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Spin axioms in relativistic continuum physics

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There is nothing so annnoying
as a good example

Mark Twain

Abstract

The 24 components of the relativistic spin tensor consist of $3+3$ basic spin fields and $9+9$ constitutive fields. Empirically only 3 basic spin fields and 9 constitutive fields are known. This empirem can be expressed by two spin axioms, one of them identifying 3 spin fields, and the other one 9 constitutive fields to each other. This identification by the spin axioms is material-independent and does not mix basic spin fields with constitutive properties. The approaches to the Weyssenhoff fluid and the Dirac-electron fluid found in literature are discussed with regard to these spin axioms. The conjecture is formulated, that another reduction from 6 to 3 basic spin fields which does not obey the spin axioms introduces special material properties by not allowed mixing of constitutive and basic fields.

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1 Introduction

We investigate the constitutive theory of spin fluids in a relativistic context. General Relativity Theory (GRT) using Riemann geometry for geometrization of gravitation is currently assumed to be the most appropriate geometrization.
Here we consider in a relativistic framework the general spin balance. The systematic reduction from 6 to 3 basic spin fields is introduced by two spin axioms which prohibit the mixing between the basic spin fields and the constitutive fields. This is obvious, because no special constitutive assumptions should be introduced by this reduction. As an example the Weyssenhoff fluid and the Dirac-electron fluid with regard to the general spin balance and the spin axioms are discussed.

2 The Spin Balances

Balance equations are differential equations for the wanted basic fields. Beside these fields constitutive quantities describing the material appear in the balances. This structural distinction into basic fields and constitutive properties is independent of writing down the balance equations in relativistic or non-relativistic form.
First of all we consider the non-relativistic spin balance for characterizing, what are the basic spin fields and what the constitutive properties.

2.1 Non-relativistic spin balance

Starting out with the definition of the angular momentum as a skew-symmetric 3-tensor of second rank $(i, j, \dots = 1, 2, 3)$

$$M^{ij} = x^{[i}\rho v^{j]} + \rho s^{ij} \tag{1}$$

the balance of angular momentum is

$$\begin{aligned} \tfrac{1}{2}\partial_t(\rho s^{ij}) + \partial_t x^{[i}\rho v^{j]} &+ \\ \partial_k \left(\tfrac{1}{2}\rho s^{ij} v^k - m^{ijk} + x^{[i}\rho v^{j]} v^k - x^{[i} t^{j]k}\right) &= \\ l^{ij} + x^{[i} f^{j]} & \end{aligned} \tag{2}$$

Here the basic fields are (1) consisting of the orbital $x^{[i}\rho v^{j]}$ and the spin angular momentum densities ρs^{ij}. The constitutive properties are described in (2) by the stress tensor t^{jk} and the couple stress m^{ijk}. An external angular momentum is given by l^{ij}.
Because the specific spin $s^{ij} \Leftrightarrow s_k$ can be identified with the axial spin vector s_k in R^3 the balance of angular momentum can be written down as a vector equation. If from this balance equation the balance of momentum multiplied by $\times\underline{x}$ is subtracted, we obtain the balance of spin in vector formulation [1]

$$\partial_t(\rho s^i) + \nabla_k \left(v^k \rho s^i - w^{ik}\right) + \epsilon^{ijk} t_{jk} \;=\; \rho g^i \tag{3}$$

Here the distinction into the basic fields ρs and the constitutive quantities w and t is clear: There are 3 basic spin fields s, 9 fields of the couple stress w, and 3 fields from the balance of momentum by the skew-symmetric part $\epsilon : t$ of the stress tensor. A given external angular momentum density is ρg.

2.2 Relativistic spin balance

The non-relativistic momentum flux density ρv^j in (1) is replaced in relativistic formulations by the energy-momentum tensor $T^{\alpha\beta}$. Therefore the tensor order one of the momentum flux density is replaced by the order two of the energy-momentum tensor in a relativistic formulation. Consequently in special relativity theory the definition of skew-symmetric angular momentum becomes $(\alpha, \beta, \dots = 1, \dots, 4)$

$$M^{\alpha\beta\mu} \;:=\; x^{[\alpha} T^{\beta]\mu} + S^{\alpha\beta\mu} \tag{4}$$

and the balance results in

$$\partial_\mu M^{\alpha\beta\mu} \;=\; L^{\alpha\beta} + x^{[\alpha} f^{\beta]} \tag{5}$$

Here $S^{\alpha\beta\mu}$ is the skew-symmetric spin tensor and $L^{\alpha\beta}$ the external angular momentum. Caused by the extension of the tensor order in relativistic theories with respect to non-relativistic ones, we now have $6 \times 4 \;=\; 24$ fields of the spin tensor, including the basic spin fields and the constitutive quantities. Of how many basic and constitutive fields these 24 fields are composed, can be seen, if the spin tensor is decomposed into its spatial and time-like parts by a 3+1 split. By use of the 4-velocity u^α the 3+1 split of the spin tensor is

$$\begin{aligned} -S_{\lambda\mu}^{\cdot\cdot\nu} = S_{\mu\lambda}^{\cdot\cdot\nu} &= s_{\mu\lambda}^{\cdot\cdot\nu} + \frac{1}{c^2}\tilde{s}_{\mu\lambda}u^\nu + \frac{1}{c^2}u_{[\mu}\hat{s}_{\lambda]}^{\cdot\nu} + \\ &\quad + \frac{1}{c^2}\check{s}_{[\mu|}^{\cdot\nu}u_{|\lambda]} + \frac{1}{c^4}\tilde{s}_{[\mu}u_{\lambda]}u^\nu + \frac{1}{c^4}u_{[\mu}\hat{s}_{\lambda]}u^\nu \qquad (6) \\ &= s_{\mu\lambda}^{\cdot\cdot\nu} + \frac{1}{c^2}\tilde{s}_{\mu\lambda}u^\nu + \frac{1}{c^2}u_{[\mu}\Xi_{\lambda]}^{\cdot\nu} + \frac{1}{c^4}u_{[\mu}\Xi_{\lambda]}u^\nu \qquad (7) \end{aligned}$$

Here we have introduced the following abbreviations

$$\begin{aligned} s_{\mu\nu}^{\cdot\cdot\lambda} &:= S_{\alpha\beta}^{\cdot\cdot\gamma}h_\mu^\alpha h_\nu^\beta h_\gamma^\lambda && \text{couple stress} && (8) \\ \tilde{s}_{\mu\nu} &:= S_{\alpha\beta}^{\cdot\cdot\gamma}h_\mu^\alpha h_\nu^\beta u_\gamma && \text{spin density} && (9) \\ \hat{s}_\nu^{\cdot\lambda} &:= S_{\alpha\beta}^{\cdot\cdot\gamma}h_\nu^\beta h_\gamma^\lambda u^\alpha && && (10) \\ \check{s}_\mu^{\cdot\lambda} &:= S_{\alpha\beta}^{\cdot\cdot\gamma}h_\mu^\alpha u^\beta h_\gamma^\lambda && && (11) \\ \tilde{s}_\mu &:= S_{\alpha\beta}^{\cdot\cdot\gamma}h_\mu^\alpha u^\beta u_\gamma && && (12) \\ \hat{s}_\nu &:= S_{\alpha\beta}^{\cdot\cdot\gamma}u^\alpha h_\nu^\beta u_\gamma && && (13) \\ \Xi_\nu^{\cdot\lambda} &:= \hat{s}_\nu^{\cdot\lambda} - \check{s}_\nu^{\cdot\lambda} && \text{spin stress} && (14) \\ \Xi_\nu &:= \hat{s}_\nu - \tilde{s}_\nu && \text{spin density vector} && (15) \end{aligned}$$

The spin balance equation determines the divergence of the spin tensor

$$S_{\cdot\cdot\cdot;\mu}^{\alpha\beta\mu} = T^{[\alpha\beta]} + L^{\alpha\beta} \qquad (16)$$

Here $T^{[\alpha\beta]}$ is the skew-symmetric part of the energy momentum tensor which couples the spin balance to the balance of momentum. Beyond this source, there may be other external moments $L^{\alpha\beta}$ from (5). We now decompose (16) by its 3+1 split which results in two parts, the hh-part and the hu-part. Because of the skew-symmetry of the spin tensor the uu-part is zero. In more detail the hh-part of the 3+1 split results in 3 equations

$$
\begin{aligned}
h^{\mu}_{\alpha}h^{\nu}_{\beta}S_{\mu\nu\cdot;\lambda}^{\cdot\cdot\lambda} &= \underbrace{s_{\mu\nu\cdot;\lambda}^{\cdot\cdot\lambda}h^{\mu}_{\alpha}h^{\nu}_{\beta}}_{\spadesuit} + \underbrace{\frac{1}{c^2}\tilde{s}_{\mu\nu;\lambda}u^{\lambda}h^{\mu}_{\alpha}h^{\nu}_{\beta} + \frac{1}{c^2}\tilde{s}_{\mu\nu}h^{\mu}_{\alpha}h^{\nu}_{\beta}u^{\lambda}_{\cdot;\lambda}}_{\diamondsuit} - \\
&\quad - \underbrace{\frac{1}{c^2}u_{[\mu}\Xi_{\nu]}^{\cdot;\lambda}\left(h^{\mu}_{\alpha}h^{\nu}_{\beta}\right)_{;\lambda}}_{\triangle} - \underbrace{\frac{1}{c^4}u_{[\mu}\Xi_{\nu]}u^{\lambda}\left(h^{\mu}_{\alpha}h^{\nu}_{\beta}\right)_{;\lambda}}_{\triangle} \\
&= \underbrace{T_{[\mu\nu]}h^{\mu}_{\alpha}h^{\nu}_{\beta}}_{\sharp} + \underbrace{L_{\mu\nu}h^{\mu}_{\alpha}h^{\nu}_{\beta}}_{\natural} \qquad (17)
\end{aligned}
$$

and the 3 equations of the hu-part becomes

$$
\begin{aligned}
h^{\nu}_{\gamma}u^{\mu}S_{\mu\nu\cdot;\lambda}^{\cdot\cdot\lambda} &= \underbrace{s_{\mu\nu\cdot;\lambda}^{\cdot\cdot\lambda}h^{\nu}_{\gamma}u^{\mu} - \frac{1}{c^2}\tilde{s}_{\mu\nu}h^{\nu}_{\gamma}u^{\mu}_{\cdot;\lambda}u^{\lambda}}_{\triangle} + \\
&\quad + \underbrace{\frac{1}{2}\frac{1}{c^2}\Xi_{\nu;\lambda}u^{\lambda}h^{\nu}_{\gamma} + \frac{1}{2}\frac{1}{c^2}\Xi_{\nu}h^{\nu}_{\gamma}u^{\lambda}_{\cdot;\lambda}}_{\diamondsuit} + \underbrace{\frac{1}{2}\frac{1}{c^2}u_{\mu}\Xi_{\nu\cdot;\lambda}^{\cdot;\lambda}u^{\mu}h^{\nu}_{\gamma}}_{\spadesuit} \\
&= \underbrace{T_{[\mu\nu]}h^{\nu}_{\gamma}u^{\mu}}_{\sharp} + \underbrace{L_{\mu\nu}h^{\nu}_{\gamma}u^{\mu}}_{\natural} \qquad (18)
\end{aligned}
$$

Now we compare (17) and (18) with the non-relativistic balance of spin (3)

$$
\underbrace{\partial_t(\frac{1}{2}\rho s^i) + \nabla_k(v^k\frac{1}{2}\rho s^i)}_{\diamondsuit} - \underbrace{w^{ik}}_{\spadesuit}) = \underbrace{-\epsilon^{ikj}t_{kj}}_{\sharp} + \underbrace{\rho g^i}_{\natural} \qquad (19)
$$

The $\diamondsuit$-terms in (19), (17), and (18) are equivalent to the total time derivative, the $\spadesuit$-terms belong to the couple stress, the $\sharp$-terms to the skew-symmetric part of the energy-momentum tensor, and the $\natural$-terms to the external angular momentum. The $\triangle$-terms can be interpreted as coupling terms between the hh- and hu-part in which the other fields appear respectively. Consequently we obtain balance equations for the spin density (hh-part) and for the spin density vector (hu-part) which are coupled to each other by the $\triangle$-terms.
From (7), (17), and from (18) we can see, how the 24 components of

the spin tensor are distributed on the basic fields and the constitutive quantities. The first term in (7), the couple stress (8), is according to (19) a constitutive equation. Because the couple stress is space-like, it represents 9 fields. The second term in (7), the spin density (9), is according to (17) a basic spin field having 3 components. According to (17) and (18) there are other spin fields, the spin stress (14) and the spin density vector (15). Comparison of (17) and (18) with (19) yields, that the spin density vector is an other basic spin field and that the spin stress is a constitutive quantity. The spin density vector field has 3 components, whereas the spin stress has 9 components. Consequently we have 3+3 = 6 basic spin fields and 9+9 = 18 constitutive quantities which all together result in the 24 components of the spin tensor.

3 Spin Axioms

As discussed in the previous section there are 6 basic spin fields in relativistic theories. But up to now only 3 spin fields are known in physics, a fact which is formulated in the following

> **Empirem**
> According to (19) in non-relativistic physics only 3 basic spin fields (the spin density) and 9 constitutive functions (the couple stress) are known and measurable.
>
> □

This empirem can be interpreted in two different ways: 3 of the 6 spin fields are caused by typically relativistic effects. They are very small and not be measured up to now. Interestingly these 6 spin fields are connected with 18 constitutive couple stresses so that also materials should have relativistic properties which cannot be detected in the non-relativistic limit. If this possibility of interpretation seems to be too artificial, the other possibility remains:

> **Spin Axiom 1**
> *Also in relativistic physics only 3 basic spin fields and 9 couple stresses exist.*
>
> □

A consequence of axiom 1 is to reduce the 6 spin fields to 3 ones in relativistic theories. The question is how to reduce them.
One first possibility is to demand, that the spin tensor is totally skew-symmetric

$$S_{\alpha\beta\gamma} \stackrel{!}{=} S_{[\alpha\beta\gamma]} \tag{20}$$

In this case we obtain only 4 spin fields which may correspond to the wanted 3 basic spin fields and 1 additional constitutive field. Therefore by (20) the constitutive equations are severely restricted by setting 8 of the 9 fields of the couple stress to zero.
A second possibility of reducing fields is to cancel 3 spin fields and 9 constitututive quantities arbitrarily. But this possibility seems not to be very systematic, because the basic spin fields and the constitutive fields are properly separated from each other: There is the spin density (9) and the spin density vector (15) as basic spin fields, and the couple stress (8) and the spin stress (14) correspond to each other as constitutive fields. Because the hh-part (17) and the hu-part (18) of the spin balance are coupled to each other, a third possibility

$$\Xi_\nu \;=\; 0, \qquad \wedge \qquad \Xi_\nu^{\cdot\lambda} \;=\; 0 \tag{21}$$

or

$$\tilde{s}_{\mu\nu} \;=\; 0, \qquad \wedge \qquad s_{\mu\nu}^{\cdot\cdot\lambda} \;=\; 0 \tag{22}$$

results in differential equations which are not of balance type any more. Inserting (21) into (17) and (18) we obtain from (17) a balance equation of the spin density, whereas (18) has to be interpreted as a constraint for the constitutive quantity couple stress: The couple stress cannot be chosen arbitrarily, because its divergence is restricted by the constraints (18). The same happens to the spin stress and the spin density vector, if (22) is inserted into (17) and (18). Consequently by choice of (21) or (22) hidden material properties are introduced.
An essential point is, that the reduction of the fields should not restrict the free choice of the also reduced constitutive equations. Consequently the possibility of identifying spin density with spin density vector and couple stress with spin stress remains.

Spin Axiom 2
Spin density and spin density vector field are semi-dual to

each other, as couple stress and spin stress are, that means we identify

$$\tilde{s}_{\alpha\beta} \stackrel{!}{=} \frac{1}{2}\frac{1}{c^2}\eta_{\alpha\beta\delta\gamma}u^\delta\Xi^\gamma \quad \Leftrightarrow \quad \frac{1}{c^2}u_{[\mu}\Xi_{\nu]} = \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\alpha\beta}\tilde{s}_{\alpha\beta} \quad (23)$$

$$s_{\alpha\beta\gamma} \stackrel{!}{=} \frac{1}{2}\frac{1}{c^2}\eta_{\alpha\beta\delta\lambda}u^\delta\Xi^\lambda_{\cdot\gamma} \quad \Leftrightarrow \quad \frac{1}{c^2}u_{[\mu}\Xi_{\nu]}^{\cdot\lambda} = \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\alpha\beta}s_{\alpha\beta}^{\cdot\cdot\lambda} (24)$$

□

Here $\eta_{\alpha\beta\delta\lambda}$ is the Levi-Civita symbol. From (23) we see that $\tilde{s}_{\alpha\beta}$ and Ξ^γ are semi-dual to each other if $\tilde{s}_{\alpha\beta}$ and $u^\delta\Xi^\gamma$ are dual to each other. According to (24) the same is valid for $s_{\alpha\beta\gamma}$ and $\Xi^\lambda_{\cdot\gamma}$.

Inserting $(23)_2$ and $(24)_2$ into (7) we obtain for the spin tensor

$$\begin{aligned} S_{\mu\nu}^{\cdot\cdot\lambda} &= \left(\delta^\alpha_{[\mu}\delta^\beta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\alpha\beta}\right)\frac{1}{c^2}\tilde{s}_{\alpha\beta}u^\lambda + \\ &\quad \left(\delta^\alpha_{[\mu}\delta^\beta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\alpha\beta}\right)s_{\alpha\beta}^{\cdot\cdot\lambda} \end{aligned} \quad (25)$$

and inserting $(23)_1$ and $(24)_1$ into (7) results in

$$\begin{aligned} S_{\mu\nu}^{\cdot\cdot\lambda} &= \left(\delta^\gamma_{[\mu}\delta^\delta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\gamma\delta}\right)\frac{1}{c^4}\Xi_\gamma u_\delta u^\lambda + \\ &\quad \left(\delta^\gamma_{[\mu}\delta^\delta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\gamma\delta}\right)\frac{1}{c^2}\Xi_\gamma^{\cdot\lambda}u_\delta \end{aligned} \quad (26)$$

By adopting the spin axiom 2 (25) and (26) are different but equivalent representations of the spin tensor, (25) in spin density and couple stress, (26) in spin density vector and spin stress. From (25) and (26) we obtain

$$S_{\mu\nu}^{\cdot\cdot\lambda} = \left(\delta^\alpha_{[\mu}\delta^\beta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\alpha\beta}\right)\left(\frac{1}{c^2}\tilde{s}_{\alpha\beta}u^\lambda + s_{\alpha\beta\cdot}^{\cdot\cdot\lambda}\right) \quad (27)$$

$$= \left(\delta^\gamma_{[\mu}\delta^\delta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\gamma\delta}\right)\left(\frac{1}{c^4}u_\gamma\Xi_\delta u^\lambda + \frac{1}{c^2}u_\gamma\Xi_\delta^{\cdot\lambda}\right) \quad (28)$$

The common bracket in the representations of the spin tensor has the remarkable property

$$\frac{1}{2}\eta_{\kappa\rho}^{\cdot\,\cdot\,\mu\nu}\left(\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}+\frac{1}{2}\eta_{\mu\nu}^{\cdot\,\cdot\,\alpha\beta}\right) = \left(\delta^{\alpha}_{[\kappa}\delta^{\beta}_{\rho]}+\frac{1}{2}\eta_{\kappa\rho}^{\cdot\,\cdot\,\alpha\beta}\right) \qquad (29)$$

Consequently we have proven the

Corollary
By the spin axioms the spin tensor is self dual with respect to the first two indices

$$S_{\mu\nu}^{\cdot\,\cdot\,\lambda} = \frac{1}{2}\eta_{\mu\nu}^{\cdot\,\cdot\,\alpha\beta}S_{\alpha\beta}^{\cdot\,\cdot\,\lambda} \qquad (30)$$

From the two representations (27) and (28) for the spin tensor we obtain two equivalent versions of the spin balance

$$\begin{aligned} S_{\mu\nu\,\cdot\,;\lambda}^{\cdot\,\cdot\,\lambda} &= \left(\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}+\frac{1}{2}\eta_{\mu\nu}^{\cdot\,\cdot\,\alpha\beta}\right)\left(\frac{1}{c^2}\tilde{s}_{\alpha\beta}u^{\lambda}+s_{\alpha\beta\,\cdot}^{\cdot\,\cdot\,\lambda}\right)_{;\lambda} &(31)\\ &= \left(\delta^{\gamma}_{[\mu}\delta^{\delta}_{\nu]}+\frac{1}{2}\eta_{\mu\nu}^{\cdot\,\cdot\,\gamma\delta}\right)\left(\frac{1}{c^4}u_{\gamma}\Xi_{\delta}u^{\lambda}+\frac{1}{c^2}u_{\gamma}\Xi_{\delta}^{\cdot\,\lambda}\right)_{;\lambda} &(32)\end{aligned}$$

The 3+1 decomposition of (31) gives

<u>$h^{\mu}_{\kappa}h^{\nu}_{\sigma}$:</u>

$$\begin{aligned} h^{\mu}_{\kappa}h^{\nu}_{\sigma}S_{\mu\nu\,\cdot\,;\lambda}^{\cdot\,\cdot\,\lambda} &= h^{\mu}_{\kappa}h^{\nu}_{\sigma}\left(\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}+\frac{1}{2}\eta_{\mu\nu}^{\cdot\,\cdot\,\alpha\beta}\right)\left(\frac{1}{c^2}\tilde{s}_{\alpha\beta}u^{\lambda}+s_{\alpha\beta\,\cdot}^{\cdot\,\cdot\,\lambda}\right)_{;\lambda}\\ &= h^{\mu}_{\kappa}h^{\nu}_{\sigma}T_{[\mu\nu]}+h^{\mu}_{\kappa}h^{\nu}_{\sigma}L_{[\mu\nu]}\\ &= t_{[\kappa\sigma]}+h^{\mu}_{\kappa}h^{\nu}_{\sigma}L_{[\mu\nu]} &(33)\end{aligned}$$

<u>$u^{\mu}h^{\nu}_{\sigma}$:</u>

$$\begin{aligned} u^{\mu}h^{\nu}_{\sigma}S_{\mu\nu\,\cdot\,;\lambda}^{\cdot\,\cdot\,\lambda} &= u^{\mu}h^{\nu}_{\sigma}\left(\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}+\frac{1}{2}\eta_{\mu\nu}^{\cdot\,\cdot\,\alpha\beta}\right)\left(\frac{1}{c^2}\tilde{s}_{\alpha\beta}u^{\lambda}+s_{\alpha\beta\,\cdot}^{\cdot\,\cdot\,\lambda}\right)_{;\lambda}\\ &= u^{\mu}h^{\nu}_{\sigma}T_{[\mu\nu]}+u^{\mu}h^{\nu}_{\sigma}L_{[\mu\nu]}\\ &= (q_{\sigma}-p_{\sigma})+u^{\mu}h^{\nu}_{\sigma}L_{[\mu\nu]} &(34)\end{aligned}$$

And the 3+1 decomposition of (32) results in
$\underline{h^\mu_\kappa h^\nu_\sigma}$**:**

$$\begin{aligned} h^\mu_\kappa h^\nu_\sigma S_{\mu\nu\cdot;\lambda}^{\cdot\cdot\lambda} &= h^\mu_\kappa h^\nu_\sigma \left(\delta^\gamma_{[\mu}\delta^\delta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\gamma\delta}\right)\left(\frac{1}{c^4}u_\gamma \Xi_\delta u^\lambda + \frac{1}{c^2}u_\gamma \Xi_\delta^{\cdot\lambda}\right)_{;\lambda} \\ &= h^\mu_\kappa h^\nu_\sigma T_{[\mu\nu]} + h^\mu_\kappa h^\nu_\sigma L_{[\mu\nu]} \\ &= t_{[\kappa\sigma]} + h^\mu_\kappa h^\nu_\sigma L_{[\mu\nu]} \end{aligned} \tag{35}$$

$\underline{u^\mu h^\nu_\sigma}$**:**

$$\begin{aligned} u^\mu h^\nu_\sigma S_{\mu\nu\cdot;\lambda}^{\cdot\cdot\lambda} &= u^\mu h^\nu_\sigma \left(\delta^\gamma_{[\mu}\delta^\delta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\gamma\delta}\right)\left(\frac{1}{c^4}u_\gamma \Xi_\delta u^\lambda + \frac{1}{c^2}u_\gamma \Xi_\delta^{\cdot\lambda}\right)_{;\lambda} \\ &= u^\mu h^\nu_\sigma T_{[\mu\nu]} + u^\mu h^\nu_\sigma L_{[\mu\nu]} \\ &= (q_\sigma - p_\sigma) + u^\mu h^\nu_\sigma L_{[\mu\nu]} \end{aligned} \tag{36}$$

Equations (33) and (35) are identical, as well as equations (34) and (36). The reduction from 6 to 3 equations (spin balances) requires, that also equations (33) and (34) are dependent on each other, as well as equations (35) and (36). This leads to the requirement that also $(q_\sigma - p_\sigma)$ and $t_{\sigma\kappa}$ are semi-dual to each other:

$$t_{\kappa\sigma} = \frac{1}{2}\frac{1}{c^2}\eta_{\kappa\sigma}^{\cdot\cdot\beta\alpha}u_\beta(q_\alpha - p_\alpha) \tag{37}$$

The particular meaning of the spin axioms (23) and (24) can be characterized by the following proposition which we will prove later

> **Conjecture**
> All reductions from 6 + 18 to 3 + 9 basic and constitutive spin fields which do not use spin axiom 2 are introducing a specially chosen material.
> □

If this conjecture is true, the reduction of the spin fields has to obey the spin axiom 2, because a reduction has to be performed before constitutive properties are introduced into the theoretical considerations. With respect to the spin axioms we now will shortly discuss two examples of spin fluids well-known from the literature.

4 Examples

There are two spin fluids which are typically different from each other: The Weyssenhoff fluid which is a classical one, and the Dirac-electron fluid which represents the classical description of a quantum-fluid. We now will discuss these fluids with respect to the spin axioms introduced above.

4.1 Weyssenhoff fluid

According to [2, 3] the spin tensor of the Weyssenhoff fluid reads:

$$S_{\alpha\beta}^{\cdot\cdot\mu} \doteq \frac{1}{c^2}\tilde{s}_{\alpha\beta}u^\mu \tag{38}$$

This choice of the spin tensor can be interpreted in two ways: Comparing (38) with (25) we could state, that the couple stress $s_{\alpha\beta}^{\cdot\cdot\lambda}$, the spin density vector Ξ_ν, and the spin stress $\Xi_\nu^{\cdot\lambda}$ are chosen to zero and that therefore the choice (38) of the spin tensor does not satisfy the spin axiom 2, because in (38) the η-term of the first bracket in (25) is missing. Consequently we obtain from (17) the balance of the spin density

$$\underbrace{\frac{1}{c^2}\tilde{s}_{\mu\nu;\lambda}u^\lambda h_\alpha^\mu h_\beta^\nu + \frac{1}{c^2}\tilde{s}_{\mu\nu}h_\alpha^\mu h_\beta^\nu u^\lambda_{\cdot;\lambda}}_{\diamondsuit} = \underbrace{T_{[\mu\nu]}h_\alpha^\mu h_\beta^\nu}_{\sharp} + \underbrace{L_{\mu\nu}h_\alpha^\mu h_\beta^\nu}_{\natural} \tag{39}$$

and from (18) the constraint for the spin density

$$\underbrace{-\frac{1}{c^2}\tilde{s}_{\mu\nu}h_\gamma^\nu u^\mu_{\cdot;\lambda}u^\lambda}_{\triangle} = \underbrace{T_{[\mu\nu]}h_\gamma^\nu u^\mu}_{\sharp} + \underbrace{L_{\mu\nu}h_\gamma^\nu u^\mu}_{\natural} \tag{40}$$

which only appears, if the spin axiom 2 is not taken into account.
If the spin axiom 2 is taken into account, (38) is compatible with (25), if

$$\frac{1}{2c^2}\eta_{\mu\nu}^{\cdot\cdot\alpha\beta}\tilde{s}_{\alpha\beta}u^\lambda + \left(\delta^\alpha_{[\mu}\delta^\beta_{\nu]} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\alpha\beta}\right)s_{\alpha\beta}^{\cdot\cdot\lambda} = 0 \tag{41}$$

or

$$s_{\mu\nu}^{\cdot\,\cdot\,\lambda} \;=\; -\frac{1}{2}\eta_{\mu\nu}^{\cdot\,\cdot\,\alpha\beta}\left(\frac{1}{c^2}\tilde{s}_{\alpha\beta}u^\lambda + s_{\alpha\beta}^{\cdot\,\cdot\,\lambda}\right) \tag{42}$$

is valid. This is a constitutive equation of the couple stress which is not zero, if the spin axiom 2 is accepted. From (42) follows by the

Proposition

$$s_{\mu\nu}^{\cdot\,\cdot\,\lambda} \;=\; \frac{1}{c^2}\left(\delta^\alpha_{[\mu}\delta^\beta_{\nu]} - \eta_{\mu\nu}^{\cdot\,\cdot\,\alpha\beta}\right)\tilde{s}_{\alpha\beta}u^\lambda \tag{43}$$

the couple stress as a function of the spin density.
Without accepting spin axiom 2 (38) describes an ideal spin fluid, but taking spin axiom 2 into account the spin fluid is not an ideal one, because the couple stress does not vanish according to (43).
According to (25) the spin tensor of an ideal spin fluid is

$$S_{\mu\nu}^{\cdot\,\cdot\,\lambda} \;=\; \left(\delta^\alpha_\mu\delta^\beta_\nu + \frac{1}{2}\eta_{\mu\nu}^{\cdot\,\cdot\,\alpha\beta}\right)\frac{1}{c^2}\tilde{s}_{\alpha\beta}u^\lambda \tag{44}$$

if the spin axiom 2 is accepted.

4.2 Dirac-electron fluid

The following ansatz is made by Bäuerle und Haneveld [4]:

$$S_{\mu\nu}^{\cdot\,\cdot\,\kappa}g_{\kappa\lambda} \;=\; \frac{1}{c^2}\tau_{[\mu\nu}u_{\lambda]} \quad \text{with} \quad \tau_{\mu\nu} \;:=\; S_{\mu\nu}^{\cdot\,\cdot\,\lambda}u_\lambda \tag{45}$$

From this ansatz follows:

$$\tau_{[\mu\nu}u_{\lambda]} = S_{[\mu\nu\lambda]} \;=\; \frac{1}{c^2}\tilde{s}_{[\mu\nu]}u_\lambda + \frac{1}{c^2}\tilde{s}_{[\lambda\mu]}u_\nu + \frac{1}{c^2}\tilde{s}_{[\nu\lambda]}u_\mu \tag{46}$$

$$= \frac{1}{c^2}\tilde{s}_{[\mu\nu]}u_\lambda + \frac{1}{c^2}u_{[\mu}\Xi_{\nu]\lambda} \tag{47}$$

$$\text{with} \qquad \Xi_{\nu\lambda} \;\stackrel{!}{=}\; \tilde{s}_{\nu\lambda} \equiv \tilde{s}_{[\nu\lambda]} \tag{48}$$

Here one can see that a special material function is assumed, equation (48) is a constitutive relation that determines the spin stress. The total antisymmetrisation mixes basic fields and constitutive functions which is rather strange from a constitutive point of view.

The interpretation of this spin tensor by use of the spin axioms is difficult.
In equation (45) the couple stress is set to zero

$$s_{\mu\nu\lambda} \doteq 0 \tag{49}$$

by use of the spin axioms also the spin stress vanishes

$$\Xi_{\mu\lambda} = 0 \tag{50}$$

According to the total antisymmetrisation (46) the spin density vector is zero

$$\Xi_{\mu} \doteq 0 \tag{51}$$

this implies by use of the spin axioms that

$$\tilde{s}_{\mu\nu} = 0 \tag{52}$$

This would means that the spin tensor vanishes.

As the total antisymmetrisation seems only to be necessary for the reduction from the 6 fields in (45) to 3, there is another way using the spin axioms. Starting out with $(45)_2$

$$S_{\mu\nu}^{\cdot\cdot\lambda} = \tau_{\mu\nu}u^{\lambda} \tag{53}$$

and then using spin axiom 2 one gets the following spin tensor

$$S_{\mu\nu}^{\cdot\cdot\lambda} = \left(\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta} + \frac{1}{2}\eta_{\mu\nu}^{\cdot\cdot\alpha\beta}\right)\frac{1}{c^2}\tilde{s}_{\alpha\beta}u^{\lambda} \tag{54}$$

which is the same as (44), and represents an ideal fluid.

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Aksiome o spinu u relativističkoj fizici kontinuuma

UDK 530.12, 531.01

24 komponente relativističkog tenzora spina se sastoje od 3+3 osnovna polja spina i 9 + 9 konstitutivnih polja. Empirički samo 3 osnovna polja spina i 9 konstitutivnih polja su poznata. Ovaj "empirem" može se izraziti sa dve aksiome spina pri čemu jedna od njih identifikuje 3 polja spina dok druga identifikuje ostalih 9 polja. Ova identifikacija je objektivna i ne meša osnovna polja spina sa konstitutivnim osobinama. Pristupi Weyssenhoffovom fluidu i Dirac-ovom elektronskom fluidu nadjeni u literaturi se diskutuju pomoću ovih aksioma spina. Formulišemo, zatim, lemu da jedna drukčija redukcija sa 6 na 3 osnovna polja spina koja ne zadovoljavaju aksiome spina uvodi specijalne materijalne osobine ne dozvoljavajući mešanje konstitutivnih i osnovnih polja.