of a second gradient HSM starting from a classical heterogeneous material by means of polynomial expansions and a first attempt to homogenize Cosserat media were proposed. In the present work, the polynomial expansion method is applied to derive a micromorphic HSM starting from a classical heterogenous medium subjected to slowly–varying mean fields. In the second part, the multiscale asymptotic method is applied to determine the nature of a mixture of micromorphic continua.

Characteristic lengths play a major role in the understanding of the proposed homogenization schemes. The size l of the heterogeneities of a random material or the size of the unit cell of a periodic material, must be compared to the wavelength L of variation of the macroscopic fields. Classical homogenization methods are used in the case $l \ll L$. The first part of this work deals with the case $l \sim L$. In the second part, the intrinsic lengths involved by the local micromorphic constituents must be compared to l and L. The hierarchy of these characteristic lengths determines the nature of the resulting homogeneous equivalent medium.

1.2 Notations

A wide use of the nabla operator $\boldsymbol{\nabla}$ is made in the sequel. The notation used for the gradient and divergence operators are the following :

$$a\boldsymbol{\nabla} = a_{,i}\underline{\boldsymbol{e}}_i, \quad \underline{\boldsymbol{a}}\otimes\boldsymbol{\nabla} = a_{i,j}\,\underline{\boldsymbol{e}}_i\otimes\underline{\boldsymbol{e}}_j, \quad \boldsymbol{\nabla}\otimes\underline{\boldsymbol{a}} = a_{j,i}\,\underline{\boldsymbol{e}}_i\otimes\underline{\boldsymbol{e}}_j, \quad \underset{\sim}{\boldsymbol{a}}.\boldsymbol{\nabla} = a_{ij,j}\,\underline{\boldsymbol{e}}_i$$

where $a, \underline{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{a}}$ respectively denote a scalar, first and second rank tensors. The comma denotes partial derivation. The $(\underline{\boldsymbol{e}}_i)_{i=1,2,3}$ are the vectors of an orthonormal basis of space and the associated Cartesian coordinates are used. Third and fourth rank tensors are respectively denoted by $\underset{\approx}{\boldsymbol{a}}$ (or $\underset{\simeq}{\boldsymbol{a}}$) and $\underset{\approx}{\underset{\sim}{\boldsymbol{a}}}$. Indices can be contracted as follows :

$$\underset{\sim}{\boldsymbol{a}} : \underset{\sim}{\boldsymbol{b}} = a_{ij}b_{ij}, \quad \underset{\simeq}{\boldsymbol{a}} : \underset{\sim}{\boldsymbol{b}} = a_{ijk}b_{jk}\underline{\boldsymbol{e}}_i, \quad \underset{\approx}{\underset{\sim}{\boldsymbol{a}}} : \underset{\sim}{\boldsymbol{b}} = a_{ijkl}b_{kl}\underline{\boldsymbol{e}}_i\otimes\underline{\boldsymbol{e}}_j$$

$$\underset{\sim}{\boldsymbol{a}} : \underset{\approx}{\boldsymbol{A}} : \underset{\sim}{\boldsymbol{b}} = a_{ij} A_{ijkl} b_{kl}$$

Six–rank tensors $\underset{\approx\atop\sim}{\boldsymbol{a}}$ will be used also with the following contraction rules :

$$\underset{\approx\atop\sim}{\boldsymbol{a}} \vdots \underset{\simeq}{\boldsymbol{x}} = a_{ijkpqr} x_{pqr} \, \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \otimes \underline{\boldsymbol{e}}_k \tag{1}$$

The third rank permutation tensor reads $\underset{\simeq}{\boldsymbol{\epsilon}}$ (or $\underset{\approx}{\boldsymbol{\epsilon}}$), its component ϵ_{ijk} being the signature of permutation (i, j, k) : $\epsilon_{ijk} = 1$ for an even permutation of $(1, 2, 3)$, -1 for an odd permutation and 0 otherwise. A skew–symmetric tensor $\underset{\sim}{\boldsymbol{a}}$ can be represented by the (pseudo–) vector $\overset{\times}{\underline{\mathbf{a}}}$:

$$\overset{\times}{\underline{\mathbf{a}}} = -\frac{1}{2} \underset{\simeq}{\boldsymbol{\epsilon}} : \underset{\sim}{\boldsymbol{a}}, \quad \underset{\sim}{\boldsymbol{a}} = -\underset{\approx}{\boldsymbol{\epsilon}} \cdot \overset{\times}{\underline{\mathbf{a}}}$$

Similarly, a rotation tensor $\underset{\sim}{\boldsymbol{R}}$ can be represented by a rotation vector $\underline{\boldsymbol{\Phi}}$ such that :

$$\underset{\sim}{\boldsymbol{R}} = \underset{\sim}{\mathbf{1}} - \underset{\approx}{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\Phi}} \quad \text{or} \quad \underline{\boldsymbol{\Phi}} = -\frac{1}{2} \underset{\simeq}{\boldsymbol{\epsilon}} : \underset{\sim}{\boldsymbol{R}}$$

in the case of small rotations. Symmetrized and skew–symmetrized tensor products are introduced :

$$\underline{\boldsymbol{a}} \overset{s}{\otimes} \underline{\boldsymbol{b}} = (\underline{\boldsymbol{a}} \otimes \underline{\boldsymbol{b}} + \underline{\boldsymbol{b}} \otimes \underline{\boldsymbol{a}})/2, \quad \underline{\boldsymbol{a}} \overset{a}{\otimes} \underline{\boldsymbol{b}} = (\underline{\boldsymbol{a}} \otimes \underline{\boldsymbol{b}} - \underline{\boldsymbol{b}} \otimes \underline{\boldsymbol{a}})/2, \tag{2}$$

2 Micromorphic overall modelling of heterogeneous materials

In this part, the local properties of the constituents are described by the classical Cauchy continuum. The aim is to derive a micromorphic macroscopic homogeneous substitution medium (HSM) endowed with effective properties. The local and global coordinates are denoted by $\underline{\boldsymbol{y}}$ and $\underline{\boldsymbol{x}}$ respectively.

2.1 Classical homogenization scheme

Within the framework of classical homogenization theory, a representative volume element Ω is defined that contains the relevant aspects of material microstructure. The following boundary value problem must then be solved on the volume element :

$$\begin{cases} \underset{\sim}{\boldsymbol{\varepsilon}} = \frac{1}{2}(\underline{\boldsymbol{u}} \otimes \boldsymbol{\nabla} + \boldsymbol{\nabla} \otimes \underline{\boldsymbol{u}}) \\ \textit{constitutive equations} \\ \underset{\sim}{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} = 0 \quad \forall \underline{\boldsymbol{y}} \in \Omega \\ \textit{boundary conditions} \end{cases} \tag{3}$$

where $\underline{\boldsymbol{u}}$ is the displacement field and $\underset{\sim}{\boldsymbol{\sigma}}$ the classical stress tensor. In the case of random materials, volume Ω must contain enough material heterogeneities to be actually representative of the microstructure. Homogeneous boundary conditions can then be applied according to :

$$\underline{\boldsymbol{u}} = \underset{\sim}{\boldsymbol{E}}.\underline{\boldsymbol{y}} \qquad \forall \underline{\boldsymbol{y}} \in \partial\Omega \qquad \text{or} \quad \underset{\sim}{\boldsymbol{\sigma}}.\underline{\boldsymbol{n}} = \underset{\sim}{\boldsymbol{\Sigma}}.\underline{\boldsymbol{n}} \qquad \forall \underline{\boldsymbol{y}} \in \partial\Omega \tag{4}$$

so that $\underset{\sim}{\boldsymbol{E}} =< \underset{\sim}{\boldsymbol{\varepsilon}} >$, $\underset{\sim}{\boldsymbol{\Sigma}} =< \underset{\sim}{\boldsymbol{\sigma}} >$ are the average macroscopic strain and stress tensors. If the microstructure is periodic, the knowledge of a unit cell Y is sufficient and the displacement field takes the form :

$$\underline{\boldsymbol{u}} = \underset{\sim}{\boldsymbol{E}}.\underline{\boldsymbol{y}} + \underline{\boldsymbol{v}} \tag{5}$$

where $\underline{\boldsymbol{v}}$ is periodic and it takes the same value at homologous points of $\partial\Omega$. For all considered boundary conditions, the following so–called Hill–Mandel condition can be shown to hold :

$$< \underset{\sim}{\boldsymbol{\sigma}}^* : \underset{\sim}{\boldsymbol{\varepsilon}}' >=< \underset{\sim}{\boldsymbol{\sigma}}^* >:< \underset{\sim}{\boldsymbol{\varepsilon}}' > \tag{6}$$

where $\underset{\sim}{\boldsymbol{\sigma}}^*$ is a divergence–free stress field and $\underset{\sim}{\boldsymbol{\varepsilon}}'$ a compatible strain field.

This homogenization procedure requires that the characteristic size l of the heterogeneities is much smaller than the size L of the computed structure or more precisely of the wave length of variation of the macroscopic fields : $l \ll L$. It means that the mean fields do not vary at the scale of Ω or Y, so that $\underset{\sim}{\boldsymbol{\Sigma}}$ and $\underset{\sim}{\boldsymbol{E}}$ can be regarded as constant over Ω.

2.2 Heterogeneous material subjected to slowly varying mean fields

When stronger deformation gradients exist in the structure so that the mean fields slowly vary from unit cell to unit cell, non–homogeneous boundary conditions have been worked out to account for them. One of them is the following quadratic extension of (4) :

$$\underline{u} = \underset{\sim}{E}.\underline{y} + \frac{1}{2}\underset{\approx}{D} : (\underline{y} \otimes \underline{y}) \qquad \forall \underline{y} \in \partial\Omega \tag{7}$$

where $\underset{\approx}{D}$ is a constant third–rank tensor [19, 14]. One can also consider a special case for which the curvature part $\underset{\sim}{K}$ only of deformation gradient $\underset{\approx}{D}$ is taken into account :

$$\underline{u} = \underset{\sim}{E}.\underline{y} + \frac{2}{3}\underset{\approx}{\epsilon} : ((\underset{\sim}{K}.\underline{y}) \otimes \underline{y}) \quad \text{with} \quad \text{trace}(\underset{\sim}{K}) = 0 \tag{8}$$

The expression of macroscopic deformation and its curvature is then :

$$\underset{\sim}{F} = < \underline{u} \otimes \nabla_y >, \qquad \overset{\times}{\underline{\mathbf{F}}} = -\frac{1}{2}\underset{\approx}{\epsilon} : \underset{\sim}{F}, \qquad \overset{\times}{\underline{\mathbf{F}}} \otimes \nabla_x = \underset{\sim}{K} \tag{9}$$

where y (resp. x) denotes the local (resp. macroscopic) coordinates. It is therefore possible to subject the unit cell to a prescribed curvature in addition to the mean strain $\underset{\sim}{E}$ [13]. In the case of periodic media, the non–homogeneous boundary condition (8) can be extended as follows :

$$\underline{u} = \underset{\sim}{E}.\underline{y} + \frac{2}{3}\underset{\approx}{\epsilon} : ((\underset{\sim}{K}.\underline{y}) \otimes \underline{y}) + \underline{v} \tag{10}$$

and fluctuation $\underline{v}$ takes the same value at homologous points of the unit cell boundary [13].

For both cases (8) and (10), an extended form of Hill–Mandel relation can be worked out :

$$\begin{aligned} < \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} > \;&=\; \underset{\sim}{\Sigma} : \underset{\sim}{E} + \underset{\sim}{M} : \underset{\sim}{K} \quad \text{with} \\ & \underset{\sim}{\Sigma} = < \underset{\sim}{\sigma} >, \quad M_{ij} = \frac{2}{3} < \epsilon_{ikl} x_k \sigma_{lj}. > \end{aligned} \tag{11}$$

In this expression, one recognizes the work of internal forces of a Cosserat effective medium, $\underset{\sim}{\Sigma}$ and $\underset{\sim}{M}$ being the corresponding force and couple stress tensors. More precisely, it is a Cosserat medium with internal constraint (*couple stress theory* [20]), since the force stress tensor still is symmetric and the trace of curvature and couple stresses vanishes.

The solution of the boundary value problem with prescribed curvature provides for instance an additional elastic modulus, namely the bending modulus of the heterogeneous material. Such Cosserat additional constants have also been determined in [21] and [22] according to a different method, and used for subsequent structural computations.

Note that the choice of the previous polynomial expansions is straightforward but remains heuristic. A more systematic way of constructing such expansions is described in the next section.

2.3 Definition of macroscopic generalized degrees of freedom

The aim here is to replace the heterogeneous material subjected to slowly–varying mean fields by a homogeneous micromorphic substitution medium. For that purpose the additional degrees of freedom must be defined as functions of the local fields on Ω. The macroscopic displacement field $\underline{\boldsymbol{U}}(\underline{\boldsymbol{x}})$ and the micro–deformation $\underset{\sim}{\boldsymbol{\chi}}(\underline{\boldsymbol{x}})$ are interpreted as the best approximation at position $\underline{\boldsymbol{x}}$ of the local displacement field by a homogeneous rotation and deformation :

$$(\underline{\boldsymbol{U}}(\underline{\boldsymbol{x}}), \underset{\sim}{\boldsymbol{\chi}}(\underline{\boldsymbol{x}})) = \min_{(\underline{\boldsymbol{U}}, \underset{\sim}{\boldsymbol{\chi}})} <\mid \underline{\boldsymbol{u}}(\underline{\boldsymbol{y}}) - \underline{\boldsymbol{U}} - \underset{\sim}{\boldsymbol{\chi}}.(\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}) \mid^2>_\Omega \qquad (12)$$

The solutions of this minimizing process are :

$$\underline{\boldsymbol{U}}(\underline{\boldsymbol{x}}) =< \underline{\boldsymbol{u}}(\underline{\boldsymbol{y}}) >, \quad \underset{\sim}{\boldsymbol{\chi}}(\underline{\boldsymbol{x}}) =< \underline{\boldsymbol{u}} \otimes (\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}) > .\underset{\sim}{\boldsymbol{A}}^{-1} \qquad (13)$$

$$\text{with} \quad \underset{\sim}{\boldsymbol{A}} =< (\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}) \otimes (\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}) > \qquad (14)$$

In the classical approach, the approximation simply is the mean displacement $\underline{\boldsymbol{U}}$. The gradients of the previous degrees of freedom are computed as :

$$\underline{\boldsymbol{U}} \otimes \boldsymbol{\nabla}_x =< \underline{\boldsymbol{u}} \otimes \boldsymbol{\nabla}_y >, \quad \underset{\sim}{\boldsymbol{\chi}} \otimes \boldsymbol{\nabla}_x =< (\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{y}}) \otimes \boldsymbol{\nabla}_y > .\underset{\sim}{\boldsymbol{A}}^{-1} - \underline{\boldsymbol{U}} \otimes \underset{\sim}{\boldsymbol{A}}^{-T} \qquad (15)$$

2.4 Two–dimensional Cosserat homogeneous substitution medium

The special case of a Cosserat homogeneous substitution medium is investigated in [15]. This corresponds to a skew–symmetric micro–deformation

$\underset{\sim}{\chi}$ and to the approximation of the local field by a rigid body motion in equation (12).

Let us consider for simplicity a quadratic unit cell Y of a two–dimensional periodic medium, with edge length l. The macroscopic degrees of freedom are the displacement $\underline{\boldsymbol{U}}$ and the micro–rotation $\underline{\boldsymbol{\Phi}}$. They are related to local quantities inside Y by expressions deduced from (14) :

$$\underline{\boldsymbol{U}}(\underline{\boldsymbol{x}}) = <\underline{\boldsymbol{u}}>, \quad \underline{\boldsymbol{\Phi}}(\underline{\boldsymbol{x}}) = \frac{6}{l^2} < (\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}) \times \underline{\boldsymbol{u}} > \tag{16}$$

The corresponding macroscopic deformation measures are the classical strain tensor $\underset{\sim}{\boldsymbol{E}}$, the relative rotation $\underline{\boldsymbol{\Omega}} - \underline{\boldsymbol{\Phi}}$ and the curvature $\underset{\sim}{\boldsymbol{K}}$, $\underline{\boldsymbol{\Omega}}$ being the axial material rotation vector associated with the skew–symmetric part of the deformation gradient :

$$\underset{\sim}{\boldsymbol{E}} = \frac{1}{2}(\boldsymbol{\nabla}_x \otimes \underline{\boldsymbol{U}} + \underline{\boldsymbol{U}} \otimes \boldsymbol{\nabla}_x)(\underline{\boldsymbol{x}}) \tag{17}$$

$$\underline{\boldsymbol{\Omega}}(\underline{\boldsymbol{x}}) = \frac{1}{2}(\frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2})\underline{\boldsymbol{e}}_3 = \Omega(\underline{\boldsymbol{x}})\underline{\boldsymbol{e}}_3 \tag{18}$$

$$\begin{aligned} \underline{\boldsymbol{K}} &= \frac{\partial \Phi_3}{\partial x_1}\underline{\boldsymbol{e}}_1 + \frac{\partial \Phi_3}{\partial x_2}\underline{\boldsymbol{e}}_2 \\ &= \frac{6}{l^2} < ((\underline{\boldsymbol{y}} \times \underline{\boldsymbol{u}}).\underline{\boldsymbol{e}}_3)\boldsymbol{\nabla}_y >_\Omega - \frac{6}{l^2}((\underset{\sim}{\boldsymbol{1}} \times \underline{\boldsymbol{U}}).\underline{\boldsymbol{e}}_3)\boldsymbol{\nabla}_x \end{aligned} \tag{19}$$

where $(\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3)$ denotes a direct basis of orthogonal unit vectors. The previous quantities are now evaluated in the case of a polynomial displacement field of the form :

$$\begin{aligned} u_i &= A_i + B_{i1}\tilde{y}_1 + B_{i2}\tilde{y}_2 + C_{i1}\tilde{y}_1^2 + C_{i2}\tilde{y}_2^2 + 2C_{i3}\tilde{y}_1\tilde{y}_2 \\ &+ D_{i1}\tilde{y}_1^3 + D_{i2}\tilde{y}_2^3 + 3D_{i3}\tilde{y}_1^2\tilde{y}_2 + 3D_{i4}\tilde{y}_1\tilde{y}_2^2 \end{aligned} \tag{20}$$

with $(i = 1, 2), (\tilde{y}_1 = y_1/l, \tilde{y}_2 = y_2/l)$. The coefficients of the polynomial can be identified under the conditions $E_{ij}(\underline{\boldsymbol{x}} = 0) = B_{ij}$ and of constant curvature $\underline{\boldsymbol{K}}$:

$$(\Phi - \Omega)(\underline{\boldsymbol{x}}) = \frac{D_{12}}{10l}, \qquad \underline{\boldsymbol{K}}(\underline{\boldsymbol{x}}) = \frac{C_{21} - C_{13}}{l^2}\underline{\boldsymbol{e}}_1 + \frac{C_{23} - C_{12}}{l^2}\underline{\boldsymbol{e}}_2 \tag{21}$$

The final form of the polynomial for the determination of the properties of a two–dimentional Cosserat HSM is [15] :

$$\left\{ \begin{array}{ll} u_1^* & = B_{11}\tilde{y}_1 + B_{12}\tilde{y}_2 - C_{23}\tilde{y}_2^2 + 2C_{13}\tilde{y}_1\tilde{y}_2 + D_{12}(\tilde{y}_2^3 - 3\tilde{y}_1^2\tilde{y}_2) \\ \\ u_2^* & = B_{12}\tilde{y}_1 + B_{22}\tilde{y}_2 - C_{13}\tilde{y}_1^2 + 2C_{23}\tilde{y}_1\tilde{y}_2 - D_{12}(\tilde{y}_1^3 - 3\tilde{y}_1\tilde{y}_2^2) \end{array} \right. \tag{22}$$

The solution of the boundary value problem on the unit cell Y is searched under the form $\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}^* + \underline{\boldsymbol{v}}$, with periodicity conditions for the fluctuation $\underline{\boldsymbol{v}}$ and anti-periodic conditions for the traction vector $\underset{\sim}{\boldsymbol{\sigma}}.\underline{\boldsymbol{n}}$. The quadratic term in equation (22) is identical to the proposal (8), whereas the cubic ansatz is necessary in order to evaluate the effect of relative rotation. The found HSM is a full Cosserat–medium without constraint. The advantages of using a Cosserat HSM instead of the classical Cauchy homogenized medium were investigated in specific examples assuming linear elasticity in [13, 15]. It can also be used in the nonlinear case like elastoplasticity as done in [23].

2.5 Three–dimensional Cosserat homogeneous substitution medium

A straightforward extension of the form (20) of the polynomial to the three–dimensional case could read :

$$\begin{array}{rcl} u_i & = & B_{i1}\tilde{y}_1 + B_{i2}\tilde{y}_2 + B_{i3}\tilde{y}_3 \\ & + & C_{i1}\tilde{y}_1^2 + C_{i2}\tilde{y}_2^2 + C_{i3}\tilde{y}_3^2 + 2C_{i4}\tilde{y}_1\tilde{y}_2 + 2C_{i5}\tilde{y}_2\tilde{y}_3 + 2C_{i6}\tilde{y}_3\tilde{y}_1 \\ & + & D_{i1}\tilde{y}_1^3 + D_{i2}\tilde{y}_2^3 + D_{i3}\tilde{y}_3^3 \\ & + & 3D_{i4}\tilde{y}_1^2\tilde{y}_2 + 3D_{i5}\tilde{y}_1\tilde{y}_2^2 + 3D_{i6}\tilde{y}_2^2\tilde{y}_3 \\ & + & 3D_{i7}\tilde{y}_2\tilde{y}_3^2 + 3D_{i8}\tilde{y}_3^2\tilde{y}_1 + 3D_{i9}\tilde{y}_3\tilde{y}_1^2 + E_i\tilde{y}_1\tilde{y}_2\tilde{y}_3 \end{array}$$

According to equation (19), the diagonal components of the curvature tensors become :

$$\underset{\sim}{\boldsymbol{K}} = < (\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}) \times \underline{\boldsymbol{u}} > \otimes \boldsymbol{\nabla}_x$$

$$K_{11} = C_{34} - C_{26} + 3(D_{34} - D_{29})y_1 + (3D_{25} - \frac{E_1}{2})y_2 + (\frac{E_3}{2} - 3D_{28})y_3$$

$$K_{22} = C_{15} - C_{34} + (\frac{E_1}{2} - 3D_{34})y_1 + 3(D_{16} - D_{35})y_2 + (3D_{17} - \frac{E_3}{2})y_3$$

$$K_{33} = C_{26} - C_{15} + (3D_{29} - \frac{E_1}{2})y_1 + (\frac{E_2}{2} - 3D_{16})y_2 + (3D_{28} - 3D_{17})y_3$$

However it turns out that the trace of the curvature is always vanishing. It means that the previous form of the polynomial does not allow to prescribe a spherical curvature to the unit cell. This becomes possible once a polynomial expansion up to the fourth degree is introduced :

$$\begin{aligned}
u_1 &= E_{11}y_1 + E_{12}y_2 + E_{31}y_3 \\
&- k_{31}y_1y_2 - \frac{k_{32}}{2}y_2^2 + k_{21}y_1y_3 + \frac{k_{23}}{2}y_3^2 + 2(k_{22}^{dev} - k_{33}^{dev})y_2y_3 \\
&+ 10\Theta_3(y_2^3 - 3y_1^2y_2) - 10\Theta_2(y_3^3 - 3y_1^2y_3) + \frac{10\text{trace}(\underset{\sim}{\boldsymbol{k}})}{3}(y_2^2 - y_3^2)y_2y_3 \\
u_2 &= E_{12}y_1 + E_{22}y_2 + E_{23}y_3 \\
&+ k_{32}y_1y_2 + \frac{k_{31}}{2}y_1^2 - k_{12}y_2y_3 - \frac{k_{13}}{2}y_3^2 - 2(k_{11}^{dev} + k_{33}^{dev})y_1y_3 \\
&- 10\Theta_3(y_1^3 - 3y_1y_2^2) + 10\Theta_1(y_3^3 - 3y_2^2y_3) + \frac{10\text{trace}(\underset{\sim}{\boldsymbol{k}})}{3}(y_3^2 - y_1^2)y_1y_3 \\
u_3 &= E_{31}y_1 + E_{23}y_2 + E_{33}y_3 \\
&- k_{23}y_1y_3 - \frac{k_{21}}{2}y_1^2 + k_{13}y_2y_3 + \frac{k_{12}}{2}y_2^2 + 2(k_{11}^{dev} + k_{22}^{dev})y_1y_2 \\
&+ 10\Theta_2(y_1^3 - 3y_1y_3^2) - 10\Theta_1(y_2^3 - 3y_2y_3^2) + \frac{10\text{trace}(\underset{\sim}{\boldsymbol{k}})}{3}(y_1^2 - y_2^2)y_1y_2
\end{aligned}$$

where $\underset{\sim}{\boldsymbol{k}}^{dev}$ is the deviatoric part of $\underset{\sim}{\boldsymbol{k}}$. It follows that :

$$\underline{\boldsymbol{\Phi}} - \underline{\boldsymbol{\Omega}} = \underline{\boldsymbol{\Theta}} + \frac{\text{trace}(\underset{\sim}{\boldsymbol{k}})}{3}\underline{\boldsymbol{x}} \qquad K_{ij} = k_{ij} - 10\text{trace}(\underset{\sim}{\boldsymbol{k}})x_ix_j \quad \text{with } i \neq j \tag{23}$$

$$K_{11} = k_{11}^{dev} + \frac{\text{trace}(\underset{\sim}{\boldsymbol{k}})}{3} + 10\text{trace}(\underset{\sim}{\boldsymbol{k}})x_1^2 - 5\text{trace}(\underset{\sim}{\boldsymbol{k}})(x_2^2 + x_3^2) \tag{24}$$

3 Homogenization of micromorphic media

In contrast to the previous part, the heterogeneous medium is now a mixture of micromorphic constituents, i.e. a heterogeneous micromorphic medium. One investigates the nature of the resulting homogeneous equivalent medium by means of asymptotic methods.

3.1 Balance and constitutive equations of the micromorphic continuum

The motion of a micromorphic body Ω is described by two independent sets of degrees of freedom : the displacement $\underline{\boldsymbol{u}}$ and the micro–deformation $\underset{\sim}{\boldsymbol{\chi}}$ attributed to each material point. The micro–deformation accounts for the rotation and distorsion of a triad associated with the underlying microstructure [2]. The micro–deformation can be split into its symmetric and skew–symmetric parts :

$$\underset{\sim}{\boldsymbol{\chi}} = \underset{\sim}{\boldsymbol{\chi}}^{s} + \underset{\sim}{\boldsymbol{\chi}}^{a} \tag{25}$$

that are called respectively the micro–strain and the Cosserat rotation. The associated deformation fields are the classical strain tensor $\underset{\sim}{\boldsymbol{\varepsilon}}$, the relative deformation $\underset{\sim}{\boldsymbol{e}}$ and the micro–deformation gradient tensor $\underset{\approx}{\boldsymbol{\kappa}}$ defined by :

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{u}} \overset{s}{\otimes} \boldsymbol{\nabla}, \quad \underset{\sim}{\boldsymbol{e}} = \underline{\boldsymbol{u}} \otimes \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\chi}}, \quad \underset{\approx}{\boldsymbol{\kappa}} = \underset{\sim}{\boldsymbol{\chi}} \otimes \boldsymbol{\nabla} \tag{26}$$

The symmetric part of $\underset{\sim}{\boldsymbol{e}}$ corresponds to the difference of material strain and micro–strain, whereas its skew-symmetric part accounts for the relative rotation of the material with respect to microstructure. The micro–deformation gradient can be split into two contributions :

$$\underset{\approx}{\boldsymbol{\kappa}} = \underset{\approx}{\boldsymbol{\kappa}}^{s} + \underset{\approx}{\boldsymbol{\kappa}}^{a}, \quad \text{with} \quad \underset{\approx}{\boldsymbol{\kappa}}^{s} = \underset{\sim}{\boldsymbol{\chi}}^{s} \otimes \boldsymbol{\nabla}, \quad \underset{\approx}{\boldsymbol{\kappa}}^{a} = \underset{\sim}{\boldsymbol{\chi}}^{a} \otimes \boldsymbol{\nabla} \tag{27}$$

In this work, the analysis is restricted to small deformations, small micro–rotations, small micro–strains and small micro–deformation gradients. The statics of the micromorphic continuum is described by the symmetric force-stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$, the generally non-symmetric relative force–stress tensor $\underset{\sim}{\boldsymbol{s}}$ and third–rank stress tensor $\underset{\approx}{\boldsymbol{m}}$. These tensors must fulfill the local form of the balance equations in the static case, in the absence of body simple nor double forces for simplicity :

$$(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}) \cdot \boldsymbol{\nabla} = 0, \quad \underset{\approx}{\boldsymbol{m}} \cdot \boldsymbol{\nabla} + \underset{\sim}{\boldsymbol{s}} = 0 \text{ on } \Omega \tag{28}$$

The constitutive equations for linear elastic centro-symmetric micromorphic materials read :

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{a}} : \underset{\sim}{\boldsymbol{\varepsilon}}, \quad \underset{\sim}{\boldsymbol{s}} = \underset{\approx}{\boldsymbol{b}} : \underset{\sim}{\boldsymbol{e}}, \quad \underset{\approx}{\boldsymbol{m}} = \underset{\approx}{\underset{\approx}{\boldsymbol{c}}} \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}} \tag{29}$$

The elasticity tensors display the major symmetries :

$$a_{ijkl} = a_{klij}, \quad b_{ijkl} = b_{klij}, \qquad c_{ijkpqr} = c_{pqrijk} \tag{30}$$

and $\underset{\approx}{\boldsymbol{a}}$ has also the usual minor symmetries. It will be assumed that the last constitutive law takes the form :

$$\underset{\sim}{\boldsymbol{m}} = \underset{\approx\approx}{\boldsymbol{c}}^s \vdots \underset{\sim}{\boldsymbol{\kappa}}^s + \underset{\approx\approx}{\boldsymbol{c}}^a \vdots \underset{\sim}{\boldsymbol{\kappa}}^a \tag{31}$$

and that the tensors $\underset{\approx\approx}{\boldsymbol{c}}^s$ and $\underset{\approx\approx}{\boldsymbol{c}}^a$ fulfill the conditions :

$$c^s_{ijkpqr} = c^s_{jikpqr}, \quad c^a_{ijkpqr} = -c^a_{jikpqr} \tag{32}$$

thus assuming that there is no coupling between the contributions of the symmetric and skew–symmetric parts of $\underset{\sim}{\boldsymbol{\chi}}$ to the third–rank stress tensor.

The setting of the boundary value problem on body Ω is then closed by the boundary conditions. In the following, Dirichlet boundary conditions are considered of the form :

$$\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}) = 0, \quad \underset{\sim}{\boldsymbol{\chi}}(\underline{\boldsymbol{x}}) = 0, \quad \forall \underline{\boldsymbol{x}} \in \partial\Omega \tag{33}$$

where $\partial\Omega$ denote the boundary of Ω. The equations (26), (28), (29) and (33) define the boundary value problem $\mathcal{P}$.

The next sections of this work are restricted to micromorphic materials with periodic microstructure. The heterogeneous material is then obtained by space tesselation with cells translated from a single cell Y^l. The period of the microstructure is described by three dimensionless independent vectors $(\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \underline{\boldsymbol{a}}_3)$ such that :

$$Y^l = \left\{ \underline{\boldsymbol{x}} = x_i \underline{\boldsymbol{a}}_i,\ |x_i| < \frac{l}{2} \right\}$$

where l is the characteristic size of the cell. We call $\underset{\approx}{\boldsymbol{a}}^l, \underset{\approx}{\boldsymbol{b}}^l$ and $\underset{\approx\approx}{\boldsymbol{c}}^l$ the elasticity tensor fields of the periodic Cosserat material. They are such that :

$$\forall \underline{\boldsymbol{x}} \in \Omega, \forall\, (n_1, n_2, n_3) \in Z^3 / \quad \underline{\boldsymbol{x}} + l(n_1 \underline{\boldsymbol{a}}_1 + n_2 \underline{\boldsymbol{a}}_2 + n_3 \underline{\boldsymbol{a}}_3) \in \Omega$$

$$\underset{\approx}{\boldsymbol{a}}^l(\underline{\boldsymbol{x}}) = \underset{\approx}{\boldsymbol{a}}^l(\underline{\boldsymbol{x}} + l(n_1 \underline{\boldsymbol{a}}_1 + n_2 \underline{\boldsymbol{a}}_2 + n_3 \underline{\boldsymbol{a}}_3)), \quad \underset{\approx}{\boldsymbol{b}}^l(\underline{\boldsymbol{x}}) = \underset{\approx}{\boldsymbol{b}}^l(\underline{\boldsymbol{x}} + l(n_1 \underline{\boldsymbol{b}}_1 + n_2 \underline{\boldsymbol{b}}_2 + n_3 \underline{\boldsymbol{b}}_3))$$

$$\underset{\approx\approx}{\boldsymbol{c}}^l(\underline{\boldsymbol{x}}) = \underset{\approx\approx}{\boldsymbol{c}}^l(\underline{\boldsymbol{x}} + l(n_1 \underline{\boldsymbol{a}}_1 + n_2 \underline{\boldsymbol{a}}_2 + n_3 \underline{\boldsymbol{a}}_3))$$

3.2 Dimensional analysis

The size L of body Ω is defined for instance as the maximum distance between two points. Dimensionless coordinates and displacements are introduced :

$$\underline{\boldsymbol{x}}^* = \frac{\underline{\boldsymbol{x}}}{L}, \quad \underline{\boldsymbol{u}}^*(\underline{\boldsymbol{x}}^*) = \frac{\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})}{L}, \quad \underset{\sim}{\boldsymbol{\chi}}^*(\underline{\boldsymbol{x}}^*) = \underset{\sim}{\boldsymbol{\chi}}(\underline{\boldsymbol{x}}) \tag{34}$$

The corresponding strain measures are :

$$\underset{\sim}{\boldsymbol{\varepsilon}}^*(\underline{\boldsymbol{x}}^*) = \underline{\boldsymbol{u}}^* \overset{s}{\otimes} \boldsymbol{\nabla}^* \underset{\sim}{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{x}}), \quad \underset{\sim}{\boldsymbol{e}}^*(\underline{\boldsymbol{x}}^*) = \underline{\boldsymbol{u}}^* \otimes \boldsymbol{\nabla}^* - \underset{\sim}{\boldsymbol{\chi}}^* = \underset{\sim}{\boldsymbol{e}}(\underline{\boldsymbol{x}}) \tag{35}$$

$$\underset{\approx}{\boldsymbol{\kappa}}^*(\underline{\boldsymbol{x}}^*) = \underset{\sim}{\boldsymbol{\chi}}^* \otimes \boldsymbol{\nabla}^* = L\, \underset{\approx}{\boldsymbol{\kappa}}(\underline{\boldsymbol{x}}) \tag{36}$$

and similarily

$$\underset{\approx}{\boldsymbol{\kappa}}^{s*}(\underline{\boldsymbol{x}}^*) = \underset{\sim}{\boldsymbol{\chi}}^{s*} \otimes \boldsymbol{\nabla}^* = L\, \underset{\approx}{\boldsymbol{\kappa}}^{s}(\underline{\boldsymbol{x}}), \quad \underset{\approx}{\boldsymbol{\kappa}}^{a*}(\underline{\boldsymbol{x}}^*) = \underset{\sim}{\boldsymbol{\chi}}^{a*} \otimes \boldsymbol{\nabla}^* = L\, \underset{\approx}{\boldsymbol{\kappa}}^{a}(\underline{\boldsymbol{x}}) \tag{37}$$

with $\underline{\boldsymbol{\nabla}}^* = \left(\dfrac{\partial\, \cdot}{\partial x_i^*}\right) \underline{\boldsymbol{e}}_i = L\boldsymbol{\nabla}$. It is necessary to introduce next a norm of the elasticity tensors :

$$A = \underset{\underline{\boldsymbol{x}} \in Y^l}{\text{Max}} \left(\left| a^l_{ijkl}(\underline{\boldsymbol{x}}) \right|, \left| b^l_{ijkl}(\underline{\boldsymbol{x}}) \right| \right)$$

$$C^s = \underset{\underline{\boldsymbol{x}} \in Y^l}{\text{Max}} \left| c^{sl}_{ijkpqr}(\underline{\boldsymbol{x}}) \right|, \quad C^a = \underset{\underline{\boldsymbol{x}} \in Y^l}{\text{Max}} \left| c^{al}_{ijkpqr}(\underline{\boldsymbol{x}}) \right| \tag{38}$$

whereby characteristic lengths l_s and l_a can be defined :

$$C^s = A l_s^2, \quad C^a = A l_a^2 \tag{39}$$

The definition of dimensionless stress and elasticity tensors follows :

$$\underset{\sim}{\boldsymbol{\sigma}}^*(\underline{\boldsymbol{x}}^*) = A^{-1} \underset{\sim}{\boldsymbol{\sigma}}(\underline{\boldsymbol{x}}), \quad \underset{\sim}{\boldsymbol{s}}^*(\underline{\boldsymbol{x}}^*) = A^{-1} \underset{\sim}{\boldsymbol{s}}(\underline{\boldsymbol{x}}), \quad \underset{\approx}{\boldsymbol{m}}^*(\underline{\boldsymbol{x}}^*) = (AL)^{-1} \underset{\approx}{\boldsymbol{m}}(\underline{\boldsymbol{x}}) \tag{40}$$

$$\underset{\approx}{\boldsymbol{a}}^*(\underline{\boldsymbol{x}}^*) = A^{-1} \underset{\approx}{\boldsymbol{a}}^l(\underline{\boldsymbol{x}}), \quad \underset{\approx}{\boldsymbol{b}}^*(\underline{\boldsymbol{x}}^*) = A^{-1} \underset{\approx}{\boldsymbol{b}}^l(\underline{\boldsymbol{x}}), \tag{41}$$

$$\underset{\approx\approx}{\boldsymbol{c}}^{s*}(\underline{\boldsymbol{x}}^*) = (A l_c^2)^{-1} \underset{\approx\approx}{\boldsymbol{c}}^{sl}(\underline{\boldsymbol{x}}), \quad \underset{\approx\approx}{\boldsymbol{c}}^{a*}(\underline{\boldsymbol{x}}^*) = (A l_c^2)^{-1} \underset{\approx\approx}{\boldsymbol{c}}^{al}(\underline{\boldsymbol{x}}) \tag{42}$$

Since the initial tensors $\underset{\approx}{\boldsymbol{a}}^l, \underset{\approx}{\boldsymbol{b}}^l$ and $\underset{\approx\approx}{\boldsymbol{c}}^l$ are Y^l-periodic, the dimensionless counterparts are Y^*-periodic :

$$Y^* = \frac{l}{L} Y, \quad Y = \left\{ \underline{\boldsymbol{y}} = y_i \underline{\boldsymbol{a}}_i, \; |y_i| < \frac{1}{2} \right\} \tag{43}$$

Y is the (dimensionless) unit cell used in the present asymptotic analyses. As a result, the dimensionless stress and strain tensors are related by the following constitutive equations :

$$\underset{\sim}{\boldsymbol{\sigma}}^* = \underset{\approx}{\boldsymbol{a}}^* : \underset{\sim}{\boldsymbol{\varepsilon}}^*, \quad \underset{\sim}{\boldsymbol{s}}^* = \underset{\approx}{\boldsymbol{b}}^* : \underset{\sim}{\boldsymbol{e}}^*, \quad \underset{\approx}{\boldsymbol{m}}^* = \left(\frac{l_s}{L}\right)^2 \underset{\approx\approx}{\boldsymbol{c}}^{s*} \vdots \underset{\approx}{\boldsymbol{\kappa}}^{s*} + \left(\frac{l_a}{L}\right)^2 \underset{\approx\approx}{\boldsymbol{c}}^{a*} \vdots \underset{\approx}{\boldsymbol{\kappa}}^{a*} \tag{44}$$

The dimensionless balance equations read :

$$\forall \underline{\boldsymbol{x}}^* \in \Omega^*, \quad (\underset{\sim}{\boldsymbol{\sigma}}^* + \underset{\sim}{\boldsymbol{s}}^*) \cdot \boldsymbol{\nabla}^* = 0, \quad \underset{\approx}{\boldsymbol{m}}^* \cdot \boldsymbol{\nabla}^* + \underset{\sim}{\boldsymbol{s}}^* = 0 \tag{45}$$

A boundary value problem $\mathcal{P}^*$ can be defined using equations (36), (44) and (45), complemented by the boundary conditions :

$$\forall \underline{\boldsymbol{x}}^* \in \partial\Omega^*, \quad \underline{\boldsymbol{u}}^*(\underline{\boldsymbol{x}}^*) = 0, \quad \underset{\sim}{\boldsymbol{\chi}}^*(\underline{\boldsymbol{x}}^*) = 0 \tag{46}$$

3.3 The homogenization problem

The boundary value problem $\mathcal{P}^*$ is treated here as an element of a series of problems $(\mathcal{P}_\epsilon)_{\epsilon>0}$ on Ω^*. The homogenization problem consists in the determination of the limit of this series when the dimensionless parameter ϵ, regarded as small, tends towards 0. The series is chosen such that

$$\mathcal{P}_{\epsilon=\frac{l}{L}} = \mathcal{P}^*$$

The unknowns of boundary value problem $\mathcal{P}_\epsilon$ are the displacement and micro–deformation fields $\underline{\boldsymbol{u}}^\epsilon$ and $\underset{\sim}{\boldsymbol{\chi}}^\epsilon$ satisfying the following field equations on Ω^* :

$$\underset{\sim}{\boldsymbol{\sigma}}^\epsilon = \underset{\approx}{\boldsymbol{a}}^\epsilon : (\underline{\boldsymbol{u}}^\epsilon \overset{s}{\otimes} \boldsymbol{\nabla}^*), \quad \underset{\sim}{\boldsymbol{s}}^\epsilon = \underset{\approx}{\boldsymbol{b}}^\epsilon : (\underline{\boldsymbol{u}}^\epsilon \otimes \boldsymbol{\nabla}^* - \underset{\sim}{\boldsymbol{\chi}}^\epsilon), \quad \underset{\approx}{\boldsymbol{m}}^\epsilon = \underset{\approx\approx}{\boldsymbol{c}}^\epsilon \vdots (\underset{\sim}{\boldsymbol{\chi}}^\epsilon \otimes \boldsymbol{\nabla}^*) \tag{47}$$

$$(\underset{\sim}{\boldsymbol{\sigma}}^\epsilon + \underset{\sim}{\boldsymbol{s}}^\epsilon) \cdot \boldsymbol{\nabla}^* = 0, \quad \underset{\approx}{\boldsymbol{m}}^\epsilon \cdot \boldsymbol{\nabla}^* + \underset{\sim}{\boldsymbol{s}}^\epsilon = 0 \tag{48}$$

Different cases must now be distinguished depending on the relative position of the constitutive lengths l_s and l_a with respect to the characteristic lengths l and L of the problem. Four special cases are relevant for the present asymptotic analysis. The first case corresponds to a limiting process for which l_s/l and l_a/l remain constant when l/L goes to zero. The second case corresponds to the situation for which l_s/L and l_a/L remain constant when l/L goes to zero. The third (resp. fourth) situation assumes that l_s/l and l_a/L (resp. l_s/L and l_a/l) remain constant when l/L goes to zero. These assumptions lead to four different homogenization schemes labelled $HS1$ to $HS4$ in the sequel. The homogenization scheme 1 (resp. 2) will be relevant when the ratio l/L is small enough and when l_s, l_a and l (resp. L) have the same order of magnitude.
Accordingly, the following tensors of elastic moduli can be defined :

$$\underset{\approx}{\boldsymbol{a}}^{(0)}(\underline{\boldsymbol{y}}) = \underset{\approx}{\boldsymbol{a}}^{*}(\frac{l}{L}\underline{\boldsymbol{y}}), \quad \underset{\approx}{\boldsymbol{b}}^{(0)}(\underline{\boldsymbol{y}}) = \underset{\approx}{\boldsymbol{b}}^{*}(\frac{l}{L}\underline{\boldsymbol{y}}), \tag{49}$$

$$\underset{\approx}{\boldsymbol{c}}^{(1)}(\underline{\boldsymbol{y}}) = \left(\frac{l_s}{l}\right)^2 \underset{\approx}{\boldsymbol{c}}^{*}(\frac{l}{L}\underline{\boldsymbol{y}}), \quad \underset{\approx}{\boldsymbol{c}}^{(2)}(\underline{\boldsymbol{y}}) = \left(\frac{l_s}{L}\right)^2 \underset{\approx}{\boldsymbol{c}}^{*}(\frac{l}{L}\underline{\boldsymbol{y}}), \tag{50}$$

$$\underset{\approx}{\boldsymbol{c}}^{s(1)}(\underline{\boldsymbol{y}}) = \left(\frac{l_s}{l}\right)^2 \underset{\approx}{\boldsymbol{c}}^{s*}(\frac{l}{L}\underline{\boldsymbol{y}}), \quad \underset{\approx}{\boldsymbol{c}}^{a(1)}(\underline{\boldsymbol{y}}) = \left(\frac{l_a}{l}\right)^2 \underset{\approx}{\boldsymbol{c}}^{a*}(\frac{l}{L}\underline{\boldsymbol{y}}), \tag{51}$$

$$\underset{\approx}{\boldsymbol{c}}^{s(2)}(\underline{\boldsymbol{y}}) = \left(\frac{l_s}{L}\right)^2 \underset{\approx}{\boldsymbol{c}}^{s*}(\frac{l}{L}\underline{\boldsymbol{y}}), \quad \underset{\approx}{\boldsymbol{c}}^{a(2)}(\underline{\boldsymbol{y}}) = \left(\frac{l_a}{L}\right)^2 \underset{\approx}{\boldsymbol{c}}^{a*}(\frac{l}{L}\underline{\boldsymbol{y}}) \tag{52}$$

They are Y-periodic since $\underset{\approx}{\boldsymbol{a}}^*, \underset{\approx}{\boldsymbol{b}}^*$ and $\underset{\approx}{\boldsymbol{c}}^*$ are Y^*-periodic. Four different hypotheses will be made concerning the constitutive tensors of problem $\mathcal{P}_\epsilon$:

Assumption 1 : $\underset{\approx}{\boldsymbol{a}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{a}}^{(0)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$, $\underset{\approx}{\boldsymbol{b}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{b}}^{(0)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$ and $\underset{\approx}{\boldsymbol{c}}^\epsilon(\underline{\boldsymbol{x}}^*) = \epsilon^2 \underset{\approx}{\boldsymbol{c}}^{(1)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$;

Assumption 2 : $\underset{\approx}{\boldsymbol{a}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{a}}^{(0)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$, $\underset{\approx}{\boldsymbol{b}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{b}}^{(0)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$ and $\underset{\approx}{\boldsymbol{c}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{c}}^{(2)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$;

Assumption 3 : $\underset{\approx}{\boldsymbol{a}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{a}}^{(0)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$, $\underset{\approx}{\boldsymbol{b}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{b}}^{(0)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$ and $\underset{\approx}{\boldsymbol{c}}^{s\epsilon}(\underline{\boldsymbol{x}}^*) = \epsilon^2 \underset{\approx}{\boldsymbol{c}}^{s(1)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$, $\underset{\approx}{\boldsymbol{c}}^{a\epsilon}(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{c}}^{a(2)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$;

Assumption 4 : $\underset{\approx}{\boldsymbol{a}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{a}}^{(0)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$, $\underset{\approx}{\boldsymbol{b}}^\epsilon(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{b}}^{(0)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*)$ and

$$\underset{\approx}{\boldsymbol{c}}^{s\epsilon}(\underline{\boldsymbol{x}}^*) = \underset{\approx}{\boldsymbol{c}}^{s(2)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*),\ \underset{\approx}{\boldsymbol{c}}^{a\epsilon}(\underline{\boldsymbol{x}}^*) = \epsilon^2 \underset{\approx}{\boldsymbol{c}}^{a(1)}(\epsilon^{-1}\underline{\boldsymbol{x}}^*).$$

Assumptions 1 and 2 respectively correspond to the homogenization schemes HS1 and HS2. Both choices meet the requirement that

$$(\epsilon = \frac{l}{L}) \Rightarrow (\underset{\approx}{\boldsymbol{a}}^\epsilon = \underset{\approx}{\boldsymbol{a}}^* \quad \text{and} \quad \underset{\approx}{\boldsymbol{c}}^\epsilon = (\frac{l_s}{L})^2 \underset{\approx}{\boldsymbol{c}}^*)$$

Assumptions 3 and 4 respectively correspond to the homogenization schemes HS3 and HS4. Both choices meet the requirement that

$$(\epsilon = \frac{l}{L}) \Rightarrow (\underset{\approx}{\boldsymbol{a}}^\epsilon = \underset{\approx}{\boldsymbol{a}}^*, \quad \underset{\approx}{\boldsymbol{c}}^{s\epsilon} = (\frac{l_s}{L})^2 \underset{\approx}{\boldsymbol{c}}^{s*} \quad \text{and} \quad \underset{\approx}{\boldsymbol{c}}^{a\epsilon} = (\frac{l_a}{L})^2 \underset{\approx}{\boldsymbol{c}}^{a*})$$

It must be noted that, in our presentation of the asymptotic analysis, the lengths l, l_s, l_a and L are given and fixed, whereas parameter ϵ is allowed to tend to zero in the limiting process.

In the sequel, the stars $*$ are dropped for conciseness.

3.4 Multiscale asymptotic method

In the setting of the homogenization problems two space variables have been distinguished : $\underline{\boldsymbol{x}}$ describes the macroscopic scale and $\underline{\boldsymbol{y}}$ is the local variable in the unit cell Y. To solve the homogenization problem, it is resorted to the method of multiscale asymptotic developments initially introduced in [18]. According to this method, all fields are regarded as functions of both variables $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$. It is assumed that they can be expanded in a series of powers of small parameter ε. In particular, the displacement, micro-rotation, force and couple stress fields are supposed to take the form :

$$\begin{aligned}
\underline{\boldsymbol{u}}^\epsilon(\underline{\boldsymbol{x}}) &= \underline{\boldsymbol{u}}_0(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon \underline{\boldsymbol{u}}_1(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon^2 \underline{\boldsymbol{u}}_2(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \dots \\
\underset{\sim}{\boldsymbol{\chi}}^\epsilon(\underline{\boldsymbol{x}}) &= \underset{\sim}{\boldsymbol{\chi}}_1(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon \underset{\sim}{\boldsymbol{\chi}}_2(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon^2 \underset{\sim}{\boldsymbol{\chi}}_3(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \dots \\
\underset{\sim}{\boldsymbol{\sigma}}^\epsilon(\underline{\boldsymbol{x}}) &= \underset{\sim}{\boldsymbol{\sigma}}_0(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon \underset{\sim}{\boldsymbol{\sigma}}_1(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon^2 \underset{\sim}{\boldsymbol{\sigma}}_2(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \dots \\
\underset{\sim}{\boldsymbol{s}}^\epsilon(\underline{\boldsymbol{x}}) &= \underset{\sim}{\boldsymbol{s}}_0(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon \underset{\sim}{\boldsymbol{s}}_1(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon^2 \underset{\sim}{\boldsymbol{s}}_2(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \dots \\
\underset{\approx}{\boldsymbol{m}}^\epsilon(\underline{\boldsymbol{x}}) &= \underset{\approx}{\boldsymbol{m}}_0(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon \underset{\approx}{\boldsymbol{m}}_1(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \epsilon^2 \underset{\approx}{\boldsymbol{m}}_2(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) + \dots
\end{aligned} \tag{53}$$

where the coefficients $\underline{\boldsymbol{u}}_i(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$, $\underset{\sim}{\boldsymbol{\chi}}_i(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$, $\underset{\sim}{\boldsymbol{\sigma}}_i(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$, $\underset{\sim}{\boldsymbol{s}}_i(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$ and $\underset{\sim}{\boldsymbol{m}}_i(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$ are assumed to have the same order of magnitude and to be Y-periodic with respect to variable $\underline{\boldsymbol{y}}$ ($\underline{\boldsymbol{y}} = \underline{\boldsymbol{x}}/\epsilon$). The average operator over the unit cell Y is denoted by

$$\langle \cdot \rangle = \frac{1}{|Y|} \int_Y \cdot \, dV$$

As a result,

$$< \underline{\boldsymbol{u}}^\epsilon >= \underline{\boldsymbol{U}}_0 + \epsilon \underline{\boldsymbol{U}}_1 + \dots \quad \text{and} \quad < \underset{\sim}{\boldsymbol{\chi}}^\epsilon >= \underset{\sim}{\boldsymbol{\Xi}}_1 + \epsilon \underset{\sim}{\boldsymbol{\Xi}}_2 + \dots \tag{54}$$

where $\underline{\boldsymbol{U}}_i =< \underline{\boldsymbol{u}}_i >$ and $\underset{\sim}{\boldsymbol{\Xi}}_i =< \underset{\sim}{\boldsymbol{\chi}}_i >$. The gradient operator can be split into partial derivatives with respect to $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$:

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_x + \frac{1}{\epsilon} \boldsymbol{\nabla}_y \tag{55}$$

This operator is used to compute the strain measures and balance equations :

$$\begin{aligned}
\underset{\sim}{\boldsymbol{\varepsilon}}^\epsilon &= \epsilon^{-1} \underset{\sim}{\boldsymbol{\varepsilon}}_{-1} + \underset{\sim}{\boldsymbol{\varepsilon}}_0 + \epsilon^1 \underset{\sim}{\boldsymbol{\varepsilon}}_1 + \dots \\
&= \epsilon^{-1} \underline{\boldsymbol{u}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_y + (\underline{\boldsymbol{u}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \boldsymbol{\nabla}_y) \\
&+ \epsilon (\underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_2 \overset{s}{\otimes} \boldsymbol{\nabla}_y) + \dots \\
\underset{\sim}{\boldsymbol{e}}^\epsilon &= \epsilon^{-1} \underset{\sim}{\boldsymbol{e}}_{-1} + \underset{\sim}{\boldsymbol{e}}_0 + \epsilon^1 \underset{\sim}{\boldsymbol{e}}_1 + \dots \\
&= \epsilon^{-1} \underline{\boldsymbol{u}}_0 \otimes \boldsymbol{\nabla}_y + (\underline{\boldsymbol{u}}_0 \otimes \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \otimes \boldsymbol{\nabla}_y - \underset{\sim}{\boldsymbol{\chi}}_1) \\
&+ \epsilon (\underline{\boldsymbol{u}}_1 \otimes \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_2 \otimes \boldsymbol{\nabla}_y - \underset{\sim}{\boldsymbol{\chi}}_2) + \dots \\
\underset{\approx}{\boldsymbol{\kappa}}^\epsilon &= \epsilon^{-1} \underset{\approx}{\boldsymbol{\kappa}}_{-1} + \underset{\approx}{\boldsymbol{\kappa}}_0 + \epsilon^1 \underset{\approx}{\boldsymbol{\kappa}}_1 + \dots \\
&= \epsilon^{-1} \underset{\sim}{\boldsymbol{\chi}}_1 \otimes \boldsymbol{\nabla}_y + (\underset{\sim}{\boldsymbol{\chi}}_1 \otimes \boldsymbol{\nabla}_x + \underset{\sim}{\boldsymbol{\chi}}_2 \otimes \boldsymbol{\nabla}_y) \\
&+ \epsilon (\underset{\sim}{\boldsymbol{\chi}}_2 \otimes \boldsymbol{\nabla}_x + \underset{\sim}{\boldsymbol{\chi}}_3 \otimes \boldsymbol{\nabla}_y) + \dots
\end{aligned} \tag{56}$$

$$(\underset{\sim}{\boldsymbol{\sigma}}^\epsilon + \underset{\sim}{\boldsymbol{s}}^\epsilon) \cdot \boldsymbol{\nabla}_x + \epsilon^{-1} (\underset{\sim}{\boldsymbol{\sigma}}^\epsilon + \underset{\sim}{\boldsymbol{s}}^\epsilon) \cdot \boldsymbol{\nabla}_y = 0, \quad \underset{\approx}{\boldsymbol{m}}^\epsilon \cdot \boldsymbol{\nabla}_x + \epsilon^{-1} \underset{\approx}{\boldsymbol{m}}^\epsilon \cdot \boldsymbol{\nabla}_y + \underset{\sim}{\boldsymbol{s}}^\epsilon = 0 \tag{57}$$

Similar expansions are valid for the tensors $\underset{\approx}{\boldsymbol{\kappa}}^s, \underset{\approx}{\boldsymbol{\kappa}}^a$. The expansions of the stress tensors are then introduced in the balance equations (57) and the terms can be ordered with respect to the powers of ϵ. Identifying the terms of same order, we are lead to the following set of equations :

- *order* ϵ^{-1},

$$(\underset{\sim}{\boldsymbol{\sigma}}_0 + \underset{\sim}{\boldsymbol{s}}_0) \cdot \boldsymbol{\nabla}_y = 0 \quad \text{and} \quad \underset{\approx}{\boldsymbol{m}}_0 \cdot \boldsymbol{\nabla}_y = 0 \tag{58}$$

- *order* ϵ^{0},

$$(\underset{\sim}{\boldsymbol{\sigma}}_0 + \underset{\sim}{\boldsymbol{s}}_0) \cdot \boldsymbol{\nabla}_x + (\underset{\sim}{\boldsymbol{\sigma}}_1 + \underset{\sim}{\boldsymbol{s}}_1) \cdot \boldsymbol{\nabla}_y = 0 \quad \text{and} \quad \underset{\approx}{\boldsymbol{S}}_0 \cdot \boldsymbol{\nabla}_x + \underset{\approx}{\boldsymbol{S}}_1 \cdot \boldsymbol{\nabla}_y + \underset{\sim}{\boldsymbol{s}}_1 = 0 \tag{59}$$

The effective balance equations follow (59) by averaging over the unit cell Y and, at the order ϵ^0 one gets :

$$(\underset{\sim}{\boldsymbol{\Sigma}}_0 + \underset{\sim}{\boldsymbol{S}}_0) \cdot \boldsymbol{\nabla} = 0 \quad \text{and} \quad \underset{\approx}{\boldsymbol{M}}_0 \cdot \boldsymbol{\nabla} + \underset{\sim}{\boldsymbol{S}}_0 = 0 \tag{60}$$

where $\underset{\sim}{\boldsymbol{\Sigma}}_0 =< \underset{\sim}{\boldsymbol{\sigma}}_0 >$, $\underset{\sim}{\boldsymbol{S}}_0 =< \underset{\sim}{\boldsymbol{s}}_0 >$ and $\underset{\approx}{\boldsymbol{M}}_0 =< \underset{\approx}{\boldsymbol{m}}_0 >$.

3.5 Homogenization scheme HS1

For the first homogenization scheme defined in section 3.3, the equations describing the local behaviour are :

$$\underset{\sim}{\boldsymbol{\sigma}}^\epsilon = \underset{\approx}{\boldsymbol{a}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{\varepsilon}}^\epsilon, \quad \underset{\sim}{\boldsymbol{s}}^\epsilon = \underset{\approx}{\boldsymbol{b}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{e}}^\epsilon \quad \text{and} \quad \underset{\approx}{\boldsymbol{m}}^\epsilon = \epsilon^2 \underset{\approx\approx}{\boldsymbol{c}}^{(1)}(\underline{\boldsymbol{y}}) \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}^\epsilon \tag{61}$$

At this stage, the expansion (56) can be substituted into the constitutive equations (61). Identifying the terms of same order, we get :

- *order* ϵ^{-1},

$$\underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_{-1} = \underset{\approx}{\boldsymbol{a}}^{(0)} : (\underline{\boldsymbol{u}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_y) = 0, \quad \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_0 = \underset{\approx}{\boldsymbol{b}}^{(0)} : (\underline{\boldsymbol{u}}_0 \otimes \boldsymbol{\nabla}_y) = 0 \tag{62}$$

- *order* ϵ^{0},

$$\underset{\sim}{\boldsymbol{\sigma}}_0 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_0, \quad \underset{\sim}{\boldsymbol{s}}_0 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_0, \quad \underset{\approx}{\boldsymbol{m}}_0 = 0 \tag{63}$$

- *order* ϵ^{1},

$$\underset{\sim}{\boldsymbol{\sigma}}_1 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_1, \quad \underset{\sim}{\boldsymbol{s}}_1 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_1, \quad \underset{\approx}{\boldsymbol{m}}_1 = \underset{\approx\approx}{\boldsymbol{c}}^{(1)} \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}_{-1} \tag{64}$$

The equation (62) implies that $\underline{\boldsymbol{u}}_0$ does not depend on the local variable $\underline{\boldsymbol{y}}$:

$$\underline{\boldsymbol{u}}_0(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}})$$

At the order ϵ^0, the higher order stress tensor vanishes,

$$\underset{\approx}{\boldsymbol{M}}_0 = < \underset{\approx}{\boldsymbol{m}}_0 > = 0$$

Finally, the fields $(\underline{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{\chi}}_1, \underset{\sim}{\boldsymbol{\sigma}}_0, \underset{\sim}{\boldsymbol{s}}_0, \underset{\approx}{\boldsymbol{m}}_1)$ are solutions of the following auxiliary boundary value problem defined on the unit cell :

$$\begin{cases} \underset{\sim}{\boldsymbol{\varepsilon}}_0 = \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \boldsymbol{\nabla}_y, \\ \underset{\sim}{\boldsymbol{e}}_0 = \underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \otimes \boldsymbol{\nabla}_y - \underset{\sim}{\boldsymbol{\chi}}_1 \\ \underset{\approx}{\boldsymbol{\kappa}}_{-1} = \underset{\sim}{\boldsymbol{\chi}}_1 \otimes \boldsymbol{\nabla}_y \\ \underset{\sim}{\boldsymbol{\sigma}}_0 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_0, \quad \underset{\sim}{\boldsymbol{s}}_0 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_0, \\ \underset{\approx}{\boldsymbol{m}}_1 = \underset{\approx\approx}{\boldsymbol{c}}^{(1)} : \underset{\sim}{\boldsymbol{\kappa}}_{-1} \\ (\underset{\sim}{\boldsymbol{\sigma}}_0 + \underset{\sim}{\boldsymbol{s}}_0) \cdot \boldsymbol{\nabla}_y = 0, \quad \underset{\approx}{\boldsymbol{m}}_1 \cdot \boldsymbol{\nabla}_y + \underset{\sim}{\boldsymbol{s}}_0 = 0 \end{cases} \tag{65}$$

The boundary conditions of this problem are given by the periodicity requirements for the unknown fields. A series of auxiliary problems similar to (65) can be defined to obtain the solutions at higher orders. It must be noted that these problems must be solved in cascade since, for instance, the solution of (65) requires the knowledge of $\underline{\boldsymbol{U}}_0$. A particular solution $\underset{\sim}{\boldsymbol{\chi}}$ for a vanishing prescribed $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_x$ is $\underset{\sim}{\boldsymbol{\chi}} = \underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \boldsymbol{\nabla}_x$. It follows that the solution $(\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \boldsymbol{\nabla}_x - \underset{\sim}{\boldsymbol{\chi}}_1)$ to problem (65) depends linearly on $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_x$, up to a translation term, so that :

$$\underline{\boldsymbol{u}}^\epsilon = \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}}) + \epsilon(\underline{\boldsymbol{U}}_1(\underline{\boldsymbol{x}}) + \underset{\simeq}{\boldsymbol{X}}_u^{(1)}(\underline{\boldsymbol{y}}) : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla})) + \dots \tag{66}$$

$$\underset{\sim}{\boldsymbol{\chi}}^\epsilon = \underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \boldsymbol{\nabla}_x + \underset{\approx}{\boldsymbol{X}}_\chi^{(1)}(\underline{\boldsymbol{y}}) : \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla} + \dots \tag{67}$$

where concentration tensors $\underset{\simeq}{\boldsymbol{X}}_u^{(1)}$ and $\underset{\approx}{\boldsymbol{X}}_\chi^{(1)}$ have been introduced, the components of which are determined by the successive solutions of the auxiliary problem for unit values of the components of $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}$. Concentration tensor $\underset{\simeq}{\boldsymbol{X}}_u^{(1)}$ is such that its mean value over the unit cell vanishes.

The macroscopic stress tensor is given by :

$$\underset{\sim}{\boldsymbol{\Sigma}}_0 =< \underset{\sim}{\boldsymbol{\sigma}}_0 >=< \underset{\approx}{\boldsymbol{a}}^{(0)} : (\underset{\sim}{\mathbf{1}} + \boldsymbol{\nabla}_x \overset{s}{\otimes} \underset{\simeq}{\boldsymbol{X}}_u^{(1)}) >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) = \underset{\approx}{\boldsymbol{A}}_0^{(1)} : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) \tag{68}$$

Accordingly, the tensor of effective moduli possesses all symmetries of classical elastic moduli for a Cauchy medium :

$$A^{(1)}_{0\,ijkl} = A^{(1)}_{0\,klij} = A^{(1)}_{0\,jikl} = A^{(1)}_{0\,ijlk}$$

The additional second rank stress tensor can be shown to vanish :

$$\underset{\sim}{\boldsymbol{S}}_0 =< \underset{\sim}{\boldsymbol{s}}_0 >=< -\underset{\approx}{\boldsymbol{m}}_1 \cdot \boldsymbol{\nabla}_y >= 0 \tag{69}$$

The effective medium is therefore governed by the single equation :

$$\underset{\sim}{\boldsymbol{\Sigma}}_0 \cdot \boldsymbol{\nabla} = 0 \tag{70}$$

The effective medium turns out to be a Cauchy continuum with symmetric stress tensor.

3.6 Homogenization scheme HS2

For the second homogenization scheme defined in section 3.3, the equations describing the local behaviour are :

$$\underset{\sim}{\boldsymbol{\sigma}}^\epsilon = \underset{\approx}{\boldsymbol{a}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{\varepsilon}}^\epsilon, \quad \underset{\sim}{\boldsymbol{s}}^\epsilon = \underset{\approx}{\boldsymbol{b}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{e}}^\epsilon, \quad \text{and} \quad \underset{\approx}{\boldsymbol{m}}^\epsilon = \underset{\approx\!\approx}{\boldsymbol{c}}^{(2)}(\underline{\boldsymbol{y}}) \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}^\epsilon \tag{71}$$

The different steps of the asymptotic analysis are the same as in the previous section for HS1. We will only focus here on the main results. At the order ϵ^{-1}, one gets

$$\underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_{-1} = 0, \quad \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_{-1} = 0, \quad \underset{\approx\!\approx}{\boldsymbol{c}}^{(2)} \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}_{-1} = 0 \tag{72}$$

This implies that the gradients of $\underline{\boldsymbol{u}}_0$ and $\underset{\sim}{\boldsymbol{\chi}}_1$ with respect to $\underline{\boldsymbol{y}}$ vanish, so that :

$$\underline{\boldsymbol{u}}_0(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}}), \quad \underset{\sim}{\boldsymbol{\chi}}_1(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \underset{\sim}{\boldsymbol{\Xi}}_1(\underline{\boldsymbol{x}}) \tag{73}$$

The fields $(\underline{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{\chi}}_1, \underset{\sim}{\boldsymbol{\sigma}}_0, \underset{\sim}{\boldsymbol{s}}_0, \underset{\approx}{\boldsymbol{m}}_0)$ are solutions of the two following auxiliary boundary value problems defined on the unit cell :

$$\left\{\begin{array}{l} \underset{\sim}{\boldsymbol{\varepsilon}}_0 = \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \boldsymbol{\nabla}_y, \quad \underset{\sim}{\boldsymbol{e}}_0 = \underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \otimes \boldsymbol{\nabla}_y - \underset{\sim}{\Xi}_1 \\ \underset{\sim}{\boldsymbol{\sigma}}_0 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_0, \quad \underset{\sim}{\boldsymbol{s}}_0 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_0 \\ (\underset{\sim}{\boldsymbol{\sigma}}_0 + \underset{\sim}{\boldsymbol{s}}_0) \cdot \boldsymbol{\nabla}_y = 0 \end{array}\right. \tag{74}$$

$$\left\{\begin{array}{l} \underset{\approx}{\boldsymbol{\kappa}}_0 = \underset{\sim}{\Xi}_1 \otimes \boldsymbol{\nabla}_x + \underset{\sim}{\boldsymbol{\chi}}_2 \otimes \boldsymbol{\nabla}_y \\ \underset{\approx}{\boldsymbol{m}}_0 = \underset{\approx\approx}{\boldsymbol{c}}^{(2)} \vdots \underset{\approx}{\boldsymbol{\kappa}}_0, \quad \underset{\approx}{\boldsymbol{m}}_0 \cdot \boldsymbol{\nabla}_y = 0 \end{array}\right. \tag{75}$$

We are therefore left with two decoupled boundary value problems : the first one with main unknown $\underline{\boldsymbol{u}}_1$ depends linearly on $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_x$ and $\underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla}_x - \underset{\sim}{\Xi}_1$, whereas the second one with unknown $\underset{\sim}{\boldsymbol{\chi}}_2$ is linear in $\underset{\sim}{\Xi}_1 \otimes \boldsymbol{\nabla}_x$. The solutions take the form :

$$\underline{\boldsymbol{u}}^\epsilon = \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}}) + \epsilon(\underline{\boldsymbol{U}}_1(\underline{\boldsymbol{x}}) + \underset{\simeq}{\boldsymbol{X}}^{(2)}_u(\underline{\boldsymbol{y}}) : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) + \underset{\simeq}{\boldsymbol{X}}^{(2)}_e(\underline{\boldsymbol{y}}) : (\underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla} - \underset{\sim}{\Xi}_1)) + \ldots \tag{76}$$

$$\underset{\sim}{\boldsymbol{\chi}}^\epsilon = \underset{\sim}{\Xi}_1(\underline{\boldsymbol{x}}) + \epsilon(\underset{\sim}{\Xi}_2(\underline{\boldsymbol{x}}) + \underset{\approx}{\boldsymbol{X}}^{(2)}_\kappa(\underline{\boldsymbol{y}}) \vdots (\underset{\sim}{\Xi}_1 \otimes \boldsymbol{\nabla})) + \ldots \tag{77}$$

where concentration tensors $\underset{\simeq}{\boldsymbol{X}}^{(2)}_u, \underset{\simeq}{\boldsymbol{X}}^{(2)}_e$ and $\underset{\approx}{\boldsymbol{X}}^{(1)}_\kappa$ have been introduced. Their components are determined by the successive solutions of the auxiliary problem for unit values of the components of $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}, \underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla} - \underset{\sim}{\Xi}_1$ and $\underset{\sim}{\Xi}_1 \otimes \boldsymbol{\nabla}_y$. They are such that their mean value over the unit cell vanishes.

The macroscopic stress tensors and effective elastic properties are given by :

$$\begin{aligned} \underset{\sim}{\boldsymbol{\Sigma}}_0 &= <\underset{\approx}{\boldsymbol{a}}^{(0)} : (\underset{\approx}{\boldsymbol{1}} + \boldsymbol{\nabla}_y \overset{s}{\otimes} \underset{\simeq}{\boldsymbol{X}}^{(2)}_u) >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) \\ &+ <\underset{\approx}{\boldsymbol{a}}^{(0)} : (\boldsymbol{\nabla}_y \overset{s}{\otimes} \underset{\simeq}{\boldsymbol{X}}^{(2)}_e) >: (\underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla} - \underset{\sim}{\Xi}_1) \end{aligned} \tag{78}$$

$$\begin{aligned} \underset{\sim}{\boldsymbol{S}}_0 &= <\underset{\sim}{\boldsymbol{s}}_0> = <\underset{\approx}{\boldsymbol{b}}^{(0)} : (\boldsymbol{\nabla}_y \otimes \underset{\simeq}{\boldsymbol{X}}^{(2)}_u) >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) \\ &+ <\underset{\approx}{\boldsymbol{b}}^{(0)} : (\boldsymbol{\nabla}_y \otimes \underset{\simeq}{\boldsymbol{X}}^{(2)}_e) >: (\underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla} - \underset{\sim}{\Xi}_1) \end{aligned} \tag{79}$$

$$\underset{\approx}{\boldsymbol{M}}_0 = <\underset{\approx}{\boldsymbol{m}}_0> = <\underset{\approx\approx}{\boldsymbol{c}}^{(2)} \vdots (\underset{\approx\approx}{\boldsymbol{1}} + \boldsymbol{\nabla}_y \otimes \underset{\approx}{\boldsymbol{X}}^{(2)}_\kappa) > \vdots \underset{\sim}{\Xi}_1 \otimes \boldsymbol{\nabla} \tag{80}$$

None of these tensors vanishes in general, which means that the effective medium is a full micromorphic continuum governed by the balance equations (60).

3.7 Homogenization scheme HS3

For the third homogenization scheme defined in section 3.3, the equations describing the local behaviour are :

$$\underset{\sim}{\boldsymbol{\sigma}}^\epsilon = \underset{\approx}{\boldsymbol{a}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{\varepsilon}}^\epsilon, \quad \underset{\sim}{\boldsymbol{s}}^\epsilon = \underset{\approx}{\boldsymbol{b}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{e}}^\epsilon, \tag{81}$$

$$\underset{\approx}{\boldsymbol{m}}^\epsilon = \epsilon^2 \underset{\approx}{\underset{\sim}{\boldsymbol{c}}}^{s(1)}(\underline{\boldsymbol{y}}) \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}^{s\epsilon} + \underset{\approx}{\underset{\sim}{\boldsymbol{c}}}^{a(2)}(\underline{\boldsymbol{y}}) \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}^{a\epsilon} \tag{82}$$

At the order ϵ^{-1}, one gets

$$\underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_{-1} = 0, \quad \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_{-1} = 0, \quad \underset{\approx}{\underset{\sim}{\boldsymbol{c}}}^{a(2)} \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}^{a}_{-1} = 0 \tag{83}$$

This implies that the gradients of $\underline{\boldsymbol{u}}_0$ and $\underset{\sim}{\boldsymbol{\chi}}^a_1$ with respect to $\underline{\boldsymbol{y}}$ vanish, so that :

$$\underline{\boldsymbol{u}}_0(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}}), \quad \underset{\sim}{\boldsymbol{\chi}}^a_1(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \underset{\sim}{\boldsymbol{\Xi}}^a_1(\underline{\boldsymbol{x}}) \tag{84}$$

The fields $(\underline{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{\chi}}^s_1, \underset{\sim}{\boldsymbol{\chi}}^a_2, \underset{\sim}{\boldsymbol{\chi}}^a_3, \underset{\sim}{\boldsymbol{\sigma}}_0, \underset{\sim}{\boldsymbol{s}}_0, \underset{\approx}{\boldsymbol{m}}_0, \underset{\approx}{\boldsymbol{m}}_1)$ are solutions of the following auxiliary boundary value problem defined on the unit cell :

$$\left\{\begin{array}{l}
\underset{\sim}{\boldsymbol{\varepsilon}}_0 = \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \boldsymbol{\nabla}_y, \\
\underset{\sim}{\boldsymbol{e}}_0 = \underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \otimes \boldsymbol{\nabla}_y - \underset{\sim}{\boldsymbol{\Xi}}^a_1 - \underset{\sim}{\boldsymbol{\chi}}^s_1 \\
\underset{\approx}{\boldsymbol{\kappa}}^s_{-1} = \underset{\sim}{\boldsymbol{\chi}}^s_1 \otimes \boldsymbol{\nabla}_y, \; \underset{\approx}{\boldsymbol{\kappa}}^a_0 = \underset{\sim}{\boldsymbol{\Xi}}^a_1 \otimes \boldsymbol{\nabla}_x + \underset{\sim}{\boldsymbol{\chi}}^a_2 \otimes \boldsymbol{\nabla}_y, \\
\underset{\approx}{\boldsymbol{\kappa}}^a_1 = \underset{\sim}{\boldsymbol{\chi}}^a_2 \otimes \boldsymbol{\nabla}_x + \underset{\sim}{\boldsymbol{\chi}}^a_3 \otimes \boldsymbol{\nabla}_y \\
\underset{\sim}{\boldsymbol{\sigma}}_0 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_0, \quad \underset{\sim}{\boldsymbol{s}}_0 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_0 \\
\underset{\approx}{\boldsymbol{m}}_0 = \underset{\approx}{\underset{\sim}{\boldsymbol{c}}}^{a(2)} \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}^a_0, \quad \underset{\approx}{\boldsymbol{m}}_1 = \underset{\approx}{\underset{\sim}{\boldsymbol{c}}}^{s(1)} \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}^s_{-1} + \underset{\approx}{\underset{\sim}{\boldsymbol{c}}}^{a(2)} \,\vdots\, \underset{\approx}{\boldsymbol{\kappa}}^a_1 \\
(\underset{\sim}{\boldsymbol{\sigma}}_0 + \underset{\sim}{\boldsymbol{s}}_0) \cdot \boldsymbol{\nabla}_y = 0, \quad \underset{\approx}{\boldsymbol{m}}_0 . \boldsymbol{\nabla}_y = 0, \\
\underset{\approx}{\boldsymbol{m}}_0 \cdot \boldsymbol{\nabla}_x + \underset{\approx}{\boldsymbol{m}}_1 \cdot \boldsymbol{\nabla}_y + \underset{\sim}{\boldsymbol{s}}_0 = 0
\end{array}\right. \tag{85}$$

This complex problem can be seen to depend linearly on $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}, \underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\Xi}}_1^a$ and $\underset{\sim}{\boldsymbol{\Xi}}_1^a \otimes \boldsymbol{\nabla}$. The solutions take the form :

$$\underline{\boldsymbol{u}}^\epsilon = \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}}) + \epsilon(\underline{\boldsymbol{U}}_1(\underline{\boldsymbol{x}}) + \underset{\simeq}{\boldsymbol{X}}{}_u^{(3)}(\underline{\boldsymbol{y}}) : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) + \underset{\simeq}{\boldsymbol{X}}{}_e^{(3)}(\underline{\boldsymbol{y}}) : (\underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\Xi}}_1^a)) + \dots \tag{86}$$

$$\underset{\sim}{\boldsymbol{\chi}}^\epsilon = \underset{\sim}{\boldsymbol{\Xi}}_1(\underline{\boldsymbol{x}}) + \epsilon(\underset{\sim}{\boldsymbol{\Xi}}_2(\underline{\boldsymbol{x}}) + \underset{\approx}{\boldsymbol{X}}{}_\kappa^{(3)}(\underline{\boldsymbol{y}}) \vdots (\underset{\sim}{\boldsymbol{\Xi}}_1^a \otimes \boldsymbol{\nabla})) + \dots \tag{87}$$

where concentration tensors $\underset{\simeq}{\boldsymbol{X}}{}_u^{(3)}, \underset{\simeq}{\boldsymbol{X}}{}_e^{(3)}$ and $\underset{\approx}{\boldsymbol{X}}{}_\kappa^{(3)}$ have been introduced. Their components are determined by the successive solutions of the auxiliary problem for unit values of the components of $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}, \underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\Xi}}_1^a$ and $\underset{\sim}{\boldsymbol{\Xi}}_1^a \otimes \boldsymbol{\nabla}_y$. They are such that their mean value over the unit cell vanishes.

The macroscopic stress tensors and effective elastic properties are given by :

$$\begin{aligned} \underset{\sim}{\boldsymbol{\Sigma}}_0 &= \; < \underset{\approx}{\boldsymbol{a}}^{(0)} : (\underset{\approx}{\boldsymbol{1}} + \boldsymbol{\nabla}_x \overset{s}{\otimes} \underset{\simeq}{\boldsymbol{X}}{}_u^{(3)}) >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) \\ &+ \; < \underset{\approx}{\boldsymbol{a}}^{(0)} : (\boldsymbol{\nabla}_y \overset{s}{\otimes} \underset{\simeq}{\boldsymbol{X}}{}_e^{(3)}) >: (\underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\Xi}}_1^a) \end{aligned} \tag{88}$$

$$\begin{aligned} \underset{\sim}{\boldsymbol{S}}_0 &= \; < \underset{\sim}{\boldsymbol{s}}_0 > = < \underset{\approx}{\boldsymbol{b}}^{(0)} : (\boldsymbol{\nabla}_y \otimes \underset{\simeq}{\boldsymbol{X}}{}_u^{(3)}) >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) \\ &+ \; < \underset{\approx}{\boldsymbol{b}}^{(0)} : (\boldsymbol{\nabla}_y \otimes \underset{\simeq}{\boldsymbol{X}}{}_e^{(3)}) >: (\underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\Xi}}_1^a) \end{aligned} \tag{89}$$

$$\underset{\simeq}{\boldsymbol{M}}_0 = < \underset{\simeq}{\boldsymbol{m}}_0 > = < \underset{\approx\approx}{\boldsymbol{c}}^{a(2)} \vdots (\underset{\approx\approx}{\boldsymbol{1}} + \boldsymbol{\nabla}_y \otimes \underset{\approx}{\boldsymbol{X}}{}_\kappa^{(3)}) > \vdots \, \underset{\sim}{\boldsymbol{\Xi}}_1^a \otimes \boldsymbol{\nabla} \tag{90}$$

They must fulfill the balance equations (60). Note that $\underset{\simeq}{\boldsymbol{m}}_0$ and therefore $\underset{\simeq}{\boldsymbol{M}}_0$ are skew–symmetric with respect to their first two indices. The averaged equation of balance of moment of momentum implies then that $\underset{\sim}{\boldsymbol{S}}_0$ is skew–symmetric. The macroscopic degrees of freedom are the displacement field $\underline{\boldsymbol{U}}_0$ and the rotation associated to $\underset{\sim}{\boldsymbol{\Xi}}_1^a$. The found balance and constitutive equations are therefore that of a Cosserat effective medium. The more classical form of the Cosserat theory is retrieved once one rewrites the previous equations using the axial vector associated to $\underset{\sim}{\boldsymbol{\Xi}}^a$ [24].

3.8 Homogenization scheme HS4

For the last homogenization scheme defined in section 3.3, the equations describing the local behaviour are :

$$\underset{\sim}{\boldsymbol{\sigma}}^{\epsilon} = \underset{\approx}{\boldsymbol{a}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{\varepsilon}}^{\epsilon}, \quad \underset{\sim}{\boldsymbol{s}}^{\epsilon} = \underset{\approx}{\boldsymbol{b}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{e}}^{\epsilon} \tag{91}$$

$$\underset{\approx}{\boldsymbol{m}}^{\epsilon} = \underset{\approx\approx}{\boldsymbol{c}}^{s(2)}(\underline{\boldsymbol{y}}) \vdots \underset{\approx}{\boldsymbol{\kappa}}^{s\epsilon} + \epsilon^2 \underset{\approx\approx}{\boldsymbol{c}}^{a(1)}(\underline{\boldsymbol{y}}) \vdots \underset{\approx}{\boldsymbol{\kappa}}^{a\epsilon} \tag{92}$$

At the order ϵ^{-1}, one gets

$$\underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_{-1} = 0, \quad \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_{-1} = 0, \quad \underset{\approx\approx}{\boldsymbol{c}}^{s(2)} \vdots \underset{\approx}{\boldsymbol{\kappa}}^{s}_{-1} = 0 \tag{93}$$

This implies that the gradients of $\underline{\boldsymbol{u}}_0$ and $\underset{\sim}{\boldsymbol{\chi}}^s_1$ with respect to $\underline{\boldsymbol{y}}$ vanish, so that :

$$\underline{\boldsymbol{u}}_0(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}}), \quad \underset{\sim}{\boldsymbol{\chi}}^s_1(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \underset{\sim}{\boldsymbol{\Xi}}^s_1(\underline{\boldsymbol{x}}) \tag{94}$$

The fields $(\underline{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{\chi}}^a_1, \underset{\sim}{\boldsymbol{\chi}}^s_2, \underset{\sim}{\boldsymbol{\chi}}^s_3, \underset{\sim}{\boldsymbol{\sigma}}_0, \underset{\sim}{\boldsymbol{s}}_0, \underset{\approx}{\boldsymbol{m}}_0, \underset{\approx}{\boldsymbol{m}}_1)$ are solutions of the following auxiliary boundary value problem defined on the unit cell :

$$\left\{\begin{array}{l}
\underset{\sim}{\boldsymbol{\varepsilon}}_0 = \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \boldsymbol{\nabla}_y, \\
\underset{\sim}{\boldsymbol{e}}_0 = \underline{\boldsymbol{U}}_0 \otimes \boldsymbol{\nabla}_x + \underline{\boldsymbol{u}}_1 \otimes \boldsymbol{\nabla}_y - \underset{\sim}{\boldsymbol{\Xi}}^s_1 - \underset{\sim}{\boldsymbol{\chi}}^a_1 \\
\underset{\approx}{\boldsymbol{\kappa}}^a_{-1} = \underset{\sim}{\boldsymbol{\chi}}^a_1 \otimes \boldsymbol{\nabla}_y, \ \underset{\approx}{\boldsymbol{\kappa}}^s_0 = \underset{\sim}{\boldsymbol{\Xi}}^s_1 \otimes \boldsymbol{\nabla}_x + \underset{\sim}{\boldsymbol{\chi}}^s_2 \otimes \boldsymbol{\nabla}_y, \\
\underset{\approx}{\boldsymbol{\kappa}}^a_1 = \underset{\sim}{\boldsymbol{\chi}}^a_2 \otimes \boldsymbol{\nabla}_x + \underset{\sim}{\boldsymbol{\chi}}^a_3 \otimes \boldsymbol{\nabla}_y \\
\underset{\sim}{\boldsymbol{\sigma}}_0 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\varepsilon}}_0, \quad \underset{\sim}{\boldsymbol{s}}_0 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_0 \\
\underset{\approx}{\boldsymbol{m}}_0 = \underset{\approx\approx}{\boldsymbol{c}}^{s(2)} \vdots \underset{\approx}{\boldsymbol{\kappa}}^s_0, \quad \underset{\approx}{\boldsymbol{m}}_1 = \underset{\approx\approx}{\boldsymbol{c}}^{a(1)} \vdots \underset{\approx}{\boldsymbol{\kappa}}^a_{-1} + \underset{\approx\approx}{\boldsymbol{c}}^{s(2)} \vdots \underset{\approx}{\boldsymbol{\kappa}}^s_1 \\
(\underset{\sim}{\boldsymbol{\sigma}}_0 + \underset{\sim}{\boldsymbol{s}}_0) \cdot \boldsymbol{\nabla}_y = 0, \quad \underset{\approx}{\boldsymbol{m}}_0 . \boldsymbol{\nabla}_y = 0, \\
\underset{\approx}{\boldsymbol{m}}_0 \cdot \boldsymbol{\nabla}_x + \underset{\approx}{\boldsymbol{m}}_1 \cdot \boldsymbol{\nabla}_y + \underset{\sim}{\boldsymbol{s}}_0 = 0
\end{array}\right. \tag{95}$$

This complex problem can be seen to depend linearly on $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}, \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\Xi}}^s_1$ and $\underset{\sim}{\boldsymbol{\Xi}}^s_1 \otimes \boldsymbol{\nabla}$. The solutions take the form :

$$\begin{aligned}
\underline{\boldsymbol{u}}^{\epsilon} \quad = \quad & \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}}) + \epsilon(\underline{\boldsymbol{U}}_1(\underline{\boldsymbol{x}}) + \underset{\simeq}{\boldsymbol{X}}^{(4)}_u(\underline{\boldsymbol{y}}) : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) + \\
& \underset{\simeq}{\boldsymbol{X}}^{(4)}_e(\underline{\boldsymbol{y}}) : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\Xi}}^s_1)) + \dots
\end{aligned} \tag{96}$$

$$\underset{\sim}{\chi}^{\epsilon} = \underset{\sim}{\Xi}_1(\underline{\boldsymbol{x}}) + \epsilon(\underset{\sim}{\Xi}_2(\underline{\boldsymbol{x}}) + \underset{\approx}{\boldsymbol{X}}_{\kappa}^{(4)}(\underline{\boldsymbol{y}}) \vdots (\underset{\sim}{\Xi}_1^s \otimes \boldsymbol{\nabla})) + \dots \tag{97}$$

where concentration tensors $\underset{\simeq}{\boldsymbol{X}}_u^{(4)}, \underset{\simeq}{\boldsymbol{X}}_e^{(4)}$ and $\underset{\approx}{\boldsymbol{X}}_{\kappa}^{(4)}$ have been introduced. Their components are determined by the successive solutions of the auxiliary problem for unit values of the components of $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}, \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\Xi}_1^s$ and $\underset{\sim}{\Xi}_1^s \otimes \boldsymbol{\nabla}_y$. They are such that their mean value over the unit cell vanishes.

The macroscopic stress tensors and effective elastic properties are given by :

$$\begin{aligned} \underset{\sim}{\boldsymbol{\Sigma}}_0 &= \; < \underset{\approx}{\boldsymbol{a}}^{(0)} : (\underset{\approx}{\boldsymbol{1}} + \boldsymbol{\nabla}_x \overset{s}{\otimes} \underset{\simeq}{\boldsymbol{X}}_u^{(4)}) >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) \\ &+ \; < \underset{\approx}{\boldsymbol{a}}^{(0)} : (\boldsymbol{\nabla}_y \overset{s}{\otimes} \underset{\simeq}{\boldsymbol{X}}_e^{(4)}) >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\Xi}_1^s) \end{aligned} \tag{98}$$

$$\begin{aligned} \underset{\sim}{\boldsymbol{S}}_0 &= \; < \underset{\sim}{\boldsymbol{s}}_0 > = < \underset{\approx}{\boldsymbol{b}}^{(0)} : (\boldsymbol{\nabla}_y \otimes \underset{\simeq}{\boldsymbol{X}}_u^{(4)}) >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla}) + \\ & \quad < \underset{\approx}{\boldsymbol{b}}^{(0)} >: (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \boldsymbol{\nabla} - \underset{\sim}{\Xi}_1^s) \end{aligned} \tag{99}$$

$$\underset{\approx}{\boldsymbol{M}}_0 = < \underset{\approx}{\boldsymbol{m}}_0 > = < \underset{\approx}{\boldsymbol{c}}^{s(2)} \vdots (\underset{\approx}{\boldsymbol{1}} + \boldsymbol{\nabla}_y \otimes \underset{\approx}{\boldsymbol{X}}_{\kappa}^{(4)}) > \; \vdots (\underset{\sim}{\Xi}_1^s \otimes \boldsymbol{\nabla}) \tag{100}$$

They must fulfill the balance equations (60). Note that $\underset{\approx}{\boldsymbol{m}}_0$ and therefore $\underset{\approx}{\boldsymbol{M}}_0$ are symmetric with respect to their first two indices. The averaged equation of balance of moment of momentum implies then that $\underset{\sim}{\boldsymbol{S}}_0 = - < \underset{\approx}{\boldsymbol{m}}_0 > \cdot \boldsymbol{\nabla}$ is symmetric. The macroscopic degrees of freedom are the displacement field $\underline{\boldsymbol{U}}_0$ and the symmetric strain tensor $\underset{\sim}{\Xi}_1^s$. Such a continuum is sometimes called a microstrain medium [2].

4 Conclusions

Figure 1 summarizes the links between homogenization methods and the mechanics of generalized continua. The available homogenization procedures enable one to replace a heterogeneous classical or generalized medium by a homogeneous substitution medium than may be classical or generalized. A generalized HSM is necessary at the global level when the structure made of the considered composite material is subjected to strong variations of the mean fields or when the intrinsic lengths of non–classical constituents are comparable to the wavelength L of variation of the mean fields. In the other situations, a Cauchy effective medium is sufficient at the macroscopic level, even if the constituents themselves are modelled by a generalized continuum.

In the present work, a homogenization scheme based on the development of non–homogeneous boundary conditions in a polynomial form has been presented to construct a Cosserat and a micromorphic HSM starting from a classical heterogeneous material. On the other hand, the multiscale asymptotic method makes it possible to replace heterogeneous micromorphic media by Cauchy, Cosserat, microstrain or micromorphic HSM, depending on the hierarchy of characteristic lengths. The results of section 3 are summarized in table 1. The program depicted in figure 1 is now almost complete. Asymptotic methods have been used in [7, 8, 9, 12] for the transition from a classical heterogeneous material to a second gradient HSM. They have also been applied to homogenize Cosserat media [17] and in the present work extended to micromorphic continua. For random materials, perturbation, self–consistent and variational methods are applied in [4, 5, 6, 16, 10, 11] to derive fully non–local overall models, that are sometimes reduced to a second gradient theory. Developments and examples illustrating the polynomial approach can be found in [19, 13, 14, 15, 23, 25, 26] together with similar but more numerical approaches in [21, 22, 27]. Figure 1 also shows which transitions could be solved in a straightforward manner using the available tools. Other routes remain to be traced especially dealing with fully non–local media at the local level. However, the main issue is the actual use and validation of the results in specific situations which are still not numerous enough. Applications to crystal plasticity for instance can be found in [28].

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homogenization scheme	characteristic lengths	effective medium
HS1	$l_s \sim l, l_a \sim l$	Cauchy
HS2	$l_s \sim L, l_a \sim L$	micromorphic
HS3	$l_s \sim l, l_a \sim L$	Cosserat
HS4	$l_s \sim L, l_a \sim l$	microstrain

Table 1: Homogenization of heterogenous micromorphic media : Nature of the homogeneous equivalent medium depending of the values of the intrinsic lengths of the constituents.

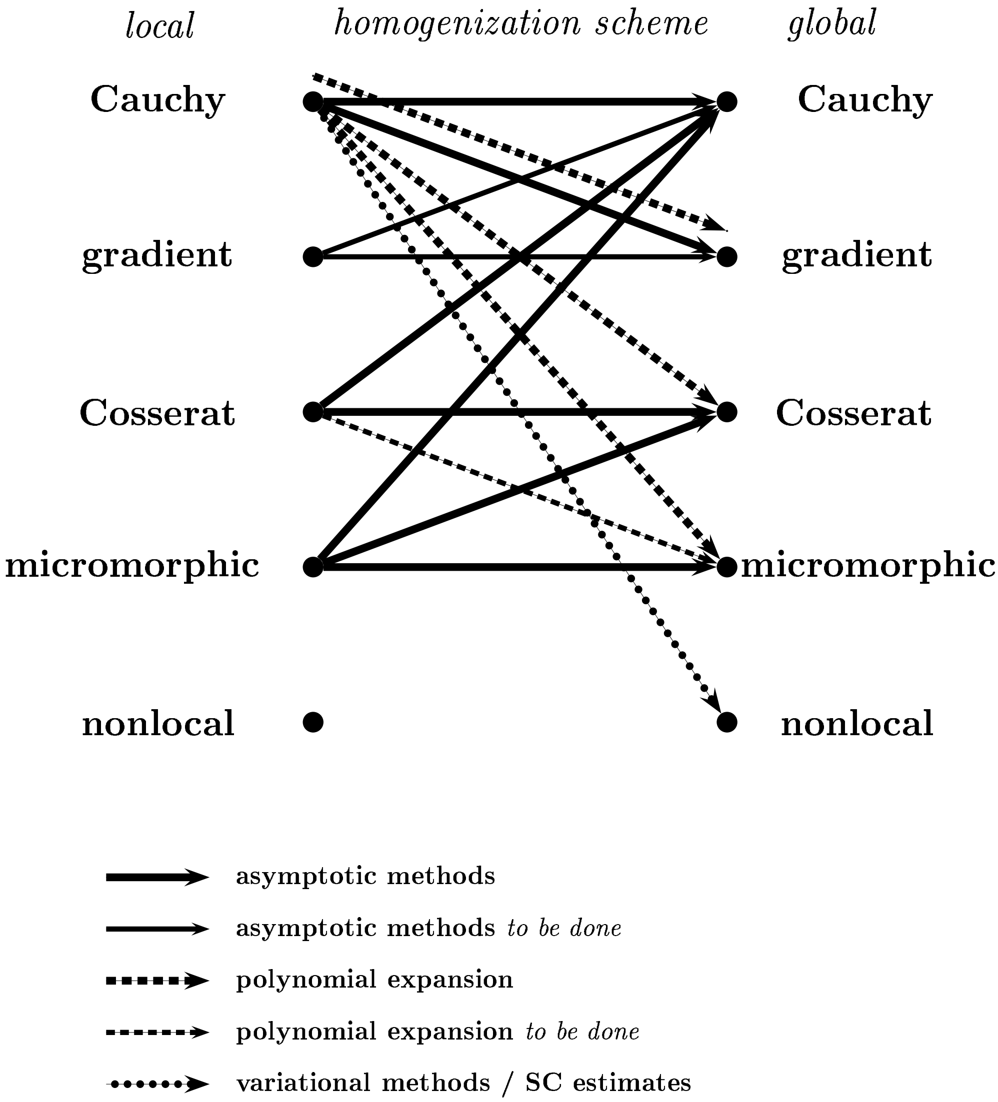

Figure 1: Homogenization schemes for generalized continua.

Metode homogenizacije i mehanika generalisanih kontinuuma - deo drugi

UDK 531.01, 539.3

Potreba za uvodjenjem generalisanih kontinuuma nastaje u više oblasti mehanike heterogenih materijala i to posebno u teoriji homogenizacije. Neki generalisani homogeni ekvivalentni medijum je na globalnom nivou potreban kada je struktura kompozita izložena oštrim promenama srednjih polja ili kada su unutrašnje (sopstvene) dužine uporedive sa talasnom dužinom promene srednjih polja. U ovom radu se sistematski metod, zasnovan na polinomijalnim razvojima, koristi za zamenu klasičnog kompozitnog materijala ekvivalentnim Cosserat ili mikromorfnim materijalom. U drugom delu smesa mikromorfnih sastojaka se homogenizuje asimptotskim metodom sa više skala. Pokazuje se da rezultujući makroskopski medijum može biti Cauchy-jev, Cosserat, mikrodeformacioni ili mikromorfni medijum zavisno od karakterističnih dužina problema.