THEORETICAL AND APPLIED MECHANICS
vol. 28-29, pp. 205-219, Belgrade 2002
Issues dedicated to memory of late Professor Rastko Stojanović

Configurational balance laws for dynamical fracture

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Abstract

In the spirit of modern continuum mechanics, global balance laws for momentum, angular momentum, energy and pseudomomentum are formulated for an elastic body in the presence of a moving crack. Upon localization, the corresponding balance equations in the bulk and at the crack tip are simultaneously obtained. The proposed framework is convenient for the derivation of the well-known formula, which relates the crack propagation velocity, the global material force and the energy release rate.

1 Introduction

This work aims at the formulation of global balance laws for an elastic body in the presence of a propagating crack through it. The standard procedure of the continuum mechanics, according to which, all the relevant equations can be produced by postulating global balance laws for mass, momentum, angular momentum and energy (in dissipative processes the second law is considered, in addition) is well-known. These laws are postulated to hold for any arbitrary part (subset) of the body, a requirement that is strong enough to provide the local equations.

Naturally, the situation in a cracked elastic solid is quite different because (i) the involved fields are not continuous across the crack and,

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particularly, they have a singularity at the crack tip (ii) new quantities are needed to describe completely the phenomenon (iii) the underlying kinematics is much more complicated due to the presence of the propagating crack. In the literature of fracture mechanics, sparse reports to equations, which can be considered as global balance laws for a fractured body, are appeared [1, 2, 3, 4, 5] but, to the best of our knowledge, there is not a complete and consistent formulation in the spirit of the modern continuum mechanics.

As in any standard text of continuum mechanics, the balance laws for momentum, angular momentum and energy are formulated. Moreover, a balance for an additional quantity, the pseudomomentum, is proposed. Although the crack propagation in a deformable body is a dissipative phenomenon, in this paper no mention is made to the second law of thermodynamics and to the subsequent discussion about constitutive relations. The analysis is based on the finite two-dimensional elasticity and on the framework of the so–called configurational (material) mechanics [1, 2, 6].

In Section 2, a properly formulated kinematics for a cracked elastic body is discussed. In Section 3, an abstract global balance law is postulated and its consequences (the local equations) are extracted in a general manner. The application to the physical fields, related to the problem under study, is made in Section 4. Finally, in Section 5, the obtained results are used to derive the relation between the energy release rate and the global material force.

2 Kinematics

Let $\mathcal{B}_R$ be the reference configuration containing a crack which is described by a smooth, non-intersecting curve C_R with its end point $\mathbf{Z}_0$ (the crack tip)(Fig.1). We consider that the crack evolves (not nesessarily in straight direction) following the "motion" of the crack tip within the body. Thus at the time t, the crack is interpreted by a smooth curve $C(t)$ belonging to a material configuration $\mathcal{B}_t$, $t \in I \subset \mathbb{R}$, where I denotes a time interval. The only difference between the reference configuration $\mathcal{B}_R$ and the material configurations $\mathcal{B}_t$ lies in the different curve they contain. Certainly, it is required for $t_1 > t_2$ to imply $C(t_2) \subset C(t_1)$.

We focus now on the end point of the crack at time t, $\mathbf{Z}(t)$. We consider that $\mathbf{Z}(t)$ is a smooth, time dependent mapping, hence its derivative

$$\mathbf{V}(t) = \frac{d\mathbf{Z}}{dt} \tag{1}$$

provides the propagation velocity of the crack.

Taking the standard view of fracture mechanics, we consider a (tip) disc of radius r centered at the crack tip $\mathbf{Z}(t)$ for any time t, denoted by $D_r(t)$:

$$D_r(t) = \{\mathbf{X} \in \mathcal{B}_t : |\mathbf{X} - \mathbf{Z}(t)| \leq r\}. \tag{2}$$

At the time t_0, the tip disc is given by:

$$D_{r_0} = \{\mathbf{Y} \in \mathcal{B}_R : |\mathbf{Y} - \mathbf{Z}_0| \leq r\}.$$

Thus, we can establish an imaginary motion of the tip disc by a mapping of the form

$$\mathbf{X} = \tilde{y}(\mathbf{Y}, t), \quad \mathbf{X} \in D_r(t), \quad \mathbf{Y} \in D_{r_0},\ t \in I \subset I\!R.$$

Also, notice that $D_{r_0} \subset \mathcal{B}_R$ and $D_r(t) \subset \mathcal{B}_t$.

Without any loss of generality, we assume that the "motion" of the tip disc is a simple translation which follows the crack tip evolution, that is

$$\tilde{y}(\mathbf{Y}, t) = \mathbf{Y} + \mathbf{Z}(t) - \mathbf{Z}_0, \quad \text{for all} \quad \mathbf{Y} \in D_{r_0}. \tag{3}$$

It is obvious that every point of D_{r_0} "moves" with the velocity of the crack tip, i.e.,

$$\frac{\partial \tilde{y}}{\partial t}(\mathbf{Y}, t) = \frac{d\mathbf{Z}}{dt} = \mathbf{V}(t), \quad \text{for all} \quad \mathbf{Y} \in D_{r_0}. \tag{4}$$

Let us assume an extension $y(\mathbf{Y}, t)$ of $\tilde{y}$ in the whole body $\mathcal{B}_R$ as follows:

$$y(\mathcal{B}_R, t) = \mathcal{B}_t, \qquad y|_{D_{r_0}} = \tilde{y}, \qquad \frac{\partial y}{\partial t} = 0, \quad \text{for all} \quad \mathbf{Y} \in \mathcal{B}_R \setminus D_{r_0}. \tag{5}$$

Consider now the deformational motion

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \mathbf{x} \in B_t, \quad \mathbf{X} \in \mathcal{B}_t,\ t \in I \subset I\!R, \tag{6}$$

which is a twice-differential and one-to-one mapping for all $\mathbf{X} \in \mathcal{B}_t \setminus C(t)$. As usual, with $\mathbf{F}(\mathbf{X},t) = \partial\chi(\mathbf{X},t)/\partial\mathbf{X}$ is denoted the deformation gradient.
We have to compose the mappings y and χ to interpret the "total motion", i.e., to interpret both the crack evolution and the motion of the body in the physical space (Fig. 1). This composition is given by the mapping

$$\widetilde{\chi} = \chi \circ y, \quad \mathbf{x} = \widetilde{\chi}(\mathbf{Y},t) = \chi(y(\mathbf{Y},t),t).$$

After all these considerations, we take

$$\overset{\circ}{\mathbf{x}} = \frac{d\chi}{dt} = \frac{\partial\chi}{\partial\mathbf{X}}(\mathbf{X},t)\ \frac{\partial y}{\partial t}(\mathbf{Y},t) + \frac{\partial\chi}{\partial t}(\mathbf{X},t), \tag{7}$$

where the differential operator d/dt denotes the total derivative with respect to time. Taking into account eqs. (4) and (5), eq. (7) takes the form

$$\overset{\circ}{\mathbf{x}} = \mathbf{F}(\mathbf{X},t)\mathbf{V}(t) + \dot{\mathbf{x}}(\mathbf{X},t) =: \widetilde{\mathbf{V}}(\mathbf{X},t), \tag{8}$$

for all $\mathbf{X} \in D_r(t) \setminus (D_r(t) \cap C(t))$, where $\dot{\mathbf{x}}(\mathbf{X},t) = \partial\chi(\mathbf{X},t)/\partial t$.
For later use, taking the view of Gurtin et al [1, 3], we assume the existence of a time–dependent function $\mathbf{U}$ such that

$$\lim_{\mathbf{X}\to\mathbf{Z}(t)} \widetilde{\mathbf{V}}(\mathbf{X},t) = \mathbf{U}(t), \quad \text{uniformly in } \ I. \tag{9}$$

3 The balance law in general form

We now proceed to the formulation of an abstract balance law. Let Ω be any smooth domain of the body in the material configuration $\mathcal{B}_t$. If the crack tip $\mathbf{Z}(t)$ is an interior point of Ω, then there exists a radius r such that $D_r(t) \subset \Omega$. In this case, we will denote with Ω_r the subset of Ω which is defined as follows (Fig. 2),

$$\Omega_r(t) = \Omega \setminus D_r(t) \ \text{ or } \ \Omega = \Omega_r(t) \cup D_r(t). \tag{10}$$

Notice that $\partial\Omega_r = \partial\Omega \cup \partial D_r(t)$. Also, the parts of the crack $C(t)$ contained in Ω_r and Ω will be denoted by γ_r and γ_Ω, respectively , that is

$$\gamma_r = C(t) \cap \Omega_r(t), \quad \gamma_\Omega = C(t) \cap \Omega.$$

In standard continuum mechanics, one has the freedom to formulate a global balance law either in the reference configuration or in the current configuration. In the proposed framework, there are three distinct configurations (Fig. 1). We choose to work on a material configuration $\mathcal{B}_t$, in which all the relevant fields should be defined. Let $\Phi(\mathbf{X}, t)$ be a scalar valued function defined in $\mathcal{B}_t$, representing some physical quantity, sufficiently smooth away from the crack tip and up to the crack $C(t)$ from either side. Let also $\mathbf{f}$ and h, are the flux and the source of Φ, respectively, which are also smooth away from the crack tip and the crack $C(t)$. Moreover, we consider a time dependent function $g(t)$ representing the source of Φ due to the crack evolution. One can write the global balance law for the quantity Φ as follows:

$$\frac{d}{dt}\int_{\Omega} \Phi(\mathbf{X}, t)\ dV = \int_{\partial\Omega} \mathbf{f}(\mathbf{X}, t)\cdot\mathbf{n}\ dS + \int_{\Omega} h(\mathbf{X}, t)\ dV + g(t), \qquad (11)$$

where $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega_r$.
According to the above considerations, the transport theorem takes the following form

$$\frac{d}{dt}\int_{\Omega_r} \Phi(\mathbf{X}, t)\ dV = \int_{\Omega_r} \frac{\partial\Phi}{\partial t}\ dV - \int_{\partial D_r} \Phi(\mathbf{V}\cdot\mathbf{n})\ dS. \qquad (12)$$

Similarly, the divergence theorem, for the problem under study, becomes

$$\int_{\Omega_r} \text{Div}\ \mathbf{f}(\mathbf{X}, t)\ dV = \int_{\partial\Omega} \mathbf{f}(\mathbf{X}, t)\cdot\mathbf{n}\ dS - \int_{\partial D_r} \mathbf{f}(\mathbf{X}, t)\cdot\mathbf{n}\ dS + \int_{\gamma_r} [\mathbf{f}(\mathbf{X}, t)]\cdot\mathbf{m}\ dl, \qquad (13)$$

where $\mathbf{m}$ is the unit normal to the crack. Also, $[\mathbf{f}]$ denotes the jump of the vector field $\mathbf{f}$ across the crack.
Recalling eqs. (10) and (12), we can write

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} \Phi(\mathbf{X}, t)\ dV &= \frac{d}{dt}\int_{\Omega_r} \Phi(\mathbf{X}, t)\ dV + \frac{d}{dt}\int_{D_r} \Phi(\mathbf{X}, t)\ dV \\ &= \int_{\Omega_r} \frac{\partial\Phi}{\partial t}\ dV - \int_{\partial D_r} \Phi(\mathbf{V}\cdot\mathbf{n})\ dS + \frac{d}{dt}\int_{D_r} \Phi(\mathbf{X}, t)\ dV. \qquad (14)\end{aligned}$$

In virtue of eqs. (10), (13) and (14), the balance law (11) can be written

$$\int_{\Omega_r} \frac{\partial \Phi}{\partial t}\, dV - \int_{\partial D_r} \Phi(\mathbf{V}\cdot\mathbf{n})\, dS + \frac{d}{dt}\int_{D_r} \Phi(\mathbf{X},t)\, dV = \int_{\Omega_r} \text{Div}\, \mathbf{f}(\mathbf{X},t)\, dV + \int_{\partial D_r} \mathbf{f}(\mathbf{X},t)\cdot\mathbf{n}\, dS - \int_{\gamma_r} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{m}\, dl + \int_{\Omega_r} h(\mathbf{X},t)\, dV + \int_{D_r} h(\mathbf{X},t)\, dV + g(t). \quad (15)$$

Without giving here the necessary details[1], we assume that the following convergences hold

$$\lim_{r\to 0}\int_{\Omega_r} \frac{\partial \Phi}{\partial t}\, dV = \int_{\Omega} \frac{\partial \Phi}{\partial t}\, dV,$$
$$\lim_{r\to 0}\int_{\Omega_r} \text{Div}\, \mathbf{f}(\mathbf{X},t)\, dV = \int_{\Omega} \text{Div}\, \mathbf{f}(\mathbf{X},t)\, dV,$$
$$\lim_{r\to 0}\frac{d}{dt}\int_{D_r} \Phi(\mathbf{X},t)\, dV = 0,$$
$$\lim_{r\to 0}\int_{\Omega_r} h(\mathbf{X},t)\, dV = \int_{\Omega} h(\mathbf{X},t)\, dV,$$
$$\lim_{r\to 0}\int_{\gamma_r} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{m}\, dl = \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{m}\, dl,$$
$$\lim_{r\to 0}\int_{D_r} h(\mathbf{X},t)\, dV = 0.$$

Consequently, we obtain, for the global balance law given by eq. (15), the following

$$\int_{\Omega}\left(\frac{\partial \Phi}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t)\right) dV + \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{m}\, dl - \lim_{r\to 0}\int_{\partial D_r} \big(\Phi(\mathbf{V}\cdot\mathbf{n}) + \mathbf{f}\cdot\mathbf{n}\big) dS - g(t) = 0, \quad (16)$$

for all Ω containing the crack tip.
We remark that in the case where Ω does not contain the crack tip and

[1]For an extensive discussion see the references [1, 3]

any part of the crack, the equation (16) takes the simpler form

$$\int_{\Omega} \left(\frac{\partial \Phi}{\partial t} - \text{Div } \mathbf{f}(\mathbf{X}, t) - h(\mathbf{X}, t) \right) dV = 0. \tag{17}$$

Thus, due to the arbitrariness of Ω, we conclude that

$$\frac{\partial \Phi}{\partial t} - \text{Div } \mathbf{f}(\mathbf{X}, t) - h(\mathbf{X}, t) = 0, \quad \text{for all } \ t \in I, \ \mathbf{X} \in \mathcal{B}_t \setminus C(t). \tag{18}$$

Similarly, we can consider Ω containing a part of the crack apart from the crack tip. In this case, the global balance law takes the form

$$\int_{\Omega} \left(\frac{\partial \Phi}{\partial t} - \text{Div } \mathbf{f}(\mathbf{X}, t) - h(\mathbf{X}, t) \right) dV + \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X}, t)] \cdot \mathbf{m} \ dl = 0. \tag{19}$$

With the aid of eq. (18), we obtain the local form of eq. (19)

$$[\mathbf{f}(\mathbf{X}, t)] \cdot \mathbf{m} = 0, \quad \text{for all } \ t \in I, \ \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}. \tag{20}$$

In the same line of argument, we obtain the localization of the balance law at the crack tip as follows:

$$g(t) = \lim_{r \to 0} \int_{\partial D_r} \big(\Phi(\mathbf{V} \cdot \mathbf{n}) + \mathbf{f} \cdot \mathbf{n} \big) dS, \quad \text{for all } \ t \in I, \text{ at } \ \mathbf{Z}(t). \tag{21}$$

In conclusion, the requirement that the balance law (11) holds for all $\Omega \in \mathcal{B}_t$ implies the local equations (18), (20) and (21).

4 The balance laws for the physical fields and the pseudomomentum

In this section, we are going to apply the procedure described in the last Section. That is, we have to substitute the relevant physical quantities at the position of the abstract function Φ. Following standard ideas, we consider the balances for momentum, angular momentum and energy. But this does not exhaust all the relevant quantities involved in our problem. We must further formulate a balance for pseudomomentum. We start with the momentum balance.

4.1 The balance of momentum

In this case, we put the physical momentum $\rho\dot{\mathbf{x}}$ instead of the function Φ in the general equation (11). The flux is given by $\mathbf{Tn}$, where $\mathbf{T}$ is the Piola–Kirchhoff stress tensor. Thus, we have

$$\frac{d}{dt}\int_{\Omega}\rho\dot{\mathbf{x}}\,dV=\int_{\partial\Omega}\mathbf{Tn}\,dS,\quad\forall\Omega\in\mathcal{B}_t. \tag{22}$$

Notice that we do not consider source term (the body force for the case under discussion) and also we do not consider source term due to the crack propagation. The latter means that there is no production of physical momentum due to the crack propagation. The localization of eq. (22), according to the procedure described in the last section (see eqs. (18), (20) and (21)), provides

$$\frac{\partial}{\partial t}(\rho\dot{\mathbf{x}})-\mathrm{Div}\mathbf{T}=0,\quad\text{for all}\quad\mathbf{X}\in\mathcal{B}_t\setminus C(t), \tag{23}$$

$$[\mathbf{T}]\cdot\mathbf{m}=0,\quad\text{for all}\quad\mathbf{X}\in C(t)\setminus\{\mathbf{Z}(t)\}, \tag{24}$$

$$\lim_{r\to 0}\int_{\partial D_r}\big(\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{n})+\mathbf{Tn}\big)\,dS=0,\quad\text{at}\quad\mathbf{Z}(t). \tag{25}$$

As we all know, eq. (23) is the equation of physical momentum, which holds in the bulk of the body. Eq. (24) represents the jump condition for the Piola-Kirchhoff stress tensor along the crack. Notice that this condition is derived, not assumed, by the general postulation we have set for the global balance law. Finally, eq. (25) is the equation of physical momentum at the crack tip and we will see later on that it is useful in the proof of the relation between the energy release rate and the global material force.

4.2 The balance of angular momentum

The global balance law for angular momentum is postulated to be of the form

$$\frac{d}{dt}\int_{\Omega}(\mathbf{x}-\mathbf{0})\times\rho\dot{\mathbf{x}}\,dV=\int_{\partial\Omega}(\mathbf{x}-\mathbf{0})\times\mathbf{Tn}\,dS,\quad\forall\Omega\in\mathcal{B}_t, \tag{26}$$

where $\mathbf{x} - \mathbf{0}$ is the position vector of $\mathbf{x}$. Upon localization, eq. (26) provides

$$\mathbf{T}\mathbf{F}^T = \mathbf{F}\mathbf{T}^T, \quad \text{for all} \quad \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{27}$$

$$\lim_{r\to 0} \int_{\partial D_r} (\mathbf{x} - \mathbf{0}) \times \big(\rho \dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{n}) + \mathbf{T}\mathbf{n}\big)\, dS = 0, \quad \text{at} \quad \mathbf{Z}(t). \tag{28}$$

Eq. (27) gives the symmetry of the Cauchy stress tensor $\mathbf{t} = J_F^{-1}\mathbf{T}\mathbf{F}^T$, where $J_F = det\mathbf{F}$. Also, eq. (28) represents the balance of angular momentum at the crack tip. We remark that the jump condition associated with the balance of angular momentum is given by [2]

$$[\mathbf{x} \times \mathbf{T}] \cdot \mathbf{m} = 0.$$

However, the above condition holds identically due to the continuity [3] of the motion $\chi(\mathbf{X}, t)$ along the curve $C(t)$ and the jump condition (24).

4.3 The balance of energy

The energy balance is the most interesting case among the physical fields, because the crack propagation process has an influence on it. More specifically, one expects some portion of the energy to be consumed by the creation of the new part of the crack. Hence, an energy source term at $\mathbf{Z}(t)$, described here by $G(t)$, must be considered. Thus, the global balance law for energy can be postulated as

$$\frac{d}{dt}\int_\Omega (W + K)\, dV = \int_{\partial\Omega} \mathbf{T}\mathbf{n}\cdot\dot{\mathbf{x}}\, dS - G, \quad \text{for all} \quad \Omega \in \mathcal{B}_t, \tag{29}$$

where W is the elastic energy density and K is the kinetic energy density. Localizing eq. (29), we obtain

$$\frac{\partial}{\partial t}(W + K) - \mathrm{Div}(\mathbf{T}^T\dot{\mathbf{x}}) = 0, \quad \text{for all} \quad \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{30}$$

$$[\mathbf{T}^T\dot{\mathbf{x}}] \cdot \mathbf{m} = 0, \quad \text{for all} \quad \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}, \tag{31}$$

$$G = \lim_{r\to 0} \int_{\partial D_r} \big((W + K)(\mathbf{V}\cdot\mathbf{n}) + \mathbf{T}^T\dot{\mathbf{x}}\cdot\mathbf{n}\big) dS, \quad \text{at} \quad \mathbf{Z}(t). \tag{32}$$

[2] $\mathbf{x} \times \mathbf{T} = e_{ijk} x_j T_{kA}$

[3] We assume that the crack faces are in perfect contact.

It is obvious that eqs. (30) and (31) are the energy equation in local form and the associated jump condition along the crack, respectively. Also, eq. (32) is nothing else but the well–known, in fracture literature, energy release rate G.

4.4 The balance of pseudomomentum

The pseudomomentum (material momentum) $\mathcal{P}$ concerns changes within the material structure. Thus, it does not make sense in classical elasticity, where only the motion in physical space is studied in the absence of any rearrangement of the material configuration. Actually, this is the first additional balance law that must be considered when one studies any kind of evolution of stuctural defects. From this point of view, it is a configurational balance law. The corresponding flux is the well–known Eshelby stress tensor, which will be denoted by $\mathbf{b}$, [7]. The source term will be the so–called material forces. Here, we consider source terms distributed throughout the body (body material forces), produced by the material inhomogeneities [2] and will be denoted by $\tilde{\mathbf{f}}$. Moreover, we consider a pseudomomentum source term produced by the crack evolution. This source term is called by Maugin global material force and it is denoted by $\mathcal{F}$ [4]. Actually, in [4], we have met for the first time the idea of a global balance law for pseudomomentum. The expressions for the pseudomomentum vector field and the Eshelby stress tensor are given through the relations [1, 2, 7]

$$\mathcal{P} = -\rho \mathbf{F}^T \dot{\mathbf{x}}, \tag{33}$$

$$\mathbf{b} = W - \frac{1}{2}\rho \dot{\mathbf{x}}^2 - \mathbf{F}^T \mathbf{T}. \tag{34}$$

Finally, we postulate for pseudomomentum

$$\frac{d}{dt}\int_\Omega \mathcal{P}\, dV = \int_{\partial\Omega} \mathbf{bn}\, dS + \int_\Omega \tilde{\mathbf{f}}\, dV - \mathcal{F}, \quad \forall \Omega \in \mathcal{B}_t. \tag{35}$$

The local equations obtained by eq. (35) are given as follows

$$\frac{\partial \mathcal{P}}{\partial t} - \mathrm{Div}\mathbf{b} - \tilde{\mathbf{f}} = 0, \quad \text{for all} \quad \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{36}$$

$$[\mathbf{b}] \cdot \mathbf{m} = 0, \quad \text{for all} \quad \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}, \tag{37}$$

$$\mathcal{F} = \lim_{r\to 0} \int_{\partial D_r} \big(\mathcal{P}(\mathbf{V}\cdot\mathbf{n}) + \mathbf{bn}\big)\, dS, \quad \text{at} \quad \mathbf{Z}(t). \tag{38}$$

Eq. (36) is the equation of pseudomomentum, which holds in the smooth part of the body and eq. (37) is the associated jump condition. In addition, eq. (38) represents the material force at the crack tip, which drives the crack evolution. Thus, one can reasonably expect that this quantity, i.e., $\mathcal{F}$ should be directly related to the energy release rate, G. This is the object of the next section.

5 The energy release rate and the global material force

In this section, we derive an already known relation (see [2, 3, 4]) by means of the kinematics formulated in Section 2 of this paper. Here, the interesting point is that a relation between the energy release rate, from the one side, and the global material force and the crack propagation velocity from the other one, can be established. It is a relation which strongly resembles the relation between the power and the corresponding classical force and velocity. This means that the notion of the global material force can reasonably be considered as the dual quantity of the crack propagation velocity. We start with the energy release rate (eq. (32)):

$$G = \lim_{r\to 0} \int_{\partial D_r} \Big((W+K)(\mathbf{V}\cdot\mathbf{n}) + \mathbf{T}^T\dot{\mathbf{x}}\cdot\mathbf{n} \Big) dS.$$

Recalling the relation (8) and the assumption (9), we can write for the last term of G

$$\begin{aligned}\lim_{r\to 0}\int_{\partial D_r} \mathbf{T}^T\dot{\mathbf{x}}\cdot\mathbf{n}\; dS &= \lim_{r\to 0}\int_{\partial D_r} \mathbf{T}^T\left(\widetilde{\mathbf{V}}(\mathbf{X},t) - \mathbf{F}\mathbf{V}(t)\right)\cdot\mathbf{n}\; dS \\ &= \mathbf{U}(t)\cdot\lim_{r\to 0}\int_{\partial D_r}\mathbf{T}\mathbf{n}\; dS - \lim_{r\to 0}\int_{\partial D_r}\mathbf{T}^T\mathbf{F}\mathbf{V}(t)\cdot\mathbf{n}\; dS.\end{aligned}$$

Similarly, the term of G concerning the kinetic energy can be written

$$\begin{aligned}\lim_{r\to 0}\int_{\partial D_r}\rho(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})(\mathbf{V}\cdot\mathbf{n})\; dS &= \lim_{r\to 0}\int_{\partial D_r}\rho\dot{\mathbf{x}}\cdot\left(\widetilde{\mathbf{V}}(\mathbf{X},t) - \mathbf{F}\mathbf{V}(t)\right)(\mathbf{V}\cdot\mathbf{n})\; dS \\ &= \mathbf{U}(t)\cdot\lim_{r\to 0}\int_{\partial D_r}\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{n})\; dS - \lim_{r\to 0}\int_{\partial D_r}\big(\rho\dot{\mathbf{x}}\cdot\mathbf{F}\mathbf{V}(t)\big)(\mathbf{V}\cdot\mathbf{n})\; dS.\end{aligned}$$

By virtue of the last calculations, the energy release rate becomes

$$\begin{aligned} G &= \lim_{r\to 0}\int_{\partial D_r}\big((W+K)(\mathbf{V}\cdot\mathbf{n})+\mathbf{T}^T\dot{\mathbf{x}}\cdot\mathbf{n}\big)dS = \\ &\lim_{r\to 0}\int_{\partial D_r}\mathbf{V}\cdot\Big(\big((W-K)\mathbf{I}-\mathbf{F}^T\mathbf{T}\big)\,\mathbf{n}-\big(\rho\mathbf{F}^T\dot{\mathbf{x}}\cdot\mathbf{V}\big)\,(\mathbf{V}\cdot\mathbf{n})\Big)\,dS + \\ &\mathbf{U}(t)\cdot\lim_{r\to 0}\int_{\partial D_r}\big(\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{n})+\mathbf{T}\mathbf{n}\big)dS, \end{aligned}$$

where $\mathbf{I}$ is the identity tensor.
Taking into account eq. (25), we obtain

$$G=\mathbf{V}\cdot\lim_{r\to 0}\int_{\partial D_r}\Big(\big((W-K)\mathbf{I}-\mathbf{F}^T\mathbf{T}\big)\,\mathbf{n}-\rho\mathbf{F}^T\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{n})\Big)\,dS.$$

Finally, using eqs. (33), (34) and (38), we conclude

$$G=\mathbf{V}\cdot\lim_{r\to 0}\int_{\partial D_r}\big(\mathbf{b}\mathbf{n}+\mathcal{P}(\mathbf{V}\cdot\mathbf{n})\big)dS=\mathbf{V}\cdot\mathcal{F},$$

which is the required relation.

6 Conclusions

The main aim of this paper was the formulation of global balance laws for a fractured elastic body. To this end, we proposed a unified framework for physical and configurational balance laws. The proposed formulation gave us the possibility to obtain, at the same time, the local equations which hold in the bulk of the body, the associated jump conditions along the crack $C(t)$ and the balance equations at the crack tip. We note that some important technicalities were omitted. In a future work, we intend to treat more rigorously the proposed framework. Moreover, we are going to apply these ideas to additional configurational balance laws.

Acknowledgements

The financial support of the GSRT (General Secretariat of Research and Technology) and the Research Committee of the University of Ioannina (contracts No 61/1124 , No 62/1220) is gratefully acknowledged.

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Konfiguracione jednačine balansa za dinamički lom

UDK 531.01, 539.421

U duhu moderne mehanike kontinuuma, globalne jednačine balansa za količinu kretanja, pseudo količinu kretanja, energiju i zamah su formulisane za neko elastično telo u prisustvu pokretne prsline. Po lokalizaciji odgovarajuće jednačine za osnovni materijal i vrh prsline su jednovremeno dobijene. Preloženi okvir je podesan za izvodjenje dobro poznatih formula koje povezuju brzinu prostiranja prsline, globalnu materijalnu silu i brzinu oslobadjanja energije.

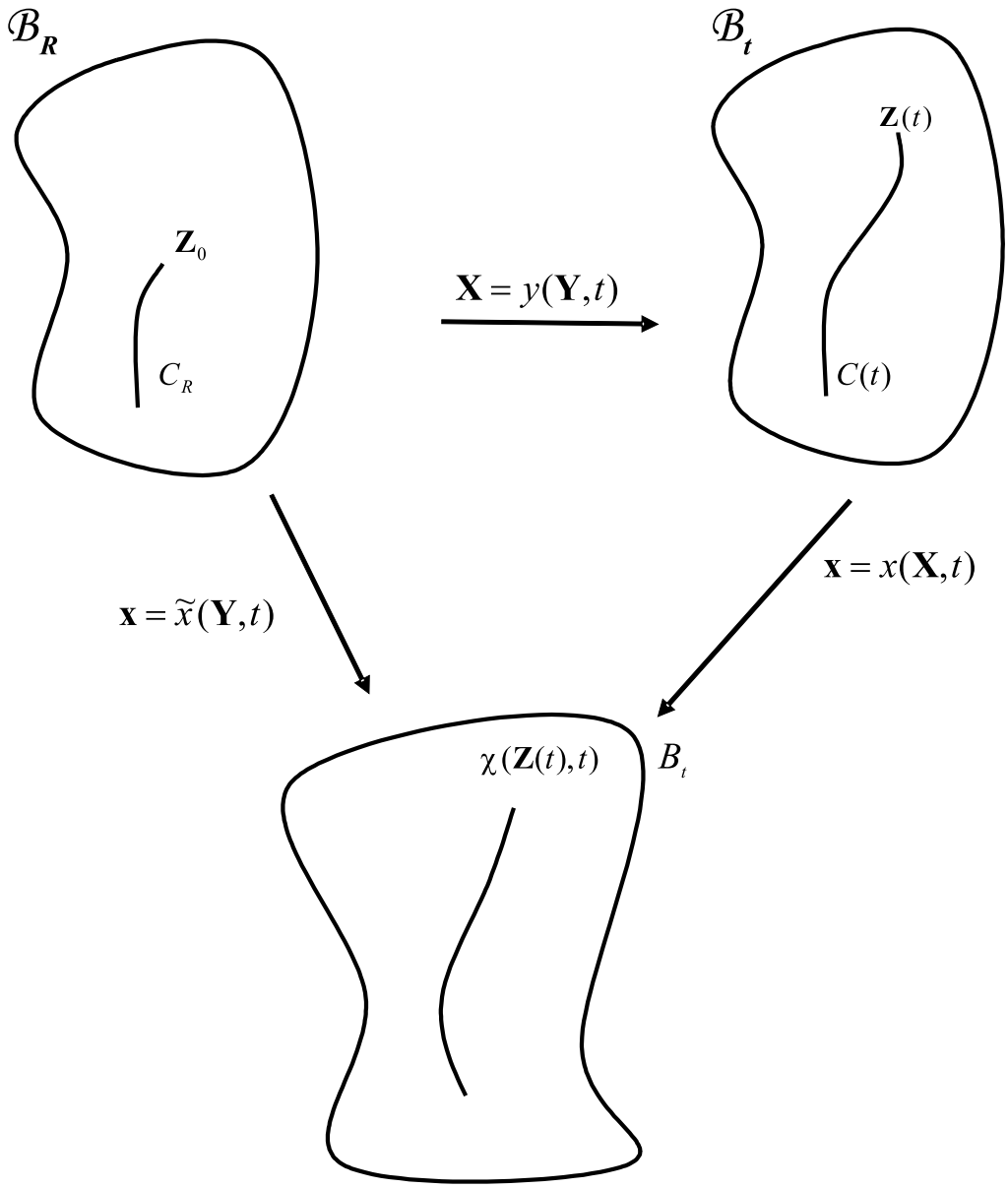
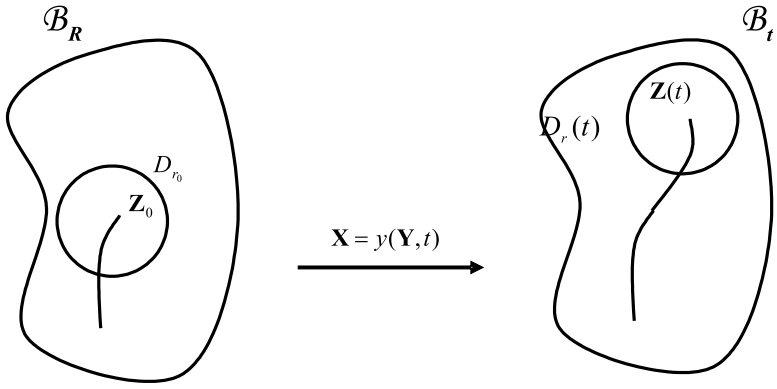


Figure 1: The configurations



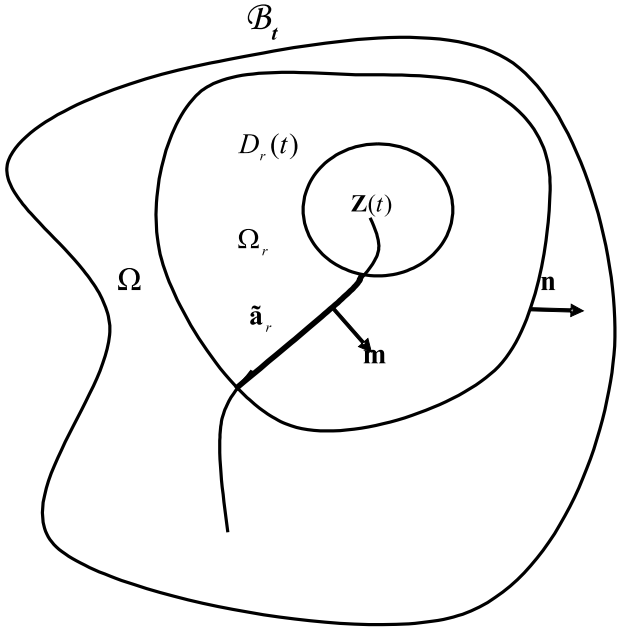


Figure 2: The "motion" of the tip disc